The Tritangent Circles of a Circular Quartic Curve

By H. W. Richmond, F.R.S., King's College, Cambridge.

(*Received* 2*nd December*, 1931. *Read* 4*th December*, 1931.)

§ 1. In a recent paper with this title Prof. W. P. Milne[1] has discussed the properties of the conics which pass through two fixed points of a plane quartic curve and touch the curve at three other points. In dealing with a numerous family of curves such as this it is very desirable to have a scheme of marks or labels to distinguish the different members of the family; Hesse's notation for the double tangents of a C_4 illustrates this. By using another line of approach to the subject, by projecting the curve of intersection of a quadric and a cubic surface from a point at which (under exceptional circumstances) the surfaces touch, I find that a fairly simple notation for the 64 conics, in harmony with that for the bitangents, can be obtained. This paper, let it be said, from start to finish is no more than an adaptation of results known for the sextic space-curve referred to; it will be sufficient therefore to state results with short explanations.

§ 2. The notation for the 28 bitangents of a plane C_4 and the modifications that arise when the curve acquires a node are well known; they must be quoted here, but as briefly as possible. There are 63 families of "contact-conics" which touch the curve at four points: in each family 6 of the conics break up into a pair of lines, bitangents of the curve. There are in all 28 bitangents, each distinguished by a label formed of two out of eight symbols, 1, 2, 3, 4, 5, 6, 7, 8; thus 12, ... 36, ... 78. The symbols may of course be permuted in any way, but this is not enough. It would appear, but it is not true, that a pair of double tangents, 24 and 48, in which a symbol 4 is repeated must differ from a pair, 24 and 58, in which the symbols are all different. Permutations do not remove the discrepancy in the notation; a further rule is required:—Cayley's

[1] *Journal of the London Mathematical Society*, 6 (1931), 90.

rule of the "bifid substitution." In the families of contact-conics the couples of bitangents which rank as degenerate conics of the family are of two types—

either (13 | 23) (14 | 24) (15 | 25) (16 | 26) (17 | 27) (18 | 28)
or (12 | 34) (13 | 24) (14 | 23) (56 | 78) (57 | 68) (58 | 67).

The first family of conics may be called the (12) family; there are 28 families of this type. The second family may be called the (1234 | 5678) family; there are 35 families of this type. The effect of an interchange of two of the eight symbols, for example 1 and 2, is to interchange the bitangents which are coupled together as degenerate conics of the (12) family, leaving the other sixteen bitangents untouched. The rule of the bifid substitution does the same for families of the other type. The eight symbols are cleft into two sets of four; the sixteen bitangents whose labels are made up of a symbol out of each set are untouched; the other six couples are interchanged. Any geometrical property of the bitangents persists after an interchange of symbols or a bifid substitution; the chief property is that the eight points of two conics of the same family lie on a conic.

When a C_4 has a node it has only 16 bitangents; twelve have given place to six tangents from the node. It can be seen what has happened and how the notation should be modified. Two of the symbols, let us say 7 and 8, become identical; the six couples of bitangents in the (78) family, 17 and 18, 27 and 28 ...., have become coincident and form the six tangents from the node, which may be called 1, 2, 3, 4, 5, 6; the other sixteen bitangents remain as bitangents of the nodal curve. Fifteen of them are denoted by two of the symbols 1, 2, 3, 4, 5, 6; the other, originally 78, is now denoted by a new symbol. The sets of four double tangents whose contacts lie on a conic, the couples which belong to each family on contact-conics, etc., can be obtained without difficulty. It does not seem necessary to carry this further.[1]

§ 3. The properties of the sextic curve which is common to a quadric and a cubic surface resemble those of the plane quartic. Associated with the curve are 255 families of "contact-quadrics," surfaces which touch the curve at six points. In each family there are 28 quadrics which break up into a pair of planes, tritangent

[1] See *Proceedings of the Edinburgh Mathematical Society* (2), 1 (1927), 31.

