FINITE SUBLATTICES OF A FREE LATTICE

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Introduction. It is known that every sublattice A of a free lattice satisfies the following conditions:

(W) *For all* $a, b, c, d \in A$, *if* $ab \leqslant c + d$, *then* $ab \leqslant c$ *or* $ab \leqslant d$ *or* $a \leqslant c+d$ *or* $b \leqslant c + d$.

(SD) *For all* $u, a, b, c \in A$, *if* $u = a + b = a + c$, *then* $u = a + bc$.

(SD') *For all* $u, a, b, c \in A$, *if* $u = ab = ac$, *then* $u = a(b + c)$.

In fact, (W) is one of the four conditions used in Whitman **(4)** to characterize free lattices, and in Jónsson **(3)** it was shown that (SD) and (SD') follow from Whitman's canonical representations of elements of a free lattice.

This note is concerned with lattices that satisfy one or more of the above conditions, and especially with finite lattices that satisfy all three conditions. It turns out that under the additional assumption of finiteness these conditions have some rather strong and unexpected consequences, notably the fact that every representation of an element as a sum or a product of five or more elements is redundant.

The results presented here may be regarded as evidence in support of the conjecture that every finite lattice that satisfies Whitman's condition (W) and the special distributive laws (SD) and (SD') is isomorphic to a sublattice of a free lattice. The corresponding statement for infinite lattices is of course false, for every sublattice A of a free lattice satisfies the following conditions:

(Δ) $$\dim A \leqslant \aleph_0.$$

(N) *For each* $u \in A$ *there exists a positive integer* $n(u)$ *such that every representation of* u *as a sum or a product of more than* $n(u)$ *elements is redundant.*

In fact, (Δ) was proved in Galvin–Jónsson **(1)**, and (N) is an immediate consequence of Whitman's canonical representation. It seems highly doubtful that even these five conditions characterize the class of all sublattices of free lattices, and at the present the problem of obtaining a characterization looks rather inaccessible. In the finite case the chances of success appear considerably better, and it is hoped that in that connection the present investigations may prove helpful. But in any case, they do give a certain amount of information about the class of lattices under consideration, and for that reason should be of some independent interest.

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1. The condition (W). Applying Jónsson (**2**, Theorem 1) with $\alpha = 0$ and P equal to an unordered set X we obtain:

THEOREM 1.1. *A lattice A generated by a set X is freely generated by X if and only if (W) holds and every non-empty finite subset of X is additively and multiplicatively irredundant.*

THEOREM 1.2. *If A is a lattice that satisfies the condition (W), and if $a_1, a_2, a_3, v \in A$ are such that*

(i) $a_1 \not\leqslant a_2 + a_3 + v$, *and cyclically,*

(ii) $v \not\leqslant a_i$ *for* $i = 1, 2, 3$,

(iii) v *is multiplicatively irreducible,*

then A contains a free sublattice with three generators.

Proof. Let $b_1 = a_1 + (a_2 + v)(a_3 + v)$, and cyclically. Clearly none of the three elements b_1, b_2, b_3 is contained in the sum of the other two, and by 1.1 we therefore need only show that none of them contains the product of the other two. By symmetry it suffices to show that the condition

$$b_1 b_2 \leqslant b_3 \tag{1}$$

leads to a contradiction.

Assume (1). Observe that neither b_1 nor b_2 is contained in the sum $b_3 = a_3 + (a_1 + v)(a_2 + v)$, and that $b_1 b_2 \not\leqslant a_3$ because $v \leqslant b_1 b_2$. Therefore, by (W), $b_1 b_2 \leqslant (a_1 + v)(a_2 + v)$. In particular, $b_1 b_2 \leqslant a_1 + v$, and applying (W) again we find that

$$b_1 b_2 \leqslant a_1 \text{ or } b_1 b_2 \leqslant v \text{ or } b_1 \leqslant a_1 + v \text{ or } b_2 \leqslant a_1 + v.$$

The first inclusion is ruled out because $v \leqslant b_1 b_2$, the second is excluded because v is multiplicatively irreducible and is strictly less than b_1 and b_2, and the fourth one cannot hold because $a_2 \leqslant b_2$. Thus the third inclusion must hold, whence it follows that

$$(a_2 + v)(a_3 + v) \leqslant a_1 + v.$$

However, this is easily seen to violate (W). For, by (i), neither factor on the left is contained in the sum on the right, by (ii) the product is not contained in a_1, and by (i) and (iii) the product is not contained in v. Thus (1) leads to a contradiction, and the proof is complete.

