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DIAMETERS IN TYPICAL CONVEX BODIES

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1. Introduction. The most usual diameters in the world are those of a sphere and they all contain its centre. More generally, a chord of a convex body in $\mathbf{R}^d$ is called a *diameter* if there are two parallel supporting hyperplanes at the two endpoints of the chord.

It is easily seen that there are points on at least two diameters. From a result of Kosiński [**6**] proved in a more general setting it follows that every convex body has a point lying on at least three diameters. Does a typical convex body behave like a sphere and contain a point on infinitely or even uncountably many diameters?

But what is a typical convex body? The space $\mathcal{K}$ of all convex bodies (d-dimensional compact convex sets) in $\mathbf{R}^d$, equipped with the Hausdorff metric, is a Baire space. Its subspace $\mathcal{K}^1$ of all convex bodies which are smooth and strictly convex is residual (see [**5**]) and is therefore itself a Baire space. We say that *typical elements* or *most elements* of a Baire space enjoy some property if all those enjoying it form a residual set.

It is known that in most planar convex bodies most points lie on infinitely many diameters ([**9**]). This is in striking contrast with the fact that in the measure-theoretical sense almost every point in any planar convex body lies on finitely many diameters [**3**] and with the fact that in many usual planar convex bodies every point lies on at most three diameters. Unfortunately, it does not seem that the method of proof used in [**9**] can be generalized to higher dimensions.

In $\mathbf{R}^d$, using other methods we prove that in most convex bodies most points lie on infinitely many diameters. Moreover, we show that in most convex bodies infinitely many points lie on uncountably many diameters. To a question of Hammer [**2**] concerning points on uncountably many diameters an answer was provided long ago by Besicovitch and Zamfirescu [**1**]. This answer will now be strengthened and put into a new light.

For a survey of results on typical convex bodies, see [**10**]. We use the following notations: xy means the line segment from x to y. $N_K(x)$, or simply $N(x)$, is the set of outer unit normals of K at $x \in \operatorname{bd} K$, where $\operatorname{bd} K$ is the boundary of K. For $K \in \mathcal{K}^1$, $N(x)$ consists of a single point which we denote by $a(x)$. The interior of K is denoted by int K.

For $M \subset \mathbf{R}^d$, diam $M = \sup\{\|x-y\| : x, y \in M\}$.

A lemma. Let $0 < \lambda < 1$ and consider a diameter xy of $K \in \mathcal{K}$. Set

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$z = \lambda x + (1-\lambda)y$ and

$$\begin{aligned}
K_\lambda &= z + \frac{1-\lambda}{\lambda}(z-K),\\
R_1 &= K \setminus \operatorname{int} K_\lambda,\\
R_2 &= K_\lambda \setminus \operatorname{int} K.
\end{aligned}$$

Actually R_1 and R_2 depend on x, y and λ, but we will suppress this dependence in the notation. Notice that $x \in R_1$ and $x \in R_2$. We call a diameter *λ–regular* if neither R_1 nor R_2 includes a continuum containing x and different $\{x\}$. A diameter is *regular* if it is λ–regular for all $\lambda \in (0,\ 1)$.

It is easy to see that no diameter of the unit ball in $\mathbf{R}^d$ is regular. We give an example of a λ–regular diameter with $\lambda = 1/2$ as follows. Define a function $f : [-1,\ 1] \longrightarrow R$ by

$$f(x) = \begin{cases} \sqrt{1-x^2} & \text{if } x \in \left[-1, \frac{1}{2}\right] \bigcup \left\{\frac{3}{4},\ \frac{7}{8},\ \frac{15}{16}, \ldots\right\} \bigcup \{1\} \\ \min\limits_i g_i(x) & \text{otherwise,} \end{cases}$$

where g_i is the linear function that coincides with f at

$$x = (2 \cdot 4^{i-1} - 1)/2 \cdot 4^{i-1} \quad \text{and} \quad x = (4^i - 1)/4^i \quad (i = 1,\ 2,\ 3, \ldots).$$

Clearly, f is concave. Now let $K \subset \mathbf{R}^d$ be the convex body obtained by rotating the graph of f around the x-axis. It follows from this construction that the interval $[-1,\ 1]$ on the x-axis is a $1/2$–regular diameter of K.

