TOPOLOGICAL TYPES OF KLEIN SURFACES WITH A MAXIMUM ORDER AUTOMORPHISM

by J. A. BUJALANCE†

(Received 4 April, 1986)

1. Introduction. If X is a Klein surface (KS) with boundary, of algebraic genus p, and Φ is an automorphism of order N, May [8] proved that $N \leq 2p + 2$ when X is orientable and p is even, and $N \leq 2p$ otherwise.

He proved also that the unique topological type of an orientable KS having an orientation-preserving automorphism of maximum order is a surface with one boundary component when p is even, with two boundary components when p is odd.

In [8] May asks for the topological types of an orientable KS with boundary having an orientation-reversing automorphism of maximum order, and also for the topological types of non-orientable KS with boundary having an automorphism of maximum order.

In this work we solve this problem. We obtain besides, in all cases, the types of NEC groups providing the automorphism of maximum order. This result is also obtained for the case that May solved in [8].

We prove also that the maximum order of an orientation-reversing automorphism of an orientable KS with boundary, of genus p odd, is $2p - 2$. This bound completes the list given by May [8] of the maximum order of automorphisms. The technique we use is the theory of NEC groups.

2. NEC groups and Klein surfaces. The first modern study of KS is due to Alling and Greenleaf [1]. They are surfaces with or without boundary endowed with a dianalytic structure; the automorphisms of these surfaces are dianalytic homeomorphisms.

In this work we study Klein surfaces using non-euclidean crystallographic groups (NEC groups) [6].

An NEC group is a discrete subgroup Γ of the group of isometries of the non-euclidean plane U ($U = \{z \in \mathbb{C}/\operatorname{Im} z > 0\}$) with compact quotient space, including orientation-reversing elements, reflections and glide-reflections.

NEC groups are classified according to their signatures [6], which are symbols of the form

$$(g; \pm; [m_1, \ldots, m_t]; \{(n_{11}, \ldots, n_{1s_1}) \ldots (n_{k1}, \ldots, n_{ks_k})\}),$$

where g is the genus of the surface U/Γ, the sign $\pm$ indicates whether the surface is orientable or not, the $m_i \geq 2$ (proper periods) represent the branching order of interior points of the surface by the canonical projection $p : U \to U/\Gamma$, the $n_{ij} \geq 2$ (periods of period-cycles) represent the branching order of boundary points of the surface by the above mapping, and k is the number of holes of the surface.

The signature of an NEC group determines a presentation of it given by generators

(i) $x_1, \ldots, x_t,$

(ii) $e_1, \ldots, e_k,$

(iii) $c_{10}, \ldots, c_{1s_1}, \ldots, c_{k0}, \ldots, c_{ks_k},$

(iv) (if the sign is $+$) $a_1, b_1, \ldots, a_g, d_g$

(if the sign is $-$) $d_1, \ldots, d_g$

and the relations

(i) $x_i^{m_i} = 1, i = 1, \ldots, t,$

(ii) $c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1 \ (i = 1, \ldots, k; j = 1, \ldots, s_i),$

(iii) $e_i^{-1} c_{i0} e_i c_{is_i} = 1, i = 1, \ldots, k,$

(iv) (if the sign is $+$)

$x_1 \ldots x_t e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_a a_g^{-1} b_g^{-1} = 1,$

(if the sign is $-$) $x_1 \ldots x_t e_1 \ldots e_k d_1^2 \ldots d_g^2 = 1.$

From now on the letters $x, a, b, c, d, e,$ will be used for these canonical generators of the group.

The relation between NEC groups and KS is a consequence of the following results of Preston [9], Alling–Greenleaf [1] and May [7].

If X is a KS of algebraic genus $p \geq 2$, then X may be represented by U/Γ, where Γ is an NEC group with signature

$$(g; \pm; [-]; \{\underbrace{(-) \ldots (-)}_{k}\}).$$

If Γ is an NEC group then the quotient space U/Γ may be endowed with a structure of KS.