COROLLARY 1.3. *If A is a finite lattice that satisfies (W), then every representation of an element of A as a sum, or a product, of more than four elements is redundant.*

Proof. If $u = a_1 + a_2 + \ldots + a_n$ with $n \geqslant 5$ were irredundant, then 1.2(i)–(iii) would be satisfied with $v = a_4 + \ldots + a_n$. As regards (iii), this is true because of the fact that in a lattice satisfying (W) every element is either additively or multiplicatively irreducible. Thus A would contain a free sublattice generated by a three-element set, which is impossible because such a lattice is infinite.

COROLLARY 1.4. *A finite lattice A that satisfies (W) and (SD′) contains at most four atoms.*

Proof. If $p_1, p_2, \ldots, p_n$ are distinct atoms of A, then $p_i p_j = 0$ for $i \neq j$, therefore $p_i(p_0 + \ldots + p_{i-1} + p_{i+1} + \ldots + p_n) = 0$ by (SD'). Consequently the atoms p_i form an additively irredundant set, so that $n \leqslant 4$ by 1.3.

2. Refinements and canonical representations. Given two sum-representations of a lattice element u,

$$u = \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j,$$

the first is said to be a *refinement* of the second if and only if each a_i is contained in some b_j. A sum-representation of u is said to be canonical if and only if it is irredundant and is a refinement of every other sum-representation of u. It is easy to show that two canonical sum-representations of the same element are identical except for the order of the terms. When applied to free lattices this notion therefore agrees with the concept of a canonical representation introduced in Whitman **(4)**.

THEOREM 2.1. *If u is an element of a lattice A, then the following conditions are equivalent:*

(i) *For all $a, b, c \in A$, $u = a + b = a + c$ implies that $u = a + bc$.*

(ii) *For any positive integers m, n, and for all $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in A$,*

$$u = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \text{implies that} \quad u = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j.$$

(iii) *Any two sum-representations of u have a common refinement.*

If u has a canonical sum-representation, then (i)–(iii) *hold. If A is finite and if* (i)–(iii) *hold, then u has a canonical sum-representation.*

Proof. Assuming (i), we shall prove that the following statement $P(m, n)$ holds whenever m and n are positive integers: *For all $v, a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in A$, if*

$$u = v + \sum_{i=1}^{m} a_i = v + \sum_{j=1}^{n} b_j,$$

then

$$u = v + \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j.$$

Since $P(1, 1)$ is precisely the hypothesis (i), we assume that $m + n > 2$, and that $P(m', n')$ holds whenever $m' + n' < m + n$.

First assume that $m = 1$, and let $b'_n = b_1 + b_2 + \ldots + b_{n-1}$. Then

$$u = (v + b'_n) + a_1 = (v + b'_n) + b_n,$$

so that by (i), $u = v + b'_n + a_1 b_n$. Therefore

$$u = (v + a_1 b_n) + a_1 = (v + a_1 b_n) + \sum_{j=1}^{n-1} b_j,$$

and it follows by $P(1, n-1)$ that

$$u = v + a_1 b_n + \sum_{j=1}^{n-1} a_1 b_j = v + \sum_{j=1}^{n} a_1 b_j.$$

Now suppose $m > 1$, and let $a'_m = a_1 + a_2 + \ldots + a_{m-1}$. Then

$$u = v + a'_m + a_m = v + a'_m + \sum_{j=1}^{n} b_j,$$

and it follows by $P(1, n)$ that

$$u = v + a'_m + \sum_{j=1}^{n} a_m b_j.$$

Therefore

$$u = \left(v + \sum_{j=1}^{n} a_m b_j\right) + \sum_{i=1}^{m-1} a_i = \left(v + \sum_{j=1}^{n} a_m b_j\right) + \sum_{j=1}^{n} b_j,$$

and we infer by $P(m-1, n)$ that

$$u = v + \sum_{j=1}^{n} a_m b_j + \sum_{i=1}^{m-1} \sum_{j=1}^{n} a_i b_j = v + \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j.$$

Thus $P(m, n)$ holds. By induction, $P(m, n)$ holds for all positive integers m and n, and therefore (ii) holds.

