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CENTRALISERS IN THE INFINITE SYMMETRIC INVERSE SEMIGROUP

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Abstract

For an arbitrary set X (finite or infinite), denote by $\mathcal{I}(X)$ the symmetric inverse semigroup of partial injective transformations on X. For $\alpha \in \mathcal{I}(X)$, let $C(\alpha) = \{\beta \in \mathcal{I}(X) : \alpha\beta = \beta\alpha\}$ be the centraliser of α in $\mathcal{I}(X)$. For an arbitrary $\alpha \in \mathcal{I}(X)$, we characterise the transformations $\beta \in \mathcal{I}(X)$ that belong to $C(\alpha)$, describe the regular elements of $C(\alpha)$, and establish when $C(\alpha)$ is an inverse semigroup and when it is a completely regular semigroup. In the case where $\mathrm{dom}(\alpha) = X$, we determine the structure of $C(\alpha)$ in terms of Green's relations.

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1. Introduction

For an element a of a semigroup S, the *centraliser* $C(a)$ of a in S is defined by $C(a) = \{x \in S : ax = xa\}$. It is clear that $C(a)$ is a subsemigroup of S. For a set X, we denote by $P(X)$ the semigroup of partial transformations on X (functions whose domain and image are included in X), where the multiplication is the composition of functions. The transformation on X with the empty set as its domain is the zero in $P(X)$, which we will denote by $\emptyset$. By a transformation semigroup, we will mean any subsemigroup S of $P(X)$. Among transformation semigroups, we have the semigroup $T(X)$ of full transformations on X (elements of $P(X)$ whose domain is X).

Numerous papers have been published on centralisers in finite transformation semigroups, for example [6, 8, 15–17, 20, 23–25, 31]. For an infinite X, the centralisers of idempotent transformations in $T(X)$ have been studied in [2, 3, 30]. The cardinalities of $C(\alpha)$, for certain types of $\alpha \in T(X)$, have been established for a countable X in [12–14]. The author has investigated the centralisers of transformations in $T(X)$ with a coauthor in [5] and in the semigroup $\Gamma(X)$ of injective elements of $T(X)$ [18, 19].

This research has been motivated by the fact that if a transformation semigroup S contains an identity 1 or a zero 0, then for any $\alpha \in S$, the centraliser $C(\alpha)$ is a generalisation of S in the sense that $S = C(1)$ and $S = C(0)$. It is therefore of interest to find out which ideas, approaches, and techniques used to study S can be extended to the centralisers of its elements, and how these centralisers differ as semigroups from S. Centralisers of transformations are also important since they appear in various areas of mathematical research, for example, in the study of automorphism groups of semigroups [4]; in the theory of unary algebras [11, 29]; and in the study of commuting graphs [1, 7, 10].

Denote by $\mathcal{I}(X)$ the symmetric inverse semigroup on a set X, which is the subsemigroup of $P(X)$ that consists of all *partial injective* transformations on X. The semigroup $\mathcal{I}(X)$ is universal for the important class of inverse semigroups (see [9, Ch. 5] and [26]) since every inverse semigroup can be embedded in some $\mathcal{I}(X)$ [9, Theorem 5.1.7]. This is analogous to the fact that every group can be embedded in some symmetric group $\mathrm{Sym}(X)$ of permutations on X. We note that $\mathrm{Sym}(X)$ is the group of units of $\mathcal{I}(X)$.

The purpose of this paper is to study centralisers in the infinite symmetric inverse semigroup $\mathcal{I}(X)$. (Centralisers in the finite $\mathcal{I}(X)$ have been studied in [22].) In Section 2 we show that any $\alpha \in \mathcal{I}(X)$ can be uniquely expressed as a join of disjoint cycles, rays and chains. This is analogous to expressing any permutation $\sigma \in \mathrm{Sym}(X)$ as a product of disjoint (finite or infinite) cycles [28, Theorem 1.3.4]. Let $\alpha \in \mathcal{I}(X)$. In Section 3 we use the decomposition theorem to characterise the transformations $\beta \in \mathcal{I}(X)$ that are members of $C(\alpha)$. In Section 4 we describe the regular elements of $C(\alpha)$ and establish when $C(\alpha)$ is an inverse semigroup and when it is a completely regular semigroup. In Section 5 we determine Green's relations in $C(\alpha)$ (including the partial orders of $\mathcal{L}$-, $\mathcal{R}$-, and $\mathcal{J}$-classes) for $\alpha \in \mathcal{I}(X)$ such that $\mathrm{dom}(\alpha) = X$.

2. Decomposition of $\alpha \in \mathcal{I}(X)$

In this section, we show that every $\alpha \in \mathcal{I}(X)$ can be uniquely decomposed into basic transformations called cycles, rays and chains.