A group H is an automorphism group of the KS U/Γ if and only if $H \cong \Gamma'/\Gamma$ where Γ' is an NEC group such that $\Gamma \triangleleft \Gamma'$ (Γ a bordered surface group).

An NEC group with orientation-reversing elements will be called a proper NEC group.

If Γ is a proper NEC group, then the Fuchsian subgroup of Γ, formed by its orientation-preserving elements, has index 2 in Γ and will be denoted by Γ^+ (canonical Fuchsian group of Γ).

We shall indicate now some results obtained in [2, 3] that will be used throughout this paper.

THEOREM 2.1 [2]. *Let Γ be an NEC group with signature*

$$(g; +; [m_1, \ldots, m_t]; \{(n_{11}, \ldots, n_{1s_1}) \ldots (n_{k1}, \ldots, n_{ks_k})\})$$

and Γ_0 a normal subgroup of Γ such that $[\Gamma:\Gamma_0] = N$, N even. We suppose that:

(i) $C = \{c_j, c_j'\}(j = 1, \ldots, p)$ is the set of pairs of reflections which are generators of Γ, not belonging to Γ_0, and such that $c_j \cdot c_j'$ is an elliptic element of order n_j;

(ii) p_i is the exponent of x_i modulo Γ_0 $(1 \leq i \leq t)$;

(iii) q_j is the exponent of $c_j \cdot c_j'$ modulo Γ_0 $(1 \leq j \leq p)$.

Then the proper periods of Γ_0 are

$$[(m_i/p_i)^{N/p_i}(i = 1, \ldots, t, p_i \neq m_i), (n_j/q_j)^{N/(2q_j)}(j = 1, \ldots, p, q_j \neq n_j)],$$

where by $(-)^r$ we mean that this proper period is repeated r times.

THEOREM 2.2 [3]. Let Γ be an NEC group with signature

$$\sigma(\Gamma) = (g; \pm; [- -]; \{\underbrace{(-) \ldots (-)}_{k}\}),$$

and Γ' an NEC group such that $\Gamma \lhd \Gamma'$ and $\Gamma'/\Gamma \cong Z_N$, N even. Then Γ' has signature

$$\sigma(\Gamma') = (g; \pm; [N/k_1, \ldots, N/k_t]; \{\underbrace{(-) \ldots (-)}_{s}\underbrace{(2, \ldots, 2)}_{r_1'}^{r_1} \ldots \underbrace{(2, \ldots, 2)}_{r_{k'}'}^{r_{k'}}\}),$$

where the numbers r_i are even and

$$k = s - p - t' + \sum_{\substack{n|N \\ n \neq N}} t_n N|n + \sum_{j=1}^{k'} r_j N/4,$$

with $t' = \sum_{\substack{n|N \\ n \neq N}} t_n$ and $t_n \geq 0$, where t_n and p are positive integers.

3. Topological types. If $X = U/\Gamma$ is a compact orientable KS of algebraic genus at least two and Φ is an automorphism of X, it is known that $\langle \Phi \rangle \cong \Gamma'/\Gamma$ where Γ' is an NEC group and $\langle \Phi \rangle$ denotes the group generated by Φ. Besides if $g\Gamma$ is a generator of Γ'/Γ, one may define

THEOREM 3.1. Let $X = U/\Gamma$ be an orientable bordered Klein surface and $G \cong \Gamma'/\Gamma$ the automorphism group of X. Then all the elements of G preserve the orientation of X if and only if all orientation-reversing elements of the canonical generating system of Γ' belong to Γ.

Proof. Let R' be a fundamental region of Γ' (see [11]) and N the order of G. Then if

$$\Gamma'/\Gamma \cong \{y_1\Gamma, y_2\Gamma, \ldots, y_{N-1}\Gamma, \Gamma\},$$

we have that

$$R = R' \cup y_1(R') \cup \ldots \cup y_{N-1}(R')$$

is a fundamental region of Γ; moreover, applying Γ to R, we obtain a tessellation of U, and $R/\Gamma \cong U/\Gamma$.