Clearly (ii) implies (iii). If (iii) holds, and if $u = a + b = a + c$, then these two representations of u have a common refinement.

$$u = \sum_{i=1}^{m} d_i,$$

Since each d_i is either contained in a or else is contained in both b and c, it follows that $u = a + bc$. Thus (iii) implies (i).

If u has a canonical sum-representation, then this is a common refinement of all sum-representations of u, and therefore the equivalent conditions (i)–(iii) hold.

If A is finite, then u has only finitely many irredundant sum-representations, and if (iii) holds, then these have a common refinement which may also be taken to be irredundant, and is therefore easily seen to be canonical.

COROLLARY 2.2. *If A is a finite lattice that satisfies (SD), then every element of A has a canonical sum-representation.*

3. The lattice B_4. In Corollaries 1.3 and 1.4 the number four in the conclusion cannot be replaced by a smaller number. In fact, in a free lattice with four generators x_1, x_2, x_3, x_4, the four atoms $p_1 = x_2 x_3 x_4, \ldots$ generate a lattice B_4 of order 22. This fact will be established below, but the main purpose of the present section and the next one is to show that this particular lattice plays a much more important role than as a mere counter-example.

LEMMA 3.1. *If a finite lattice A satisfies (W) and (SD'), and if the elements $a_1, a_2, a_3, v \in A$ are such that*

(i) $a_1 \not\leq a_2 + a_3 + v$, *and cyclically,*

(ii) $v \not\leq a_i$ *for* $i = 1, 2, 3$,

then

$$(a_2 + a_3 + v)(a_3 + a_1 + v)(a_1 + a_2 + v) = v.$$

Proof. Let $b_i = a_i + v$ for $i = 1, 2, 3$, $w_1 = (a_1 + v)(a_2 + a_3 + v)$ and cyclically, and $w = w_1 + w_2 + w_3$. Then $b_1 \not\leq b_2 + b_3 + w$ and cyclically, and it follows by 1.2 that either one of the elements b_i contains w, or else w is multiplicatively reducible and hence additively irreducible.

If $w \leqslant b_1$, then

$$(1) \qquad (a_2 + v)(a_3 + a_1 + v) \leqslant a_1 + v, \quad (a_3 + v)(a_1 + a_2 + v) \leqslant a_1 + v.$$

In the first of these formulae, neither factor on the left is contained in $a_1 + v$, and their product is not contained in a_1 because $v \not\leq a_1$. Therefore, and by similar reasoning using the second formula in (1),

$$v = (a_2 + v)(a_3 + a_1 + v) = (a_3 + v)(a_1 + a_2 + v).$$

Hence the desired formula follows by the dual of 2.1.

If w is additively irreducible, then one of the summands w_i contains the other two, say $w_2 \leqslant w_1$ and $w_3 \leqslant w_1$. Since $w_1 \leqslant b_1$, this implies that $w \leqslant b_1$, and the present case reduces to the one already considered.

LEMMA 3.2.* *If A is a lattice that satisfies (W), and if A is generated by an additively irredundant four-element set $\{p_1, p_2, p_3, p_4\}$ such that*

$$\text{(i)} \qquad (p_2 + p_3 + p_4)(p_3 + p_4 + p_1)(p_4 + p_1 + p_2) = p_4,$$

and cyclically, then the order of A is 22, and A is isomorphic to the lattice B_4 generated by the atoms in a free lattice with four generators.