Let $\gamma \in P(X)$. We denote the domain of γ by $\mathrm{dom}(\gamma)$ and the image of γ by $\mathrm{im}(\gamma)$. The union $\mathrm{dom}(\gamma) \cup \mathrm{im}(\gamma)$ will be called the *span* of γ and denoted $\mathrm{span}(\gamma)$. As in [5], we will call γ *connected* if $\gamma \neq \emptyset$ and, for all $x, y \in \mathrm{span}(\gamma)$, there are integers $k, m \geq 0$ such that $x \in \mathrm{dom}(\gamma^k)$, $y \in \mathrm{dom}(\gamma^m)$, and $x\gamma^k = y\gamma^m$, where $\gamma^0 = \mathrm{id}_X$. (We will write mappings on the right and compose from left to right; that is, for $f : A \to B$ and $g : B \to C$, we will write xf, rather than $f(x)$, and $x(fg)$, rather than $g(f(x))$.)

Let $\gamma, \delta \in P(X)$. We say that δ is *contained* in γ (or γ *contains* δ), if $\mathrm{dom}(\delta) \subseteq \mathrm{dom}(\gamma)$ and $x\delta = x\gamma$ for every $x \in \mathrm{dom}(\delta)$. We say that γ and δ are *completely disjoint* if $\mathrm{span}(\gamma) \cap \mathrm{span}(\delta) = \emptyset$.

Definition 2.1. Let M be a set of pairwise completely disjoint elements of $P(X)$. The *join* of the elements of M, denoted $\bigsqcup_{\gamma\in M} \gamma$, is the element of $P(X)$ whose domain is $\bigcup_{\gamma\in M} \mathrm{dom}(\gamma)$ and whose values are defined by

$$x\Big(\bigsqcup_{\gamma\in M} \gamma\Big) = x\gamma_0$$

where γ_0 is the (unique) element of M such that $x \in \mathrm{dom}(\gamma_0)$. If $M = \emptyset$, we define $\bigsqcup_{\gamma\in M} \gamma$ to be $\emptyset$. If $M = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ is finite, we may write the join as $\gamma_1 \sqcup \gamma_2 \sqcup \cdots \sqcup \gamma_k$.

The following result has been proved in [5].

Proposition 2.2. *Let $\alpha \in P(X)$ with $\alpha \neq \emptyset$. Then there exists a unique set M of pairwise completely disjoint, connected elements of $P(X)$ such that $\alpha = \bigsqcup_{\gamma\in M} \gamma$.*

The elements of the set M from Proposition 2.2 are called the *connected components* of α.

Definition 2.3. Let $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ be pairwise distinct elements of X. The following elements of $\mathcal{I}(X)$ will be called *basic* partial transformations on X.

- A *cycle* of length k ($k \geq 1$), written $(x_0\ x_1\ \ldots\ x_{k-1})$, is an element $\sigma \in \mathcal{I}(X)$ with $\mathrm{dom}(\sigma) = \{x_0, x_1, \ldots, x_{k-1}\}$, $x_i\sigma = x_{i+1}$ for all $0 \leq i < k-1$, and $x_{k-1}\sigma = x_0$.
- A *right ray*, written $[x_0\ x_1\ x_2 \ldots\rangle$, is an element $\eta \in \mathcal{I}(X)$ with $\mathrm{dom}(\eta) = \{x_0, x_1, x_2, \ldots\}$ and $x_i\eta = x_{i+1}$ for all $i \geq 0$.
- A *double ray*, written $\langle \ldots\ x_{-1}\ x_0\ x_1 \ldots\rangle$, is an element $\omega \in \mathcal{I}(X)$ such that $\mathrm{dom}(\omega) = \{\ldots, x_{-1}, x_0, x_1, \ldots\}$ and $x_i\omega = x_{i+1}$ for all i.
- A *left ray*, written $\langle \ldots\ x_2\ x_1\ x_0]$, is an element $\lambda \in \mathcal{I}(X)$ with $\mathrm{dom}(\lambda) = \{x_1, x_2, x_3, \ldots\}$ and $x_i\lambda = x_{i-1}$ for all $i > 0$.
- A *chain* of length k ($k \geq 1$), written $[x_0\ x_1\ \ldots\ x_k]$, is an element $\tau \in \mathcal{I}(X)$ with $\mathrm{dom}(\tau) = \{x_0, x_1, \ldots, x_{k-1}\}$ and $x_i\tau = x_{i+1}$ for all $0 \leq i \leq k-1$.

By a *ray* we will mean a double, right, or left ray.