LEMMA. *For most convex bodies $K \in \mathcal{K}$ and for most points $x \in \operatorname{bd} K$ there is a unique diameter xy and this diameter is regular.*

Proof. We show uniqueness first, which is easy. $\mathcal{K}^1$ is residual in $\mathcal{K}$ (see [**5**]). Take $K \in \mathcal{K}^1$ and $x \in \operatorname{bd} K$. Then $a(x)$ is uniquely determined and there is a unique $y \in \operatorname{bd} K$ with $a(y) = -a(x)$. Thus xy is a diameter and it is the only diameter that contains x.

Let $\alpha \in (0,\ 1/2)$, $\beta > 0$ and define, for $K \in \mathcal{K}^1$,

$$\begin{aligned}
S_\alpha(K) = &\{x \in \operatorname{bd} K : \ \text{the diameter } xy \text{ is not}\\
&\lambda\text{– regular for some } \lambda \in [\alpha, 1-\alpha]\},\\
S_{\alpha,\beta}(K) = &\{x \in S_\alpha(K) : \exists \text{ continuum } C \ni x \text{ in } R_1 \text{ or } R_2 \text{ with}\\
&\operatorname{diam} C \geqq \beta \text{ for some } \lambda \in [\alpha, 1-\alpha]\}.
\end{aligned}$$

A routine argument shows that $S_{\alpha,\beta}(K)$ is closed. Clearly

$$S(K) = \bigcup_{\alpha>0} \bigcup_{\beta>0} S_{\alpha,\beta}(K)$$

is in the set of endpoints of non-regular diameters. Set

$$\mathcal{K}^0 = \{K \in \mathcal{K}^1 : S(K) \text{ is of second category}\}.$$

Choosing sequences $\alpha_i \longrightarrow 0$ and $\beta_j \longrightarrow 0$, $S_{\alpha_i,\beta_j}(K)$ cannot all be nowhere dense if $K \in \mathcal{K}^0$. Then, for some i and j, $S_{\alpha_i,\beta_j}(K)$ must contain a "disk"

$$D = \operatorname{bd} K \bigcap B(x, \gamma),$$

where $x \in \operatorname{bd} K, \gamma > 0$ and $B(x, \gamma)$ stands for the ball with radius γ and centre x. Set

$$\mathcal{K}_{\alpha,\beta,\gamma} = \{K \in \mathcal{K}^1 : S_{\alpha,\beta}(K) \text{ contains a disk of radius } \gamma\}.$$

Clearly, for some sequence $\gamma_k \longrightarrow 0$,

$$\mathcal{K}^0 = \bigcup_i \bigcup_j \bigcup_k \mathcal{K}_{\alpha_i,\beta_j,\gamma_k}$$

Now we show that $\mathcal{K}_{\alpha,\beta\gamma}$ is nowhere dense in $\mathcal{K}$. This will prove the lemma. We (again) omit the proof of the following claim: $\mathcal{K}_{\alpha,\beta,\gamma}$ is closed.

Now let O be an open set in $\mathcal{K} \setminus \mathcal{K}_{\alpha,\beta,\gamma}$. Take a smooth and strictly convex body $K \in O$. Choose a finite set $X \subset \operatorname{bd} K$ with the property that any disk $D \subset \operatorname{bd} K$ of radius $\gamma/2$ contains a point of X. This is clearly possible. For each x, let xy be the unique diameter through x. Cut off a small cap C_x (and C_y) from K by a hyperplane orthogonal to $a(x)$ and close to x (y, respectively). These cuts can be chosen so small that the caps C_x, $C_y (x \in X)$ are all pairwise disjoint, and the diameter of each C_x and C_y is less than $\gamma/2$. Moreover, if the cuts are small enough, then

$$K' = K \setminus \bigcup_{x \in X} (C_x \cup C_y) \in O$$

and any $K^* \in \mathcal{K}$ with $\mathcal{K}' \subset K^* \subset K$ belongs to O, too. The diameter xy intersects bd K' in points u and v where $u \in C_x$ and $v \in C_y$. Let $B(u)$ $[B(v)]$ be a $(d-1)$-dimensional ball with centre $u[v]$ and radius ρ, lying in the bounding hyperplane of C_x $[C_y]$. The radius ρ can be chosen so small that $B(u) \subset C_x$ and $B(v) \subset C_y$ for all $x \in X$.