Suppose that not all reflections or glide reflections of the canonical generator system of Γ' belong to Γ; then let c be a reflection of Γ' not belonging to Γ: we may take $y_i = c$; if we write R' as $A\tau$ where τ is the side fixed by c and A denotes the other sides of the polygon, we have in R the connected region $R' \cup c(R')$. Suppose that $c\Gamma$ is the element of Γ'/Γ corresponding to an automorphism $\Phi \in G$; then Φ reverses the orientation of the interior of R since the image of R' associated to this automorphism is $c(R')$. Hence Φ reverses the orientation of U/Γ.

We would act the same way if there were a glide reflection of Γ' not belonging to Γ.

We now prove the sufficiency. Let all orientation-reversing elements of the canonical generating system of Γ' belong to Γ and suppose that $g\Gamma \in \Gamma'/\Gamma$.

Let $g = w_1 \ldots w_{k-1}w_kw_{k+1} \ldots w_r$ be an expression for g in terms of the canonical generators of Γ', where w_k reverses orientation. Then

$$g\Gamma = (w_1 \ldots w_{k-1})w_k(w_{k+1} \ldots w_r)\Gamma = (w_1 \ldots w_{k-1})w_k\Gamma w_{k+1} \ldots w_r$$
$$= w_1 \ldots w_{k-1}w_{k+1} \ldots w_r\Gamma,$$

as $w_k \in \Gamma$. Thus we can choose all the coset representatives to be orientation-preserving.

Given $\Phi \in G$, let $y_i\Gamma$ be the element of Γ'/Γ associated to Φ. Define now an orientation in the interior of y_sR'; since y_i preserves the orientation, y_iy_sR' has the same orientation. The identification of sides of y_sR' and y_iy_sR' is made by elements of Γ' preserving the orientation (non-orientable generators of Γ' belong to Γ), so Φ preserves the orientation of U/Γ.

THEOREM 3.2. *Let $X = U/\Gamma$ be an orientable KS with k boundary components of algebraic genus $p \geq 2$.*

(a) *If p is even and X has an orientation-reversing automorphism of order $2p + 2$, then $k = p + 1$. Besides, if Φ is an (orientation-reversing) automorphism of X of order $2p + 2$, the group generated by it is $\langle \Phi \rangle \cong \Gamma'/\Gamma$ where Γ' is an NEC group with signature*

$$(0; +; [p + 1]; \{(2, 2)\}).$$

(b) *If p is odd, the maximum order of an orientation-reversing automorphism of X is $2p - 2$, and this value is only obtained when $k = 2$ and $4 \nmid p - 3$ or when $k = 4$ and $4 \mid p - 3$. Besides if Φ is an orientation-reversing automorphism of X of order $2p - 2$, the group generated by it is $\langle \Phi \rangle \cong \Gamma'/\Gamma$, where Γ' is an NEC group with signature*

$$(1; -; [2]; \{(-)\}).$$

Proof. (a) If p is even, the maximum order of an orientation-reversing automorphism is $2p + 2$. If this value is achieved, there exists an NEC group Γ' with $\Gamma'/\Gamma \cong Z_{2p+2}$.

From the relation of areas of the fundamental regions (see [11]) of Γ and Γ' one has $|\Gamma'| = |\Gamma|/(2p + 2)$. Since the area of Γ (see [10]) (up to a factor 2π) is $p - 1$, one has

$$|\Gamma'| = 1/2 - 1/(p + 1). \tag{1}$$

Since Γ has period-cycles, Γ' also has period-cycles.

