ON THE PERIODICITY OF COMPOSITIONS OF ENTIRE FUNCTIONS

FRED GROSS

Introduction. For two entire functions $f(z)$ and $g(z)$ the composition $f(g(z))$ may or may not be periodic even though $g(z)$ is not periodic. For example, when $f(u) = \cos \sqrt{u}$ and $g(z) = z^2$, or $f(u) = e^u$ and $g(z) = p(z) + z$, where $p(z)$ is a periodic function of period $2\pi i$, $f(g(z))$ will be periodic. On the other hand, for any polynomial $Q(u)$ and any non-periodic entire function $f(z)$ the composition $Q(f(z))$ is never periodic **(2)**.

The general problem of finding necessary and sufficient conditions for $f(g(z))$ to be periodic is a difficult one and we have not succeeded in solving it. However, we have found some interesting related results, which we present in this paper.

THEOREM 1. *Let*

$$f(z) = \sum_{i=1}^{n} Q_i(z)e^{g_i(z)} + Q_0(z),$$

where $g_i(z) - g_j(z)$ *and* $g_i(z)$ *are non-constant and entire and where* $Q_i(z)$ *are polynomials for all* i *and* j *with* $i \neq j$. *If* $f(z)$ *is non-constant and periodic, then* $g_i(z)$ *is of the form* $p(z) + az$, *and* $Q_0(z)$ *must be a constant. Here* a *is a constant and* $p(z)$ *is periodic.*

Proof. By a well-known theorem of Borel **(1)** if $f(z + t) = f(z)$, then

$$g_1(z) = g_{i_1}(z + t) + \text{const.},$$

$$g_{i_1}(z) = g_{i_2}(z + t) + \text{const.},$$

$$\vdots$$

$$g_{i_k}(z) = g_{i_{k+1}}(z + t) + \text{const.},$$

where $\{1, i_1, \ldots, i_k\}$ is a permutation of $\{1, 2, \ldots, n\}$. Thus

$$g_1(z) = g_1(z + mt) + \text{const.}$$

for some fixed integer m and the first part of our assertion follows. If $Q_0(z)$ is non-constant, then again by Borel's theorem we would have $Q_0(z + t) = Q_0(z)$, which is impossible, and our proof is complete.

Received April 23, 1965.

https://doi.org/10.4153/CJM-1966-072-7 Published online by Cambridge University Press

Remark. Borel proved: Let $a_i(z)$ be an entire function of order at most ρ, let $g_i(z)$ also be entire and let $g_i(z) - g_j(z)$ $(i \neq j)$ be a transcendental function or polynomial of degree higher than ρ. Then

$$\sum_{i=1}^{n} a_i(z)e^{g_i(z)} = a_0(z)$$

implies that $$a_0(z) = a_1(z) = \ldots = a_n(z) = 0.$$

The following theorem yields an example of entire functions $f(z)$ and $g(z)$ such that $f(g(z))$ is periodic if and only if $g(z)$ is periodic.

For its proof we shall need the concept of order of magnitude due to Borel **(1)**.

Let $F(x)$ and $G(x)$ be two increasing functions. F and G are said to be of the same order of magnitude if

$$[G(x)]^{1-\epsilon} < F(x) < [G(x)]^{1+\epsilon}$$

whatever the positive number ϵ may be, provided that x is sufficiently large.

In a similar manner one defines what is meant by the statement that F has a greater order of magnitude than G.

Borel associates with each entire function f an increasing function $\rho_f(r)$ and defines the order of magnitude of f denoted by $O(f)$ via $\rho_f(r)$. He proves that

(i) $O(g) = O(g')$,

(ii) $O(e^g) > O(g)$,

(iii) if $O(f) > O(g)$, then $O(f+g) = O(f\cdot g) = O(f)$.

(Borel's proof is incomplete. The argument was completed by R. Nevanlinna in his book *Le théorème de Picard Borel* (Paris, 1929), who used his characteristic $T(r,f)$ as $P_f(r)$.)