Proof. Let

$$z = p_1 p_2 p_3 p_4, \qquad u = p_1 + p_2 + p_3 + p_4,$$
$$q_1 = p_2 + p_3 + p_4, \text{ and cyclically,}$$
$$a_{i,j} = p_i + p_j, \; b_{i,j} = q_i q_j \text{ for } i, j = 1, 2, 3, 4 \text{ with } i \neq j.$$

*In the original versions of Lemma 3.2 and Theorem 3.3 the lattice A was assumed to satisfy (SD) and (SD'). We are indebted to the reviewer for pointing out that in the proof of the lemma neither condition is needed, and that in the theorem only (SD') need therefore be assumed.

We shall show that these 22 elements satisfy the addition and multiplication tables indicated in Figure 1.

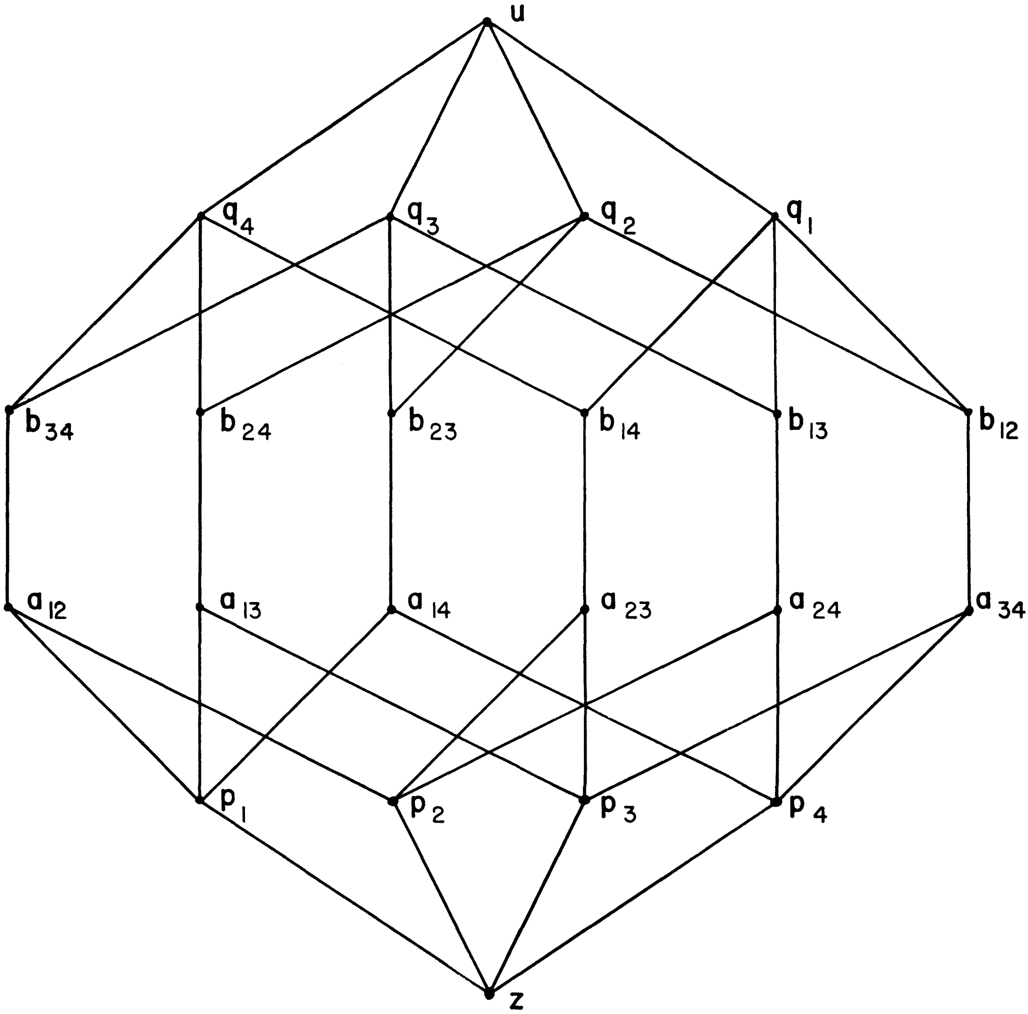

FIGURE 1. The lattice B_4.