planes of the curve. There are 120 tritangent planes in all, and the twelve points of contact of any two quadrics (proper or composite) from the same family lie on a quadric, other than the one which passes through the whole curve. Thanks chiefly to the investigations of Pascal,[1] we have a simple notation for the tritangent planes, closely resembling that of the bitangents. From ten symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, three may be selected in 120 ways; and one set of three symbols can be assigned to each tritangent plane in such a manner that the 28 couples of planes in any family are represented in one of two ways, just as was the case with the bitangents. The first is simple; the couples are

$$(134 \mid 234)\ (135 \mid 235)\ \ldots\ (1rs \mid 2rs)\ \ldots\ \ \ldots\ (190 \mid 290),$$

and the family of quadrics may obviously be called the (12) family. The other type is less easy to describe, but again it depends upon a bifid cleavage of the ten symbols into two groups, this time of four and six. Consider the expression, showing a division into four and six,

$$(1234 \mid 567890).$$

In the family of contact-quadrics which it serves to describe, the couples of tritangent planes are either two like 578 and 690 which have no common symbol and between them include all the six symbols, or else two like 813 and 824 which have one symbol out of the six common and four others which include all the four. The 255 families of conics are made up of 45 of the first type and 210 of the second. Every two tritangent planes form a couple associated together in one family and no other. Also there is a rule,[2] closely resembling Cayley's rule of the bifid substitution, which derives from any set of tritangent planes possessing a property other sets possessing the same property. Permutation of the symbols derives others.

§ 4. For our purpose the sextic curve must have a double point, and its properties are modified in much the same way as those of the plane quartic were in § 2. The quadric and cubic surfaces which intersect in the curve have the same tangent plane at the double point; by combining their equations we replace the cubic

[1] *Repertorium d. höheren Math.* II, 2, 961 : *Encyklopädie d. Math. Wiss.* III, 2, 1407.

[2] F. Bath, *Journal of the London Mathematical Society*, 3, 84.

surface by another which has a conical point there. With co-ordinates t, x, y, z, the double point being $(1, 0, 0, 0)$, the quadric and cubic surfaces have equations

$$t\, u_1(x, y, z) = u_2(x, y, z); \qquad t\, v_2(x, y, z) = v_3(x, y, z);$$

and the projection of their intersection is the quartic curve whose equation is

$$u_1 v_3 = u_2 v_2,$$

a quite general quadric on which lie the two points at which u_1 and u_2 vanish. It makes for brevity if we follow Milne and use metrical terms which apply to the case when these two points are I and J, the circular points, but bear in mind the fact that properties are projective. Consider the points cut out on the sextic curve by (i) a plane through T the double point, (ii) a plane not passing through T, (iii) a quadric passing through T, (iv) a quadric not passing through T. They project into sets of points on the quartic as follows; (i) four collinear points, (ii) six concyclic points, (iii) ten points on a circular cubic curve, (iv) twelve points on a bicircular quartic. In each case the locus on which the points lie is the projection of the intersection of the plane or quadric with the quadric on which the sextic space-curve lies: and in each case these sets of points may coincide two by two, when the plane is a tritangent plane, or the quadric a contact-quadric of the space-curve.[1] The planes lead to (i) the 28 bitangents, (ii) Milne's 64 tritangent circles; the quadrics may break up into two of the planes.

§5. Suppose now that two of the ten symbols of §3, for example 9 and 0, become indistinguishable. In one family of contact-quadrics, the (90) family, the 56 planes which are coupled in pairs to form degenerate quadrics of the family, become 28 pairs of coincident planes. They are planes whose labels include one or other of the symbols 9 and 0; the remaining 64 planes, whose labels contain both or neither of these two symbols, remain as proper tritangent planes. We conclude that the 28 coincident pairs are tritangent planes which pass through the double point and are

[1] When the plane or quadric passes through T, T is counted as a point of contact. Analogy here would suggest that two tritangent planes or contact-quadrics have become coincident, and the notation bears this out. Or we may visualize two such planes or quadrics touching a small oval on opposite sides and coinciding when the oval has shrunk to nothing.

projected into the double tangents of the quartic, while the 64 lead to the 64 tritangent circles of Milne's paper. The symbols 9 and 0 are now dispensed with (*cf.* § 3 at end), but it is not necessary to introduce a new symbol: eight of the tritangent planes (190), (290), ... (890), and the tritangent circles to which they lead are represented by a single symbol, 1, 2, ... 8; the others are represented as before by a set of three out of these symbols. The 28 bitangents, when the 9 and the 0 are dropped, are labelled by two out of these symbols; and when the sets of four whose eight points of contact lie on a conic are examined it is plain that the bitangents are denoted in the usual manner. Thus a notation for Milne's 64 tritangent circles is built on to the classical notation of Hesse for the bitangents. Eight of the circles are denoted by the single symbols 1, 2, 3, 4, 5, 6, 7, 8; the rest are denoted by sets of three of these symbols.