We are now prepared to prove

THEOREM 2. *If $g(z)$ is any non-periodic entire function, then $e^{g(z)} + g(z)$ is not periodic.*

Proof. Assume that $e^{g(z)} + g(z)$ is periodic of period t and $g(z)$ is not periodic. Thus

$$e^{g(z+t)} + g(z+t) = e^{g(z)} + g(z). \tag{1}$$

If $O(g(z+t)) > O(g(z))$, then $O(e^{g(z+t)}) > O(g(z+t))$ implies that

$$O(e^{g(z+t)} + g(z+t)) > O(e^{g(z)} + g(z)).$$

Since this contradicts (1), we must have $O(g(z+t)) \leqslant O(g(z))$. In a similar manner one shows that

$$O(g(z+t)) \geqslant O(g(z)),$$

so that $$O(g(z+t)) = O(g(z)).$$

Now differentiating both sides of (1) and substituting for $e^{g(z)}$, we obtain after simplification

$$[g'(z+t) - g'(z)]e^{g(z+t)} = g'(z)[g(z+t) - g(z)] + g'(z) - g'(z+t). \tag{2}$$

https://doi.org/10.4153/CJM-1966-072-7 Published online by Cambridge University Press

Since $O(g(z+t)) = O(g(z))$, (2) is possible only if $g(z)$ is periodic, contrary to our hypothesis. Our proof is now complete.

The following alternative proof was suggested by the referee and does not depend on Borel's order of magnitude.

Put $f(z) = e^z + z$ and assume that $g(z)$ is such that

$$f(g(z+t)) = f(g(z)) = F(z), \qquad t \neq 0.$$

Let L be the line $z_0 + \lambda t$, $-\infty < \lambda < \infty$. The periodic function $F(z)$ is bounded on L. If $g(z)$ is unbounded on L, then $g(L)$ is a path extending arbitrarily far from the origin on which $f(z)$ is bounded; but the form of $f(z)$ shows that there is no such path. Hence $g(z)$ is bounded on L. Take a value z_0 on L such that $\alpha = f(g(z_0))$ is not an algebraic singularity of the inverse function of $f(z)$ (the algebraic singularities form a countable set). Now $\{g(z_0 + nt)\}$, $n = 1, 2, \ldots$, is bounded, say $|g(z_0 + nt)| \leqslant M$, while

$$f(g(z_0 + nt)) = f(g(z_0)) = \alpha.$$

Thus all $g(z_0 + nt)$ are among the finite set of solutions of $f(w) = \alpha$ which belong to $|w| \leqslant M$. Hence for some $m \neq n$, $g(z_0 + mt) = g(z_0 + nt)$. Moreover, for all small ϵ,

$$f(g(z_0 + \epsilon + mt)) = f(g(z_0 + \epsilon + nt)) = \beta(\epsilon),$$

so that $g(z_0 + \epsilon + mt)$ and $g(z_0 + \epsilon + nt)$ are both equal to the unique root of $f(w) = \beta(\epsilon)$, which lies near $g(z_0 + mt)$. Thus we must have

$$g(z + mt) \equiv g(z + nt),$$

and $g(z)$ has period $(m - n)t$.

THEOREM 3. *If $F(z) = f(g(z)) = g(f(z))$ with $f(z)$ and $g(z)$ non-linear and $F(z)$ of finite order, then $F(z)$ cannot be periodic.*

Proof. By a theorem of Polya **(4)**, $f(z)$ and $g(z)$ are both of order zero unless one of them, say $f(z)$, is a polynomial. If, however, $f(z)$ were a polynomial and $g(z)$ not of zero order, then $f(g(z))$ and $g(f(z))$ could not be of the same order. It follows that $f(z)$ and $g(z)$ are both of zero order. Since $F(z)$ is periodic, there is a path L running to infinity on which $F(z)$ is bounded. Then either $g(z)$ is bounded on L or, if $g(L)$ is unbounded, then $f(z)$ is bounded on $g(L)$. Either case is impossible since f and g are of zero order.