It will be assumed throughout that i, j, k, l are distinct indices ranging from 1 to 4. By (i), $p_i p_j \leqslant p_k$, hence $p_i p_j = z$. Also $p_i q_i \leqslant p_j$, hence $p_i q_i = z$. Since $a_{j,k} \leqslant q_i$ and $b_{i,k} \leqslant q_i$, this yields $p_i a_{j,k} = z$ and $p_i b_{i,k} = z$. Inasmuch as $p_i \leqslant a_{i,k}$ and $p_i \leqslant b_{j,k}$, this verifies the multiplication table for the case when one of the factors is p_i.

By (i), $a_{i,j} q_i = p_j$, while $a_{i,j} \leqslant q_k$. Therefore $a_{i,j} b_{i,j} = a_{i,j} q_i q_j = p_j q_j = z$ and $a_{i,j} b_{i,k} = a_{i,j} q_i q_k = p_j q_k = p_j$, while $a_{i,j} \leqslant b_{k,l}$. Again by (i), $q_i q_j q_k = p_l$, whence it follows that $q_i q_j q_k q_l = z$. Therefore $p_i \leqslant a_{i,j} a_{i,k} \leqslant q_k q_l q_j q_l = p_i$,

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$a_{i,j}a_{i,k} = p_i$, and $a_{i,j}a_{k,l} \leqslant q_kq_lq_iq_j = z$, $a_{i,j}a_{k,l} = z$. This verifies the multiplication table for the case when one of the factors is $a_{i,j}$.

By definition, $q_iq_j = b_{i,j}$, and since it has already been observed that $q_iq_jq_k = p_l$ and $q_iq_jq_kq_l = z$, it follows that $b_{i,j}q_k = b_{i,j}b_{i,k} = p_l$ and $b_{i,j}b_{k,l} = z$, while $b_{i,j} \leqslant q_i$. This completes the checking of the multiplication table.

To verify the addition table we need only observe that the dual of the hypothesis holds with p_i replaced by q_i, and that the duals of the definitions of the 22 elements are satisfied if we interchange z and u, p_i and q_i, $a_{i,j}$ and $b_{i,j}$.

To show that the 22 elements are actually distinct, observe first that $z < p_i < a_{i,j} < b_{k,l} < q_k < u$. Therefore, if v is any one of the elements z, p_j, $a_{k,l}$, $b_{i,k}$, q_l, then $p_iv = z < p_i$, so that $p_i \not\leqslant v$. Also, if v is any element other than $a_{i,j}$, $b_{k,l}$, q_k, q_l, then $a_{i,j}v \leqslant p_i < a_{i,j}$, so that $a_{i,j} \not\leqslant v$. Similarly, if v is not one of the elements $b_{k,l}$, q_k, q_l, u, then $b_{k,l}v < b_{k,l}$, and hence $b_{k,l} \not\leqslant v$, and if $v \neq q_k, u$, then $q_kv < q_k$ and hence $q_k \not\leqslant v$. Therefore only the indicated inclusions hold, and the 22 elements are distinct.

We have shown that the hypothesis determines A up to isomorphism, and in order to prove that $A \cong B_4$, it therefore suffices to show that B_4 satisfies this hypothesis. If the generators of the free lattice are x_1, x_2, x_3, x_4, then its atoms, the generators of B_4, are $p_1 = x_2x_3x_4$, and cyclically. These four elements form an additively irredundant set because $p_i + p_j + p_k \leqslant x_l$ and $p_l \not\leqslant x_l$. Furthermore, the three factors on the left in (i) are contained in x_1, x_2, and x_3, respectively, and their product is therefore contained in p_4. Consequently (i) holds.