Define two functions f, g : $[0, \infty) \longrightarrow [0, \infty)$ by

$$f(t) = \begin{cases} t^4 & \text{if } t \leqq \epsilon, \\ t^2 - \epsilon^2 + \epsilon^4 & \text{if } t > \epsilon, \end{cases}$$
$$g(t) = t^3,$$

where ϵ is to be specified later.

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Let $z \in \mathbf{R}^d$ and write

$$z - u = z^+ + \zeta a$$

where $a = a(x)$ and z^+ is the component of $z - u$ orthogonal to a. Define the set

$$F_x = \{z \in \mathbf{R}^d : \|z^+\| \leqq \rho, 0 \leqq \zeta \leqq \nu - \mu f(\|z^+\|)\},$$

where $\mu > 0$ and $\nu > 0$ will be chosen so small that, firstly, $F_x \subset C_x$ and, secondly, no hyperplane touching F_x at some point z with $\|z^+\| < \rho$, $\zeta = \nu - \mu f(\|z^+\|)$ meets $K \backslash C_x$ (for all $x \in X$). Similarly, let $w \in R^d$ and set

$$w - v = w^+ + \omega b$$

where $b = -a$ is the outer normal to K at y and w^+ is the component of $w - v$ orthogonal to b. Define

$$G_y = \{w \in R^d : \|w^+\| \leqq \rho, 0 \leqq \omega \leqq \nu - \mu g(\|w^+\|)\},$$

where, again, $\nu > 0$ and $\mu > 0$ are chosen so that $G_y \subset C_y$ and no hyperplane, touching G_y at some point w with $\|w^+\| < \rho, \omega = \nu - \mu g(\|w^+\|)$ meets $K \backslash C_y$. Clearly, ν and μ can be chosen the same for all F_x and G_y.

Define now

$$K^* = \operatorname{conv}(K' \cup \bigcup_{x \in X}(F_x \cup G_y)).$$

Evidently $K \supset K^* \supset K'$ and so $K^* \in O$. We claim that $K^* \notin \mathcal{K}_{\alpha,\beta,\gamma}$.

Consider $x \in X$ and the points y, u, v defined above. Set

$$x^* = u + \nu a, \quad y^* = v + \nu b.$$

By the construction, $x^* y^*$ is a diameter of K^* with outer normals a and b at its endpoints. We will now show that for $x^* = \lambda x^* + (1-\lambda) y^*$ with $\alpha < \lambda < 1 - \alpha$ neither $R_1^* = K^* \backslash \operatorname{int} K_\lambda^*$ nor $R_2^* = K_\lambda^* \backslash \operatorname{int} K^*$ contains a continuum of diameter β containing x^*.

By construction, bd K^* coincides with a piece of bd F_x [bdF_y] in a neighbourhood of radius ρ of x^* [y^*]. Then bd K is rotationally symmetric in a small neighbourhood of x^* [y^*], the axis of symmetry being the line though $x^*[y^*]$ with direction a. Thus R_1^* and R_2^* are also rotationally symmetric.

It is easy to reduce our question now to a question about the functions f and g : R_1^* and R_2^* contain a continuum of diameter at least β containing x^* if and only if either

$$f(t) \geqq \frac{\lambda}{1-\lambda} g\left(\frac{1-\lambda}{\lambda} t\right)$$

on the interval $[0,\ \beta/2]$, or

$$f(t) \leqq \frac{\lambda}{1-\lambda} g\left(\frac{1-\lambda}{\lambda} t\right)$$

on the same interval. We show now that neither of these two possibilities holds. Write $\sigma = (1-\lambda)/\lambda$; then

$$\alpha/(1-\alpha) < \sigma < (1-\alpha)/\alpha.$$

Clearly

$$f(t) = t^4 < \frac{1}{\sigma} g(\sigma t) = \gamma^2 t^3$$

if $0 \leqq t < \sigma^2/(1-\alpha)^2$, and we choose $0 < \epsilon < \alpha^2/(1-\alpha)^2$. Now for $t = 2\epsilon$

$$f(t) = 3\epsilon^2 + 16\epsilon^4,$$
$$\frac{1}{\sigma} g(\sigma t) = \sigma^2 \cdot 8\epsilon^3 \leqq \left(\frac{1-\alpha}{\alpha}\right) \cdot 8\epsilon^3,$$

and so

$$\frac{1}{\sigma} g(\sigma t) < f(t) \quad \text{if } \epsilon < \frac{3}{8}\left(\frac{\alpha}{1-\alpha}\right)^2.$$