LEMMA 1. *Let $f(z)$, $\alpha(z)$, and $\beta(z)$ be any three entire functions such that $f(\alpha(z)) = f(\beta(z))$. If there exists a number z_0 such that $\alpha(z_0) = \beta(z_0)$ and $f'(\alpha(z_0)) \neq 0$, then $\alpha(z)$ is identical with $\beta(z)$.*

Proof. Our hypotheses imply that f is 1–1 on a neighbourhood of $\alpha(z_0)$ and $f(\alpha(z)) = f(\beta(z))$ in this neighbourhood. It follows that $\alpha(z)$ is identical with $\beta(z)$ on a continuum of z and hence everywhere.

LEMMA 2. *Let $f(z)$ be entire and t be any complex number. If*

$$[f(z+2t) - f(z)]\cdot[f(z+t) - f(z)]$$

has no zeros, then $f(z)$ must be of the form

$$e^{p(z)+az}\left(c + \int_k^z p^*(w)e^{-aw}\,dw\right) \tag{3}$$

where a, c, and k are constants and $p(z)$ and $p^*(z)$ satisfy $p(z+t) = p(z)$ and *$p^*(z+t) = p^*(z)$.*

Proof. We have

$$f(z+t) - f(z) = e^{\alpha(z)}, \quad f(z-t) - f(z) = -e^{\alpha(z-t)},$$

and $f(z+t) - f(z-t) = e^{\gamma(z)}$, where $\alpha(z)$ and $\gamma(z)$ are entire functions.

It follows from Borel's theorem that $\alpha(z) = \alpha(z-t) + \text{const.}$, so that

$$\alpha(z) = p(z) + az$$

where $p(z)$ has the property $p(z+t) = p(z)$. Hence $f(z+t) - f(z) = e^{p(z)+az}$.

Let $f(z) = g(z)e^{p(z)+az}$. Then

$$e^{p(z)+az} = f(z+t) - f(z) = (e^{at}g(z+t) - g(z))e^{p(z)+az}.$$

Thus $e^{at}g(z+t) - g(z) = 1$ and, differentiating, we obtain

$$e^{at}g'(z+t) - g'(z) = 0.$$

Let $p^*(z) = e^{az}g'(z)$; then

$$p^*(z+t) = e^{az}e^{at}g'(z+t) = e^{az}g'(z) = p^*(z)$$

and p^* has period t. Hence

$$g(z) = \int_k^z p^*(t)e^{-at}\,dt \quad \text{and} \quad f(z) = e^{p(z)+az}\int_k^z p^*(t)e^{-at}\,dt$$

and our lemma follows.

THEOREM 4. *Let $f(z)$ be an entire function such that $f'(z)$ is never zero. If $g(z)$ is not periodic and not of the form* (3), *then $f(g(z))$ cannot be periodic.*

Proof. Were $f(g(z+nt)) = f(g(z))$ and $f'(z) \neq 0$, it would follow by Lemma 1 that $g(z+nt) - g(z)$ has no zeros; hence, by Lemma 2, $g(z)$ must have the form (3).

We have seen in Theorem 1 that if $\phi(z)$ is entire and $e^{\phi(z)}$ is periodic, then $\phi(z) = p(z) + az$, where $p(z)$ is periodic and a is a constant.

In the opposite direction we have

THEOREM 5. *If $\phi(z)$ is periodic and non-linear, then for any given $k > 0$ there exists an $r > 0$ such that the number of zeros of $e^{\phi(z)} - w$ in a period strip of e^{ϕ} for any w, with $|w| = r$, is greater than k.*

To prove this theorem we need a lemma due to Hayman **(3)**.

LEMMA 3. *Suppose that $f(z)$ is entire and $f(z) \neq 0$ in $|z| < 1$, and that for each $r > 0$ there is a w such that the equation $f(z) = w$ has at most p roots in $|z| < 1$ and $|w| = r$, p being a fixed integer. Then we have for $|z| < 1$*

$$|f'(z)| \leqslant \frac{4(p+1)}{1-|z|^2}|f(z)|.$$

We now proceed with the proof of Theorem 5.