THEOREM 3.3. *If A is a finite lattice that satisfies (W) and (SD′), and if A contains a four-element subset that is additively irredundant, then A contains a sublattice that is isomorphic to B_4.*

4. The decomposition theorem. It will now be shown that under the hypothesis of 3.3 the given lattice A can be expressed as the union of certain sublattices, and that A is isomorphic to a sublattice of a free lattice if and only if each of the summands is isomorphic to a sublattice of a free lattice. This result can therefore be regarded as a reduction of the embedding problem. On the other hand, the process is reversible, for A is uniquely determined by the given sublattices. These results therefore provide us with a new method for constructing finite sublattices of a free lattice.

For the present purpose it is convenient to regard the empty set as a lattice, and as a sublattice of every lattice.

THEOREM 4.1. *Suppose A is a finite lattice that satisfies (W), (SD), and (SD′) and assume that the lattice B_4 in Figure 1 is a sublattice of A. Let*

$$A' = A - \{x | x \in A \text{ and } z < x < u\},$$
$$C_{i,j} = \{x | x \in A \textit{ and } a_{i,j} < x < b_{k,l}\} \textit{ for } \{i, j, k, l\} = \{1, 2, 3, 4\}.$$

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Then A' is a sublattice of A, each of the sets $C_{i,j}$ is a sublattice of A, and

$$A = A' \cup B_4 \cup C_{1,2} \cup C_{1,3} \cup C_{1,4} \cup C_{2,3} \cup C_{2,4} \cup C_{3,4}.$$

Furthermore,

(i) *If $\{i, j, k, l\} = \{1, 2, 3, 4\}$, then*

$x \in C_{i,j}$ *and* $y \in C_{i,k}$ *implies that* $x + y = q_l$ *and* $xy = p_i$,

$x \in C_{i,j}$ *and* $y \in C_{k,l}$ *implies that* $x + y = u$ *and* $xy = z$.

(ii) *If $x \in A'$ and $z \leqslant y \leqslant u$, then*

$x \not\leq z$ *implies that* $x + y = x + u$,

$x \not\geq u$ *implies that* $xy = xz$.

Proof. First observe that p_i covers z in A. In fact, p_i covers some element $d \geqslant z$, and it follows that $p_i \not\leq d + p_j$, for otherwise we would have $a_{i,k}a_{i,l} = p_i \leqslant d + p_j$, in violation of (W). Consequently $p_i(d + p_j) = d$. Similarly $p_i(d + p_k) = d$ and $p_i(d + p_l) = d$, and it follows by (SD') that $p_i(d + q_i) = d$. We now apply 1.2 with $a_1 = p_i$, $a_2 = p_k$, $a_3 = p_l$, and $v = d + p_j$. Since the conditions (i) and (ii) of 1.2 hold, but the conclusion fails, the condition (iii) must fail. Thus v is multiplicatively reducible, and is therefore additively irreducible. Inasmuch as $p_j \not< d$, this implies that $d \leqslant p_j$, $d \leqslant p_i p_j$, $d = z$. Thus p_i covers z. Dually, q_i is covered by u.

Next we show that $a_{i,j}$ covers p_i in A. If $p_i \leqslant d < a_{i,j}$, then $p_j \not\leq d$, $p_j d < p_j$, $p_j d = z$. Applying (SD') to this equation and to the equation $p_j q_j = z$, we find that $p_j(d + q_j) = z$, so that $d + q_j < u$. Since q_j is covered by u, this implies that $d \leqslant q_j$. Therefore $d \leqslant a_{i,j}q_j = p_i$, $d = p_i$. Thus $a_{i,j}$ covers p_i and dually $b_{i,j}$ is covered by q_i.

Now consider an element $v \in A - B_4$ with $z < v < u$. The lattice quotient u/z cannot have more than four atoms, and since p_1, p_2, p_3, p_4 are atoms of u/z, one of them must be contained in v, say $p_i \leqslant v$. If the remaining three atoms are not contained in v, then $p_i \leqslant a_{i,j}v < a_{i,j}$, hence $a_{i,j}v = p_i$, and similarly $a_{i,k}v = p_i$ and $a_{i,l}v = p_i$. Applying (SD') we therefore infer that

$$p_i = v(a_{i,j} + a_{i,k} + a_{i,l}) = vu = v,$$

contrary to our assumption that $v \notin B_4$. Thus at least two atoms p_i and p_j must be contained in v, and therefore $a_{i,j} \leqslant v$. Dually, v must be contained in one of the elements $b_{s,t}$, and since the only one that contains $a_{i,j}$ is $b_{k,l}$, it follows that $v \leqslant b_{k,l}$. Therefore $v \in C_{i,j}$.