We also have to have $t = 2\epsilon < \beta/2$, $t = 2\epsilon < \rho$ and $\gamma t = \gamma^2 \epsilon < \rho$. Thus, choosing

$$\epsilon = \frac{1}{2} \min \left\{ \frac{3}{8}\left(\frac{\alpha}{1-\alpha}\right)^2, \beta/4, \rho/2 \frac{\rho\alpha}{2(1-\alpha)} \right\}$$

there will be no continuum, of diameter at least β and containing x^* in either of the sets R_1^* and R_2^* for all $x \in X$.

Assume now that there is a disk $D^* \subset S_{\alpha,\beta}(K^*)$ of radius Γ, with centre $z^* \in K^*, z^* \in C_x$ (or C_y) would imply $x^* \in D^*$ (or $y^* \in D^*$) because C_x and C_y have diameter less then $\beta/2$. But $x^* \in D^* \subset S_{\alpha,\beta}(K^*)$ contradicts what we just established. So $xz^* \in \text{bd } K$, and, by the choice of X, $\|z^* - x\| < \gamma/2$ for some $x \in X$. The diameter of C_x is less than $\gamma/2$, so $\|x - x^*\| < \gamma/2$ whence

$$\|z^* - x^*\| \leqq \|z^* - x\| + \|x - x^*\| < \gamma$$

and $x^* \in D^* \subset S_{\alpha,\beta}(k^*)$, a contradiction.

Points on infinitely many diameters.

THEOREM 1. *In most convex bodies, on most diameters, each point belongs to infinitely many diameters.*

Proof. Let K be a smooth convex body with the properties of the lemma. Also, let M_α be the set of points in $\mathbf{R}^d$ lying on a least α diameters of K.

Consider a regular diameter xy of K. For $\lambda \in (0,\ 1)$, let

$$z = \lambda x + (1-\lambda)y,$$
$$K_\lambda = z + \frac{1-\lambda}{\lambda}(z - K).$$

For $v \in \operatorname{bd} K$ close to x, let vw be a diameter of K and

$$v' = z + \frac{1-\lambda}{\lambda}(z - w).$$

Put

$$f_z(v) = \|v - z\| - \|v' - z\|.$$

From now on we assume $d \geqq 3$; the case $d = 2$ can be dealt with analogously (or see [**9**]). By the definition of a regular diameter, in any neighbourhood of x there is a component V^+ of $f_z^{-1}(\mathbf{R}^+)$ and a component V^- of $f_z^{-1}(\mathbf{R}^-)$ both surrounding x, i.e., such that x lies in a bounded component V'^+ of $\complement V^+$ and in a bounded component V'^- of $\complement V^-$. Now ρ_z has in every component V^+ a local maximum. Since every local maximum is an endpoint of a diameter through z, $z \in M_{\aleph_0}$. By the Lemma, the set of regular diameters is residual. Thus the theorem is proved.

THEOREM 2. *In most convex bodies, most points belong to infinitely many diameters.*

Proof. As a consequence of the proof of Theorem 1, we find k components $V_1^+, \dots, V_k^+$ of $f_z^{-1}(\mathbf{R}^+)$ and $k-1$ components $V_1^-, \dots, V_{k-1}^-$ of $f_z^{-1}(\mathbf{R}^-)$ such that $V_i^- \subset V_i'^+$ and $V_{i+1}^+ \subset V_i'^-$ $\quad (i = 1, \dots, k-1)$. Since v' depends continuously on v and v and z, so does $f_z(v)$, which implies that for some neighbourhood N_i of z we have, for any $u \in N_i$, two components U_i^+, U_{i+1}^+ of $f_u^{-1}(\mathbf{R}^+)$ and a component U_i^- of $f_u^{-1}(\mathbf{R}^-)$ such that U_i^- lies in a bounded component $U_i'^+$ of $\complement U_i^+$ and U_{i+1}^+ lies in a bounded component $U_i'^-$ of $\complement U_i^-$. Thus, for any $u \in U_{i=1}^k N_i$, $U_i^- \subset U_i'^+$ and $U_{i+1}^+ \subset U_i'^-$ $\quad (i = 1, \dots, k-1)$. It follows that, for any such u, $f_u(v)$ has at least k local maxima (in the variable v). Every such local maximum is the endpoint of a diameter through u. This means that