If Γ' has sign $+$ in its signature, by 2.2 the signature of Γ' has the form

$$(g'; +; [(2p+2)/k_1, \ldots, (2p+2)/k_t]; \{\underbrace{(-)\ldots(-)}_{s}\underbrace{(2,\ldots,2)^{r_1}}_{r_i'}\ldots\underbrace{(2,\ldots,2)^{r_{k'}}}_{r_{k'}'}\}),$$

and so its area is, up to a factor 2π,

$$|\Gamma'| = k_0 + 2g' - 2 + 1/2\sum_{i=1}^{k'}\sum_{j=1}^{r_ir_i'}(1-1/2) + \sum_{i=1}^{t}(1-k_i/(2p+2)), \qquad (2)$$

where k_0 is the number of period-cycles in the signature of Γ', and so $k_0 \geq 1$.

By (1), $|\Gamma'| < 1/2$ and as $k_0 \geq 1$ we have $g' = 0$. We then see that $k_0 = 1$ and now from (1) and (2)

$$(i) \quad k' = 1, r_1'r_1 = 0, t = 2$$

or

$$(ii) \quad k' = 1, r_1'r_1 = 2, t = 1.$$

In case (i) we have from (1) and (2)

$$1/2 - 1/(p+1) = -1 + \sum_{i=1}^{2}(1 - k_i/2(p+1))$$

and hence $k_1 + k_2 - 3 = p$. As k_1 and k_2 both divide $2(p+1)$ the only solution is $k_1 = p+1$, $k_2 = 2$.

Analogously in case (ii) one obtains $k_1 = 2$.

Hence the signatures of Γ' are

$$(\alpha) \quad (0; +; [2, p+1]; \{(-)\})$$

or

$$(\beta) \quad (0; +; [p+1]; \{(2, 2)\}).$$

If the signature of Γ' is (α) then there is just one conjugacy class of reflections in Γ'. As U/Γ has a boundary, Γ also has reflections. Thus if $c \in \Gamma'/\Gamma$ is a reflection and $\theta : \Gamma' \to Z_{2p+2}$ is the homomorphism then $\theta(c) = \bar{0}$ so by Theorem 3.1 the automorphism associated to θ is orientable. In case (β) there exists an epimorphism

$$\theta : \Gamma' \to Z_{2p+2}$$

with $\ker\theta \cong \Gamma$, such that

$$\left\{\begin{array}{l} \theta(c_1) = \overline{p+1}, \\ \theta(c_2) = \bar{0}, \\ \theta(c_3) = \overline{p+1} \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} \theta(c_1) = \bar{0}, \\ \theta(c_2) = \overline{p+1}, \\ \theta(c_3) = \bar{0}. \end{array}\right\}$$

By 2.1 $\theta(x_1) = \bar{a}$ has order $p+1$, and so $\theta(e_1) = \overline{2p+2-a}$, since $\theta(x_1e_1)$ is $\bar{0}$; hence by

Theorem 3.1 the automorphism associated to θ reverses the orientation. As $c_1 c_3 \in \ker \theta$, by [4, 3.1] $k = (2p + 2)/2 = p + 1$.

If Γ' has sign $-$ in its signature, then by 2.2 it has the form

$$(g'; -; [(2p + 2)/k_1, \ldots, (2p + 2)/k_t]; \{\underbrace{(-) \ldots (-)}_{s} \underbrace{(2, \ldots, 2)^{r_1}}_{r_i'} \ldots \underbrace{(2, \ldots, 2)^{r_k}}_{r'_{k'}}\}),$$

and so the area of the associated fundamental region (up to factor 2π) is

$$|\Gamma'| = k_0 + g' - 2 + 1/2 \sum_{i=1}^{k'} \sum_{j=1}^{r_i r_i'} (1 - 1/2) + \sum_{i=1}^{t} (1 - k_i/(2p + 2)), \tag{3}$$

where k_0 is the number of period-cycles of the signature of Γ' and so $k_0 \geq 1$. As $g' \geq 1$, then $|\Gamma'| \geq 1/2$ (since r' is even), and so (3) is never equal to (1).

(b) If p is odd, then by [8] the maximum possible order of an orientation-reversing automorphism of X is $2p$. If this bound is attained, there exists an NEC group Γ' with $\Gamma'/\Gamma \cong Z_{2p}$.