Thus we see that A is the union of A' and B_4 and of the six sets $C_{i,j}$. Since $a_{i,j}$ is additively reducible, hence multiplicatively irreducible, and since, dually, $b_{i,j}$ is additively irreducible, the set $C_{i,j}$ is a sublattice of A. To complete the proof of the first part of the theorem it therefore remains only to show that A' is a sublattice of A. Actually we shall show that $A - (u/z)$ is a sublattice of A, and after (ii) has been proved, it will follow that A' is also a sublattice of A. By duality we need only verify that $A - (u/z)$ is closed under addition. This is equivalent to the assertion that, for all $c, d \in A$,

$$z \leqslant c + d \leqslant u \text{ implies that } z \leqslant c \text{ or } z \leqslant d.$$

Assume that this fails. Thus $z = q_1q_2q_3q_4$ is contained in the sum $c + d$, but is contained in neither summand, and it follows that one of the factors q_i must be contained in $c + d$. Inasmuch as u covers q_i, this implies that $c + d = u$ or $c + d = q_i$.

If $c + d = u$, then c and d cannot both be contained in q_i, and we may assume that $c \not\leqslant q_i$. Therefore $q_i + c = u$ and, since $q_i + p_i = u$, it follows by (SD) that $q_i + p_i c = u$. Since, by hypothesis, $z \not\leqslant c$, we have $p_i \not\leqslant c$ so that $p_i c < p_i$. But p_i is multiplicatively reducible and therefore additively irreducible, and p_i covers z. Therefore $p_i c \leqslant z \leqslant q_i$, which is clearly a contradiction.

If $c + d = q_i$, then c and d cannot both be contained in $b_{i,j}$, and we may assume that $c \not\leqslant b_{i,j}$. Therefore $b_{i,j} + c = q_i$, and together with $b_{i,j} + p_j = q_i$ this gives $b_{i,j} + p_j c = q_i$. But, as before, $p_j c \leqslant z$, and we obtain $b_{i,j} = q_i$, a contradiction.

If $x \in C_{i,j}$ and $y \in C_{i,k}$, then $a_{i,j} < x < b_{k,l}$ and $a_{i,k} < y < b_{j,l}$. Since $a_{i,j} + a_{i,k} = q_l = b_{k,l} + b_{j,l}$ and $a_{i,j}a_{i,k} = p_i = b_{k,l}b_{j,l}$, it follows that $x + y = q_l$ and $xy = p_i$. If $x \in C_{i,j}$ and $y \in C_{k,l}$, then $a_{i,j} < x < b_{k,l}$ and $a_{k,l} < y < b_{i,j}$. Since $a_{i,j} + a_{k,l} = u = b_{k,l} + b_{i,j}$ and $a_{i,j}a_{k,l} = z = b_{k,l}b_{i,j}$, it follows that $x + y = u$ and $xy = z$. Thus (i) holds.

Finally we show that if $x \in A'$ and $x \not\leqslant z$, then $u \leqslant x + z$. From this the first part of (ii) readily follows, and the second part can then be inferred by duality.

If $u \not\leqslant x + z$, then $u(x + z) < u$, and $u(x + z)$ is contained in some element that is covered by u. Now u covers each of the elements q_i, and by the dual of 1.4, u cannot cover more than four elements. Therefore $u(x + z)$ is contained in some q_i. It follows that $p_i(x + z) = z$. Together with $p_i q_i = z$ this yields $p_i(x + q_i) = z$, which clearly implies that $u \not\leqslant x + q_i$. Since u is the unique element that covers q_i, we infer that $x \leqslant q_i$. Thus $z < x + z \leqslant q_i$ and consequently one of the elements p_j must be contained in $x + z$. Therefore

$$u = q_j + p_j = q_j + x, \text{ hence } u = q_j + p_j x,$$

and we find that $p_j \leqslant x$. But since $x \leqslant q_i$, this contradicts the hypothesis that $x \in A'$.