$$\bigcap_{i=1}^{k} N_i \subset M_k.$$

Now let O be open in K. Since every point of K lies on a diameter, there is a diameter meeting O. By Theorem 1, there also is a regular diameter xy with a point $z \in xy \cap O$. Now, the considerations above imply that CM_k is nowhere dense, since

$$O \cap \bigcap_{i=1}^{k} N_i \subset M_k.$$

Thus

$$CM_{\aleph_0} = C \bigcap_{k=1}^{\infty} M_k = \bigcup_{k=1}^{\infty} CM_k$$

is of first category and the theorem is proved.

Points on uncountably many diameters. Theorems 1 and 2 leave open the question whether a typical convex body must have a point lying on uncountably many diameters.

For any $K \in \mathcal{K}$ and $z \in K$, let $\Omega(z)$ be the set of all unit vectors $u \in S^{d-1}$ such that the chord through z in direction u is a diameter of K. Clearly, $\Omega(z) = -\Omega(z)$.

THEOREM 3. *Let $S \subset \mathbf{R}^d$ be countable. For most convex bodies K, at each point $z \in S \cap K$, $\Omega(z)$ is perfect and nowhere dense.*

Let $z \in S$. We prove the theorem for $S = \{z\}$; since any countable intersection of residual sets is residual, the theorem in its general form will follow. We start by observing that $z \in \operatorname{bd} K$ for a nowhere dense family of sets K only. So we may assume $z \in \operatorname{int} K$. We prove that, for most $K \in \mathcal{K}$, the set $\Omega(z)$ is perfect and nowhere dense. First we remark that $\Omega(z)$ is always closed. Thus we only have to prove that $\Omega(z)$ contains no open set and no isolated element, for most K. In fact the nowhere density follows from Theorem 5. However we give here a simple direct proof.

Suppose $\Omega(z)$ includes an open set. This means that there are two diametrically opposite points x, y with $z \in xy$ and two open sets F_x, F_y around x and y on $\operatorname{bd} K$ such that F_x and F_y are inversely homothetical with respect to z. Let

$$\mathcal{K}_n = \{K \in \mathcal{K} : \exists F_x,\ F_y \text{ such that } \operatorname{diam} F_x \geqq n^{-1},\ \operatorname{diam} F_y \geqq n^{-1}\}.$$

Clearly, $\mathcal{K}_n$ is closed; it remains to show that $C\mathcal{K}_n'$ is dense. This is indeed so since any convex body is approximable by a polytope without parallel facets.

Suppose now $\Omega(z)$ has an isolated element ω. Then there is a disk $G_x \subset \operatorname{bd} K$ around an endpoint x of the diameter xy through z in direction ω such that the only diameter through z with an endpoint in G_x is xy. Let

$$\mathcal{K}_n = \{K \in \mathcal{K} : \exists G_x \text{ such that } \operatorname{diam} G_x > n^{-1}\}.$$

We show that $\mathcal{K}_n$ is nowhere dense. Let $O \subset \mathcal{K}$ be open.

Take $K' \in O$ to be smooth. $\Omega(z)$ is compact, so we may choose a finite open covering of $\Omega(z)$ with sets $D_1, \dots, D_m$ of diameter less than $(4n)^{-1}$. Consider a point

$$\omega \in S^{d-1} \setminus \bigcup_{i=1}^{m} D_i.$$

Since $\omega \notin \Omega(z)$, the chord $x_\omega y_\omega$ in direction ω satisfies

$$a(x_\omega) + a(y_\omega) \neq 0,$$

and in a small neighbourhood $D(\omega)$ of ω we have

$$\|a(x_v) + a(y_v)\| > \eta(\omega)$$

for some $\eta(\omega) > 0$ and all $v \in D(\omega)$. Choose a finite covering $D(\omega_1), \dots, D(\omega_l)$ of $S^{d-1} \setminus \bigcup_{i=1}^{m} D_i$ and let