From the relation of the areas of fundamental regions of Γ and Γ' one has that $|\Gamma'| = |\Gamma|/(2p)$.

Since the area of Γ (up to a factor 2π) is $p - 1$, we have

$$|\Gamma'| = 1/2 - 1/2p. \tag{4}$$

If the signature of Γ' has sign $+$, then by 2.2 it has the form

$$(g'; +; [2p/k_1, \ldots, 2p/k_t]; \{\underbrace{(-) \ldots (-)}_{s} \underbrace{(2, \ldots, 2)^{r_1}}_{r_i'} \ldots \underbrace{(2, \ldots, 2)^{r_k}}_{r'_{k'}}\}),$$

and so its area is

$$|\Gamma'| = k_0 + 2g' - 2 + 1/2 \sum_{i=1}^{k'} \sum_{j=1}^{r_i r_i'} (1 - 1/2) + \sum_{i=1}^{t} (1 - k_i/2p), \tag{5}$$

where $k_0 \geq 1$.

From (4) and (5) we have $g' = 0$; besides, k_0 cannot be 2 since then the remaining quantities of (5) would have a value bigger than $1/2$; if $k_0 = 1$ the only solutions for coincidence of values of (5) and (4) are:

$$(0; +; [2p]; \{(2, 2)\}),$$
$$(0; +; [2, 2p]; \{(-)\}),$$
$$(0; +; [3, 3]; \{(-)\}), p = 3,$$
$$(0; +; [3, 5]; \{(-)\}), p = 15,$$

for which there does not exist the epimorphism we are looking for.

Since there are not automorphisms with maximum order $2p$, we look for automorphisms of order $2p - 2$.

The relation of areas is now

$$|\Gamma'| = |\Gamma|/(2p+2) = (p-1)/(2p-2) = 1/2. \tag{6}$$

If Γ' has sign $+$ in its signature, then by 2.2 that signature has the form

$$(g'; +; [(2p-2)/k_1, \ldots, (2p-2)/k_t]; \{\underbrace{(-) \ldots (-)}_{s}\underbrace{(2, \ldots, 2)^{r_1}}_{r_1'} \ldots \underbrace{(2, \ldots, 2)^{r_{k'}}}_{r'_{k'}}\})$$

and so the area is (up to a factor 2π)

$$|\Gamma'| = k_0 + 2g' - 2 + 1/2 \sum_{i=1}^{k'} \sum_{j=1}^{r_i r_i'} (1 - 1/2) + \sum_{i=1}^{t} (1 - k_i/(2p-2)), \tag{7}$$

where $k_0 \geq 1$. Then the only solutions equating (6) and (7) are

  (i) $(0; +; [-]; \{(2, 2, 2, 2, 2, 2)\})$,
  (ii) $(0; +; [2]; \{(2, 2, 2, 2)\})$,
  (iii) $(0; +; [2, 2]; \{(2, 2)\})$,
  (iv) $(0; +; [2, 2, 2]; \{(-)\})$,
  (v) $(0; +; [4, 4]; \{(-)\})$,
  (vi) $(0; +; [3, 6]; \{(-)\})$,
  (vii) $(0; +; [2]; \{(-)(-)\})$,
  (viii) $(0; +; [-]; \{(2, 2)(-)\})$.

In cases (i), (ii), (iii), (iv), if there exists an epimorphism $\theta : \Gamma' \to Z_{2p-2}$ with any of these signatures, then by [4, 3.1] and 2.1 the group Z_{2p-2} would have only generators of order 2, whence $p = 2$, which is impossible.

 In case (v) there exists no epimorphism whose associated automorphism is non-orientable.