THEOREM 4.2. *Under the hypothesis of Theorem* 4.1, *if* A' *and all the lattices* $C_{i,j}$ *are isomorphic to sublattices of free lattices, then so is* A.

Proof. First observe that u/z is isomorphic to a sublattice of a free lattice. In fact, consider a free lattice F with infinitely many generators, and let f be an isomorphism of B_4 into F. By Jónsson (**3**, Lemma 2.3) each of the intervals $f(b_{k,l})/f(a_{i,j})$ contains as a sublattice a free lattice with infinitely many generators, and therefore there exists an isomorphism $f_{i,j}$ of $C_{i,j}$ into this interval. Let g be the function that agrees with f on B_4 and with $f_{i,j}$ on $C_{i,j}$. It is then easy to observe that g maps u/z isomorphically into F.

Now consider an isomorphism h of A' into F. Then $h(u)/h(z)$ contains as a sublattice a free lattice with infinitely many generators and we may therefore assume that the above function g maps u/z into $h(u)/h(z)$. Let k be the function that agrees with h on the lattice $A - (u/z) = A' - \{u, z\}$, and with g on u/z.

Clearly k maps $A - (u/z)$ and u/z isomorphically into F. Now consider $x \in A - (u/z)$ and $y \in u/z$. If $x \leqslant z$, then $x \leqslant y$ and

$$k(x) = h(x) \leqslant h(z) \leqslant g(z) \leqslant g(y) = k(y),$$

but if $x \not\leqslant z$, then

$$x + y = x + z = x + u,$$
$$h(x) + h(z) = h(x) + h(u) = h(x + u),$$

and since $h(z) \leqslant k(y) \leqslant h(u)$, it follows that

$$k(x + u) = k(x) + k(y).$$

Thus in either case $k(x + y) = k(x) + k(y)$. Therefore, and by duality, k is an isomorphism.

5. Dimension and order. In Jónsson **(3)** it was shown that if a finite dimensional lattice A satisfies the conditions (W), (SD), and (SD'), then it is finite. In fact, the argument used there shows that if the dimension of A is n, then the order of A is less than $2 \cdot (n!)$. Actually, this estimate can be considerably improved:

THEOREM 5.1. *Let f_n be the maximum order of an n dimensional lattice that satisfies (W), (SD), and (SD'). Then** $(\sqrt{2})^n \leqslant f_n \leqslant 2^n$.

Proof. Suppose A_n is an n-dimensional lattice of order f_n that satisfies the given conditions. By a method given in Jónsson **(3,** Theorem 2.5**)** we can then construct a lattice A_{n+2} of dimension $n + 2$ and order $2f_n + 2$ that also satisfies these conditions. Hence $f_{n+2} > 2f_n$, and since $f_0 = 1$ and $f_1 = 2$, this yields the lower bound for f_n.

If p is an atom of A_n, then each of the sets

$$\{x | p \leqslant x \in A_n\}, \qquad \{x | x \in A_n \text{ and } px = 0\}$$

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*The lower limit for f_n was obtained by Patrick R. Ahern. By a more involved argument than the one given here, and using various results contained in this paper, the second author has obtained a considerably sharper upper bound. In fact, he has shown that

$$\limsup_{n\to\infty} \sqrt[n]{f_n} \leqslant \rho$$

where ρ is the positive root of the equation $\rho^5 - \rho^4 - \rho^2 - 1 = 0$. Easy calculations show that $\rho < 1.571$. There is considerable evidence to indicate that

$$\lim_{n\to\infty} \sqrt[n]{f_n} = \sqrt{2}.$$

is a sublattice of A_n. In the case of the latter set, this is a consequence of (SD'). Since the dimension of each of these sublattices is at most $n - 1$, we infer that $f_n \leqslant 2f_{n-1}$. This gives the desired upper bound for f_n.

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