Suppose that for every r there is a w, with $|w| = r$, such that the number of zeros of $e^\phi - w$ in a period strip is less than k. Then the number of zeros of $e^\phi - w$ in $|z| < \rho/\sqrt{2}$ is less than $p = c\rho$, where c is some constant. Thus letting $f(z) = e^{\phi(z)}$ and applying Lemma 4, we have

$$\left|\frac{f'(z)}{f(z)}\right| < \frac{8(c\rho+1)}{\rho} < c'$$

for sufficiently large ρ. Here c' is a constant. Hence $(d(\log f(z)))/dz$ is a constant and consequently $f(z) = e^{cz+b}$, contrary to our hypotheses.

The proof of the following theorem was communicated to the author by I. N. Baker.

THEOREM 6. *If $p(z)$ is a polynomial of degree k greater than 2 and $f(z)$ is any non-constant entire function, then $f(p(z))$ is not periodic.*

Proof. We may assume that $f(z)$ is transcendental.

Suppose that (the necessarily non-constant function) $F(z) = f(p(z))$ is periodic and that the period is i (this may be achieved by a linear change of variable, if necessary). Since the strip $S: -\frac{1}{2} < \operatorname{Im} z \leqslant +\frac{1}{2}$ is a period strip, it follows that $\max|F(z)| = M_F(r)$ must be attained on that part of the circle $|z| = r$ which lies in S. Let z_0 be a point where $|F(z)|$ attains its maximum on $|z| = r$. Solve $p(z_0) = p(z')$, taking that solution z' for which

$$\text{(4)} \qquad \begin{cases} \arg z' \approx \arg z_0 + 2\pi/k, \\ |z| \approx |z_0| = r \end{cases}$$

as $r \to \infty$. Now

$$M_F(r) = |F(z_0)| = |F(z')|$$

and, since

$$F(z') = F(x' + iy') = F(x' + i\tilde{y})$$

for some point

$$x' + i\tilde{y} \quad \text{with } x' = \operatorname{Re} z', |\tilde{y}| < \tfrac{1}{2},$$

we have

$$M_F(r) = |F(z')| = |F(x' + i\tilde{y})| \leqslant M_F(x' + 1)$$

and by (4), since $\arg z_0 \approx 0$ or π,

$$|x' + 1| = |1 + |z'| \cos(\arg z')| \leqslant 1 + |z'|\{\cos(2\pi/k) + \epsilon\} \leqslant \gamma r$$

for some γ with $\cos(2\pi/k) < \gamma < 1$.

Thus for large r we have $M_F(r) \leqslant M_F(\gamma r)$, which can only occur (by the maximum modulus theorem) if $F(z)$ is constant. Thus we have a contradiction.

For $p(z)$ a polynomial of degree 2, Theorem 6 does not hold. The first of our examples in the Introduction illustrates this fact. We can, however, prove the following:

THEOREM 7. *If $p(z)$ is a polynomial of degree 2 and $f(z)$ is periodic, then $f(p(z))$ is not periodic.*

Proof. Let us first prove this for $p(z) = z^2$. We note that if $f(z)$ and $f(z^2)$ are entire periodic functions with periods τ_1 and τ_2 respectively, then there exists an entire function $F(z)$ such that $F(z)$ and $F(z^2)$ are periodic and have the same period τ_2^2/τ_1. For let

$$F(z) = f\left(\frac{\tau_1^2}{\tau_2^2} z\right).$$

Then

$$F\left(z + \frac{\tau_2^2}{\tau_1}\right) = f\left(\frac{\tau_1^2}{\tau_2^2} z + \tau_1\right) = f\left(\frac{\tau_1^2}{\tau_2^2} z\right) = F(z),$$

$$F\left(\left(z + \frac{\tau_2^2}{\tau_1}\right)^2\right) = f\left(\left(\frac{\tau_1}{\tau_2} z + \tau_2\right)^2\right) = f\left(\frac{\tau_1^2}{\tau_2^2} z^2\right) = F(z^2).$$