 In case (vii) if there exists $\theta : \Gamma' \to Z_{2p-2}$ with ker $\theta \cong \Gamma$, then by [3, 3.1] and 2.1 one has

$$\theta(x_1) = \overline{p-1} \text{ of order } 2,$$
$$\theta(c_1) = \bar{0} \text{ or } p - 1,$$
$$\theta(c_2) = \bar{0} \text{ or } p - 1,$$
$$\theta(e_1) = \bar{b},$$
$$\theta(e_2) = \theta(e_1^{-1}x_1^{-1}).$$

Since the automorphism associated to θ is non-orientable, then by Theorem 3.1 $\theta(c_1) = \overline{p-1}$ or $\theta(c_2) = \overline{p-1}$, whence $\theta(x_1 c_1) = \bar{0}$ or $\theta(x_1 c_2) = \bar{0}$, and hence (see [5, Theorem 2]) Γ would have sign $-$ in its signature, which is impossible.

In case (viii), by [**3**, 3.1] one has

$$
\begin{cases}
\theta(c_{10}) = \overline{p-1}, \\
\theta(c_{11}) = \bar{0}, \\
\theta(c_{12}) = \overline{p-1}
\end{cases}
\quad \text{or} \quad
\begin{cases}
\theta(c_{10}) = \bar{0}, \\
\theta(c_{11}) = \overline{p-1}, \\
\theta(c_{12}) = \bar{0},
\end{cases}
$$

$$
\theta(c_{20}) = \bar{0} \quad \text{or} \quad \overline{p-1}.
$$

Since $\theta(e_2) = \theta(e_1^{-1})$, one has $\theta(e_1)$ and $p-1$ generate Z_{2p-2}; as $2(p-1)$ is a multiple of 4, $\theta(e_1)$ must have order $2(p-1)$, since otherwise the two elements would not generate the group. Supposing $\theta(e_1) = \bar{1}$ (in other cases we act analogously) one obtains $\theta(e_1^{p-1}c_{10}) = \bar{0}$, and so ker θ has sign $-$ in its signature (which is not possible).

If Γ' has sign $-$ in its signature, then by 2.2 the signature of Γ' has the form

$$
(g; -; [(2p-2)/k_1, \ldots, (2p-2)/k_t]; \{\underbrace{(-) \ldots (-)}_{s}\underbrace{(2, \ldots, 2)}_{r_1'}\^{r_1'} \ldots \underbrace{(2, \ldots, 2)}_{r_{k'}'}\^{r_{k'}'}\}),
$$

and so the area is (up to a factor 2π)

$$
|\Gamma'| = k_0 + g' - 2 + 1/2 \sum_{i=1}^{k'} \sum_{j=1}^{r_i r_i'} (1 - 1/2) + \sum_{i=1}^{t} (1 - k_i/(2p-2)), \tag{8}
$$

where $k_0 \geq 1$ and $g' \geq 1$. The only solutions for (6) equating with (8) are:

$$
\text{(i)} \quad (1; -; [-]; \{(2,2)\}),
$$

$$
\text{(ii)} \quad (1; -; [2]; \{(-)\}).
$$

If Γ' has signature (i) and there exists an epimorphism $\theta : \Gamma' \to Z_{2p-2}$ with ker $\theta \cong \Gamma$, we have, by [**3**, 3.1]

$$
\begin{cases}
\theta(c_{10}) = \overline{p-1}, \\
\theta(c_{11}) = \bar{0}, \\
\theta(c_{12}) = \overline{p-1}
\end{cases}
\quad \text{or} \quad
\begin{cases}
\theta(c_{10}) = \bar{0}, \\
\theta(c_{11}) = \overline{p-1}, \\
\theta(c_{12}) = \bar{0}
\end{cases}
$$

with $\theta(e_1) = \theta(d_1^{-2})$; so $p-1$ and $\theta(d_1)$ generate Z_{2p-2}. An argument similar to the case (viii) above allows us, without loss of generality, to suppose that $\theta(d_1) = \bar{1}$; from which one deduces $\theta(d_1^{p-1}c_{10}) = \bar{0}$ and $d_1^{p-1}c_{10}$ is a non-orientable element since $p-1$ is even; hence ker θ would have sign $-$ in its signature and it would not be isomorphis to Γ.