$$\eta = \min\{\eta(\omega_i) : \; i = 1, \dots, l\}.$$

Now choose a polytope P very close to K', so that

$$\|v' + v''\| \geqq \eta/2$$

for any $v' \in N_p(x_v)$, $v'' \in N_p(y_v)$, and any

$$v \in \bigcup_{i=1}^{l} D(\omega_i),$$

and so that P has a diameter $x_i y_i$ in a direction $u_i \in D_i$ for any $i = 1, \dots, m$. These diameters can be chosen so that x_i is a vertex of P and y_i is in the relative interior of a facet F_i of P. Now fix i and denote $x_i y_i$ and F_i by xy and F respectively (for simplicity of notation). Choose a line segment $x^1 x^2$ parallel to F, so that x is the midpoint of $x^1 x^2$. Let $y^1 [y^2]$ be the point in $\operatorname{bd} P$ close to y and collinear with x^1 and $z [x^2$ and $z]$.

We choose the line segment $x^1 x^2$ so short that y^1, $y^2 \in F$ and that $P' = \operatorname{conv}(P \cup \{x^1, x^2\})$ is as close as we wish to P.

Now we take $n^1 \in \operatorname{int} N_{P'}(x^1)$ and $n^2 \in \operatorname{int} N_{P'}(x^2)$ very close to $-a(y)$ and we cut off from P' two caps by hyperplanes through y orthogonal to n^1, respectively n^2. We get in this way a convex polytope Q as close to P' as we wish.

Denote by $\bar{y}^j$ the point in $\operatorname{bd} Q$ collinear with x^j and z $\quad (j = 1,\ 2)$. It is clear that if the segment $x^1 x^2$ is short enough then $x^1 \bar{y}^1$ and $x^2 \bar{y}^2$ will be diameters of Q.

Let H_x, H_y be the hyperplanes through x, respectively y, both parallel to F. Now define a function f_Q on H_x: for $t \in H_x$, let $x(t) \in \operatorname{bd} Q$ be the point close to x with $x(t)-t$ orthogonal to H_x (or zero) then determine $y(t) \in \operatorname{bd} Q$ collinear with z and $x(t)$. Next determine $s(t) \in H_y$ as the orthogonal projection of $y(t)$ to H_y. Finally let

$$f_Q(t) = \|x(t) - t\| - \frac{\|x(t) - z\|}{\|y(t) - z\|} \|y(t) - s(t)\|.$$

By the construction, this function f_Q has strict local minima at x^1 and x^2. Now we do this same thing successively with all diameters $x_i y_i$ $(i = 1, \ldots, m)$, not disturbing the properties of the polytopes and functions already constructed, which are of interest for our purposes.

We eventually get a polytope $P^* \in O$. We claim that, if K is in a neighbourhood $\mathcal{N}$ of P^* which is small enough, then for every diameter of K through z there will be another one with angular distance less than n^{-1}. If K is close enough to P^* then K cannot have a diameter through z in any direction from $\bigcup_{i=1}^{1} D(\omega_i)$. So every diameter of K through z is defined by a direction from D_i, for some $i \in \{1, \ldots, m\}$. We show that there are two directions of diameters through z in every D_i and this will prove the claim. Consider the function f_K defined in the same way as before, but with K instead of Q.

If K is close enough to P^*, then f_K will be very close to f_{P^*} and so f_K will have at least two local minima t_1 and t_2, close to x^1 and x^2. At these local minima, the corresponding chords $x_K(t_j)y_K(t_j) (j = 1, 2)$ will be diameters of K. Thus

$$O \cap \mathcal{N} \cap \mathcal{K}_n = \emptyset$$

and the proof is finished.

In 1965 Hammer [**2**] raised the question whether there exist a convex body K and a point $z \in \operatorname{int} K$ such that the set $R(z)$ of all ratios into which z divides the various diameters through z is uncountable. Besicovitch and Zamfirescu [**1**] answered the question by providing such a convex body and such an interior point.