Thus we may assume that for some θ

$$f((z + \theta)^2) = f(z^2) \quad \text{and} \quad f(z + \theta) = f(z).$$

Hence

$$f((z + n\theta)^2 + m\theta) = f(z^2), \qquad m \text{ and } n \text{ integers.} \tag{5}$$

It follows from Lemma 1 that

$$f'\left(\left(\frac{n^2\theta + m}{2n}\right)^2\right) = 0 \qquad \text{for all integers } m, n \ (n \neq 0).$$

Differentiating (5), we obtain

$$2(z + n\theta)f'((z + n\theta)^2 + m\theta) = 2zf'(z^2).$$

We now show that there exists a dense set of values z for which $(z + n_0\theta)^2 + m_0\theta$ is of the form $[(n^2\theta + m)/2n]^2$ for appropriate integers n_0 and m_0.

Setting

$$(z + n_0\theta)^2 + m_0\theta = \left[\frac{n^2\theta + m}{2n}\right]^2, \qquad m_0 = 0 \text{ and } n_0 = \tfrac{1}{2}n, \tag{6}$$

we get

$$z + \frac{n\theta}{2} = \pm\tfrac{1}{2}\left[\frac{n^2\theta + m}{n}\right],$$

so that $z = \frac{1}{2}m/n$ is a solution of (6). Hence $f'(m/2n) \neq 0$ for some integers m and n with

$$(z + n\theta) \neq 0 \quad \text{and} \quad f'\left(\left(\frac{n^2\theta + m}{2n}\right)^2\right) \neq 0,$$

which gives us a contradiction.

To complete the proof, we note that for any polynomial of degree 2

$$az^2 + bz + c = a((z + \tfrac{1}{2}b)^2 + k), \qquad k = c - (\tfrac{1}{2}b)^2.$$

Thus $f(p(z)) = g((z + \frac{1}{2}b)^2)$ where $g(u) = f(a(u + k))$. One can easily verify that if $f(z)$ and $f(p(z))$ are periodic, then the same is true of $g(z)$ and $g(z^2)$ and our proof is complete.

Finally we have

THEOREM 8. *Let $f(z)$ and $g(z)$ be two entire functions with $f(z)$ periodic and $g(z)$ non-linear. If $f(g(z))$ is of finite lower order, then it cannot be periodic.*

Proof. From Polya's theorem **(4)** it follows that either $f(z)$ is of lower order zero or $g(z)$ is a polynomial. By a generalization of Wiman's theorem **(5)**, however, $f(z)$ cannot be of lower order zero, so that $g(z)$ is a polynomial. Our conclusion now follows from Theorems 6 and 7.

We have already seen in the Introduction that if the lower order is infinite, then both $f(z)$ and $f(g(z))$ can be periodic.

Using the arguments of the alternative proof of Theorem 2 and the proof of Theorem 8, one obtains

THEROEM 9. *Let $g(z)$ be non-periodic and not a polynomial of degree $\leqslant 2$. If $f(g)$ is of finite lower order, then it cannot be periodic.*

REFERENCES

1. E. Borel, *Sur les zéros des fonctions entières*, Acta Math. *20* (1897), 357–396.
2. F. Gross, *Entire solutions of the functional equation $\alpha(\beta(z)) = \alpha(\gamma(z)) + c$*, to appear.
3. W. K. Hayman, *Symmetrization in the theory of functions*, Stanford University Tech. Rep. No. 11 (1950), 21.
4. G. Polya, *On an integral function of an integral function*, J. London Math. Soc., *1* (1926), 12.
5. J. M. Whittaker, *The lower order of integral functions*, J. London Math. Soc., *8* (1933), 20–27.

U.S. Naval Research Laboratory,
Washington, D.C.