If Γ' has signature (ii), any epimorphism $\theta : \Gamma' \to Z_{2p-2}$ with ker $\theta \cong \Gamma$ should satisfy

$$
\theta(x_1) = \overline{p-1},
$$
$$
\theta(c_{10}) = \bar{0},
$$
$$
\theta(e_1) = \theta(x_1^{-1}d_1^{-2}).
$$

As $\overline{p-1}$ and $\theta(d_1)$ generate Z_{2p-2} we may suppose $\theta(d_1) = \bar{1}$, and so $\theta(e_1) = \overline{p-3}$. By [**5**, Theorem 2] $U/\ker \theta$ is an orientable KS, and the automorphism associated to θ is non-orientable.

Now we look for the number of boundary components of $U/\ker \theta$; by [**4**, Lemma 1] it is enough to know the minimum power of e_1 belonging to $\ker \theta$. If it is α, then $\alpha(p-3) = n(2p-2)$, n being an integer number, and α divides $p-1$, since $p-3$ is even. Hence $\alpha = 2n(p-1)/(p-3) = 2n + 2(2n/(p-3))$. As $\alpha \leq p-1$, we have

$$2n(p-1)/(p-3) \leq p-1,$$

and so $2n/(p-3) \leq 1$. As α is an integer, we have

$$2n/(p-3) = 1 \quad \text{or} \quad 2n/(p-3) = 1/2,$$

and so $p = 3 + 2n$ or $p = 3 + 4n$.

If $p = 3 + 2n$, we have $\alpha = p-1$. If $p = 3 + 4n$ we have $\alpha = (p-1)/2$. So, if $p - 3 = 4$, the minimum power α of e_1 such that $\theta(e_1^\alpha) = \bar{0}$ is $(p-1)/2$. Hence by [**4**, Lemma 1], $k = 2(p-1)/(p-1) = 2$.

If $p - 3 \neq 4$, the minimum power α of e_1 with $(e_1^\alpha) = \bar{0}$ is $p-1$, and by [**4**, Lemma 1] we have $k = 2(p-1)/(p-1) = 2$.

In a similar way we have the following theorems; part of the next theorem appears in May [**8**].

THEOREM 3.3. *Let $X = U/\Gamma$ be an orientable KS with k boundary components of algebraic genus $p \geq 2$.*

(a) *If p is even and X has an orientation-preserving automorphism of order $2p + 2$, then $k = 1$. Besides, if Φ is an automorphism of X of order $2p + 2$, the group generated by it is $\langle \Phi \rangle \cong \Gamma'/\Gamma$, where Γ' is an NEC group with signature*

$$(0; +; [2, p+1]; \{(-)\}).$$

(b) *If p is odd and X has an orientation-preserving automorphism of order $2p$, then $k = 2$. Besides, if Φ is an automorphism of X of order $2p$, the group generated by it is $\langle \Phi \rangle \cong \Gamma'/\Gamma$, and Γ' is an NEC group with signature*

$$(0; +; [2, 2p]; \{(-)\}).$$

THEOREM 3.4. *Let $X = U/\Gamma$ be a non-orientable KS with k boundary components, of algebraic genus $p \geq 2$.*

If X has an automorphism of order $2p$, then $k = p$. Besides, if Φ is an automorphism of X of order $2p$, the group generated by it is $\langle \Phi \rangle \cong \Gamma'/\Gamma$, and Γ' is an NEC group with signature

$$(0; +; [2p]; \{(2, 2)\}).$$

This paper forms part of the doctoral thesis of the author, written under the direction of Prof. E. Bujalance. I express to him my acknowledgement.

The author wishes to thank the referee for his helpful suggestions.

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J. A. BUJALANCE
DEP. DE MATEMATICAS
CENTRO UNIVERSITARIO DE TOLEDO
TOLEDO
SPAIN