THEOREM 4. *Let $S \subset \mathbf{R}^d$ be countable. For most convex bodies K, at each point $z \in S \cap K$, $R(z)$ is uncountable.*

Proof. Let $z \in S$. We prove the theorem for $S = \{z\}$. First we remark that $\{K \in \mathcal{K} : z \in \operatorname{bd} K\}$ is nowhere dense. We prove that, for most $K \in \mathcal{K}$ with $z \in \operatorname{int} K$, $R(z)$ is perfect. Clearly, $R(z)$ is closed. Thus we only have to show that $R(z)$ contains no isolated point, for most $K \in \mathcal{K}$. The proof parallels the last part of the proof of Theorem 3, where it is shown that, typically, $\Omega(z)$ has no isolated point.

The only change which has to be made is the following: Using the notations of the mentioned proof, we have to arrange, when choosing the unit vectors n^1 and n^2 in int $N_{P'}(x^1)$ and $\operatorname{int} N_{P'}(x^2)$ respectively, that

$$\frac{\|x^1 - z\|}{\|\bar{y}^1 - z\|} \neq \frac{\|x^2 - z\|}{\|\bar{y}^2 - z\|}.$$

Directions of concurrent diameters. In Theorem 3 it is shown that, for countably many points z, the set $\Omega(z)$ is nowhere dense. This can be considerably improved by allowing z to be any point in K and by replacing nowhere density by uniform porosity.

A set M in a metric space (X,δ) is called *uniformly porous* if there exists a constant $\alpha > 0$ such that in any ball $B(x, r) \subset X$ there is a point y such that

$$B(y, \alpha\delta(x, y)) \cap M = \emptyset.$$

THEOREM 5. For most convex bodies $K \in \mathcal{K}$ and for all points $z \in K$, $\Omega(z)$ is uniformly porous.

Proof. Suppose $d \geqq 3$. The proof for $d = 2$ is analogous and simpler. Define $K(\epsilon)$ to be the inner parallel body to K at distance ϵ. We denote the angle between the vectors $u, v \in S^{d-1}$ by $\sphericalangle\, uv$. Let $i(S)$ and $\mu(S)$ be the inradius and the $(d-2)$-dimensional measure of the convex set $S \subset S^{d-2}$ (the convexity is understood on S^{d-2}). Since $i(S) \longrightarrow 0$ implies $\mu(S) \longrightarrow 0$, the number

$$\alpha = \inf\{i(S) :\ \mu(S) = d^{-2}\mu(S^{d-2})\}$$

is positive. Set now

$$\mathcal{K}_n = \left\{ K \in \mathcal{K} :\ \exists z \in K(n^{-1}) \text{ and } \exists\, u \in S^{d-1} \text{such that } \forall\, v \in S^{d-1} \right.$$
$$\left. \text{with } \sphericalangle\, uv \leqq n^{-1}, \exists\, w \in \Omega(z) \text{ with } \sphericalangle\, wv \leqq \frac{\alpha}{8} \cdot \sphericalangle\, uv \right\}.$$

Obviously, $\mathcal{K}_n$ is closed.

We show now that $\mathcal{K}_n$ is nowhere dense. Let $O \subset \mathcal{K}$ be an open set. If there is no $K \in O$ with $K(n^{-1}) \neq \emptyset$ then $O \cap \mathcal{K}_n = \emptyset$ and we are done. If this is not the case, then there is a simple polytope $P \in O$ without parallel facets or facets of diameter at least n^{-1}, and such that $P(n^{-1}) \neq \emptyset$. Consider now $z \in P(n^{-1})$ and a chord xy of P through z with direction u. If xy is not a diameter of P then the chords through z that are close enough to xy are not diameters either.

So assume xy is a diameter. Then x or y or both are contained in faces of P, each of dimension less than $d-1$, for P has no parallel facets. Let $F_1, \ldots, F_k$ and $G_1, \ldots, G_l$ be the facets containing x and y respectively. Let H be the hyperplane through x with normal u. Project $F_1, \ldots, F_k$ from z to H, then reflect $G_1, \ldots, G_l$ through z and project the resulting sets from z to H. In this way we get $(d-1)$-dimensional polytopes $F'_1, \ldots, F'_k$ and $G'_1, \ldots, G'_l$ in H (because the diameter of each facet is less than n^{-1}). Let