The other 254 families of contact-quadrics of the sextic curve are found to fall into two categories. In the original form of § 3, an interchange of 9 and 0 sometimes changes a family into a different family; when 9 and 0 become indistinguishable these two families coincide. In these, every composite member of the family consists of a tritangent plane which passes through the double point of the curve and another which does not; in fact all quadrics of the family pass through the double point. In the second category are the families which are unaffected by interchange of 9 and 0; in these 16 couples of planes are proper tritangent planes, and the other 12 coincide in pairs into six pairs of planes through the double point. When projected they give six pairs of bitangents which are the composite conics of one family of conics out of the 63 spoken of in section 2. Examples of families belonging to the two categories are these:—

First Category (19); which is identical with (10):
(1239 | 456780); which is identical with (1230 | 456789).

Second Category (12); (1234 | 567890); (1290 | 345678).

It is now possible to tabulate the sets of four tritangent circles whose twelve points of contact lie on a bicircular quartic, and the sets of two tritangent circles and two bitangent lines whose ten points of contact lie on a circular cubic. For the bitangents it is sufficient to give the family of contact-conics to which the two belong, because any other pair of bitangents included in that family will serve equally well. In the Tables, a, b, c, d, e, f, g, h stand for the symbols 1, 2, 3, 4, 5, 6, 7, 8 in any order.

TABLE I.

SETS OF FOUR TRITANGENT CIRCLES WHOSE POINTS OF CONTACT LIE ON A BICIRCULAR QUARTIC.

(1)	(abd)	(abe)	(acd)	(ace)	(4)	(abc)	(ade)	(afg)	(h)
(2)	(abc)	(def)	(abd)	(cef)	(5)	(abc)	(ade)	(fgh)	(a)
(3)	(abc)	(aef)	(dbf)	(dec)	(6)	(acd)	(bcd)	(a)	(b)

TABLE II.

SETS OF TWO BITANGENT LINES AND TWO TRITANGENT CIRCLES WHOSE POINTS OF CONTACT LIE ON A CIRCULAR CUBIC.

Bitangents of the family (ab)
Tritangent Circles

(7)	(acd)	(bcd)
(8)	(cde)	(fgh)
(9)	(abc)	(c)
(10)	(a)	(b)

Bitangents of the family $(abcd \mid efgh)$
Tritangent Circles.

(11)	(abe)	(cde)
(12)	(abc)	(d)

§6. From any one tritangent conic of a plane quartic curve, *i.e.* a conic which touches the quartic at three points and cuts it at two others (which it will be convenient to denote by H and K), it is known to be possible to derive all the conics which cut the curve at H and K and touch it at three further points. For a cubic drawn through H, K, the three points of contact of the first tritangent conic and the four points of contact of any two bitangents or any contact-conic out of a family of contact-conics, cuts out a set of three further points which are the points of contact of one of the conics wanted; and as there are 63 families of contact-conics, 63 other conics are found. From the results embodied in the second Table, the conics can be named in such a way that all the results in the two Tables hold good.

Let the first tritangent conic be denoted by (1), and let a, b, c, d, e, f, g denote the symbols 2, 3, 4, 5, 6, 7, 8 in any order. Apply the construction given above, taking a contact-conic from the $(1a)$ family. It gives a tritangent conic to which we assign the label (a); we thus obtain seven tritangent conics (2), (3), (4), (5), (6), (7), (8). Take the contact-conic from the (ab) family and we obtain the 21 tritangent conics labelled $(1ab)$. Lastly take the contact-conic from the $(1abc \mid defg)$ family, and we obtain 35 more labelled (abc), completing the system of sixty-four.

https://doi.org/10.1017/S001309150001378X Published online by Cambridge University Press