$$F''_i = \{\|x' - x\|^{-1}(x' - x) :\ x' \in F'_1\} \quad (i = 1, \ldots, k)$$

and define G_j'' $(j = 1, \ldots, l)$ analogously. These sets are convex in S^{d-2}. Clearly, the convex sets $F_i'' \cap G_j''$ $(i = 1, \ldots, k,\ j = 1, \ldots, l)$ form a subdivision of S^{d-2}. Now P is simple, so $k \leqq d$ and $l \leqq d$. Thus there are i and j so that

$$\mu(F_i'' \cap G_j'') \geqq d^{-2}\mu(S^{d-2}).$$

Then $F_i'' \cap G_j''$ contains a $(d-2)$-dimensional (spherical) disk of radius α and centre c. Consider the point $c_\lambda = x + \lambda c$. For small enough $\lambda > 0$, the open $(d-1)$-dimensional ball D in H around c_λ of radius $\alpha\|c_\lambda - x\|/2$ lies in $F_i' \cap G_j'$. Choose

$$v = \|c_\lambda - z\|^{-1}(c_\lambda - z).$$

We claim that, for any $w \in S^{d-1}$ with $\sphericalangle\, vw \leqq (\alpha/8) \sphericalangle\, uv$, the halfline $\{z + \lambda w : \lambda \geqq 0\}$ intersects H inside D.

Indeed, let S^* be the sphere with centre z which passes through x and let v' be the intersection of zc_λ with S^*. Let w' and w'' be the intersections of the ray from z in direction w with S^* and H, respectively. Let w_0 be the point of zw'' nearest to v'. Then

$$\frac{\|v' - w_0\|}{\|c_\lambda - x\|} = \cos \sphericalangle\, uw \cdot \cos \sphericalangle\, uv \geqq \cos n^{-1} \cdot \cos(2n)^{-1} > \cos^2 1 > \frac{1}{4}.$$

But

$$\frac{\|v' - w_o\|}{\|c_\lambda - x\|} \leqq \frac{v'w'}{v'x} \leqq \frac{\sphericalangle\, vw}{\sphericalangle\, uv} \leqq \frac{\alpha}{8}.$$

Thus

$$\|c_\lambda - w''\| < 4\|v' - w_0\| \leqq \frac{\alpha}{2}\|c_\lambda - x\|,$$

which verifies our claim.

Now, $D \subset F_i' \cap G_j'$. Since F_i and G_j are not parallel, $w \notin \Omega(z)$. Hence $P \subseteq \mathcal{K}_n$ and $\mathcal{K}_n$ is nowhere dense.

It follows that

$$\mathcal{K}^1 \cap \left(\mathcal{K} \setminus \bigcup_{n=1}^{\infty} \mathcal{K}_n \right)$$

is residual. Let K be an element of this set and take $z \in K$. If $z \in \operatorname{int} K$, then $z \in K(n^{-1})$ for some n and $\Omega(z)$ is uniformly porous. If $z \in \operatorname{bd} K$, then $\Omega(z)$ consists of two (antipodal) points only. The theorem is proved.

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Remark. It follows from Theorem 5 and from Lebesgue's density theorem that, for most $K \in \mathcal{K}$ and any $z \in K$, $\Omega(z)$ has measure zero on S^{d-1}.

Normals to typical convex surfaces. We shall now consider normal lines to convex surfaces. Heil [**4**] proved that any convex surface in $\mathbf{R}^d$ admits a point lying on at least 6 normals to the surface for any dimension $d \geqq 3$. For a typical convex surface and any $d \geqq 2$ the following holds.

THEOREM. ([**7**], [**8**]) *For most convex surfaces, most points in $\mathbf{R}^d$ lie on infinitely many normals.*

We are now able to complete the generic description of normals to convex surfaces.

Let $\Phi(z)$ denote the set of directions of normals through the point $z \in \mathbf{R}^d$ to a given convex surface.

THEOREM 6. *Let $S \subset \mathbf{R}^d$ be countable. For most convex surfaces in $\mathbf{R}^d$ the following is true:*

(i) *For any point $z \in S, \Phi(z)$ is perfect in S^{d-1}.*

(ii) *For any point $z \in \mathbf{R}^d$, $\Phi(z)$ is uniformly porous in S^{d-1}.*

Theorem 6 admits a proof which is very similar and slightly simpler than those of Theorems 3 and 5 and will therefore be omitted.

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