We note the following:

- The span of a basic partial transformation is exhibited by the notation. For example, the span of the right ray $[1\ 2\ 3 \ldots\rangle$ is $\{1, 2, 3, \ldots\}$.
- The left bracket in '$\varepsilon = [x \ldots$' indicates that $x \notin \mathrm{im}(\varepsilon)$; while the right bracket in '$\varepsilon = \ldots\ x]$' indicates that $x \notin \mathrm{dom}(\varepsilon)$. For example, for the chain $\tau = [1\ 2\ 3\ 4]$, $\mathrm{dom}(\tau) = \{1, 2, 3\}$ and $\mathrm{im}(\tau) = \{2, 3, 4\}$.
- A cycle $(x_0\ x_1\ \ldots\ x_{k-1})$ differs from the corresponding cycle in the symmetric group of permutations on X in that the former is undefined for every $x \in X \setminus \{x_0, x_1, \ldots, x_{k-1}\}$, while the latter fixes every such x.

It is clear that the connected components of $\alpha \in \mathcal{I}(X)$ are precisely the basic partial transformations contained in α. Thus, the following decomposition result follows immediately from Proposition 2.2.

Proposition 2.4. *Let $\alpha \in \mathcal{I}(X)$ with $\alpha \neq \emptyset$. Then there exist unique sets A of right rays, B of double rays, C of cycles, P of left rays, and Q of chains such that the transformations in $A \cup B \cup C \cup P \cup Q$ are pairwise disjoint and*

$$\alpha = \bigsqcup_{\eta\in A} \eta \sqcup \bigsqcup_{\omega\in B} \omega \sqcup \bigsqcup_{\sigma\in C} \sigma \sqcup \bigsqcup_{\lambda\in P} \lambda \sqcup \bigsqcup_{\tau\in Q} \tau. \tag{2.1}$$

We will call the join (2.1) the *ray–cycle–chain decomposition* of α. We note the following:

- if $\alpha \in \mathrm{Sym}(X)$, then $\alpha = \bigsqcup_{\omega\in B} \omega \sqcup \bigsqcup_{\sigma\in C} \sigma$ (since $A = P = Q = \emptyset$), which corresponds to the decomposition given in [28, 1.3.4];
- if $\mathrm{dom}(\alpha) = X$, then $\alpha = \bigsqcup_{\eta\in A} \eta \sqcup \bigsqcup_{\omega\in B} \omega \sqcup \bigsqcup_{\sigma\in C} \sigma$ (since $P = Q = \emptyset$), which corresponds to the decomposition given in [21];
- if X is finite, then $\alpha = \bigsqcup_{\sigma\in C} \sigma \sqcup \bigsqcup_{\tau\in Q} \tau$ (since $A = B = P = \emptyset$), which is the decomposition given in [22, Theorem 3.2].

Remark 2.5. Let $\alpha \in \mathcal{I}(X)$ with the ray–cycle–chain decomposition as in (2.1). Then, for every $x \in X$:

(1) if $\sigma \in A$ and $x \in \mathrm{span}(\sigma)$, then $x\alpha^p = x$ for some $p \geq 1$;

(2) if $\lambda \in P$, $\tau \in Q$, and $x \in \mathrm{span}(\lambda) \cup \mathrm{span}(\tau)$, then $x\alpha^p \notin \mathrm{dom}(\alpha)$ for some $p \geq 0$.

3. Members of $C(\alpha)$

In this section, for an arbitrary $\alpha \in \mathcal{I}(X)$, we determine which transformations $\beta \in \mathcal{I}(X)$ belong to $C(\alpha)$. For $\alpha \in P(X)$ and $x, y \in X$, we write $x \xrightarrow{\alpha} y$ if $x \in \mathrm{dom}(\alpha)$ and $x\alpha = y$. The following proposition applies to any semigroup of partial transformations.

Proposition 3.1. *Let S be any subsemigroup of $P(X)$, $\alpha \in S$, and $C(\alpha) = \{\beta \in S : \alpha\beta = \beta\alpha\}$. Then for every $\beta \in S$, $\beta \in C(\alpha)$ if and only if for all $x, y \in X$, the following conditions are satisfied.*

(1) *If $x \xrightarrow{\alpha} y$ and $y \in \mathrm{dom}(\beta)$, then $x \in \mathrm{dom}(\beta)$ and $x\beta \xrightarrow{\alpha} y\beta$.*

(2) *If $x \xrightarrow{\alpha} y$, $x \in \mathrm{dom}(\beta)$, and $y \notin \mathrm{dom}(\beta)$, then $x\beta \notin \mathrm{dom}(\alpha)$.*

(3) *If $x \notin \mathrm{dom}(\alpha)$ and $x \in \mathrm{dom}(\beta)$, then $x\beta \notin \mathrm{dom}(\alpha)$.*

Proof. Suppose that $\beta \in C(\alpha)$, that is, $\alpha\beta = \beta\alpha$. Let $x \xrightarrow{\alpha} y$ and $y \in \mathrm{dom}(\beta)$. Then $x \in \mathrm{dom}(\alpha\beta) = \mathrm{dom}(\beta\alpha) \subseteq \mathrm{dom}(\beta)$. Further, $y\beta = (x\alpha)\beta = (x\beta)\alpha$, and so $x\beta \xrightarrow{\alpha} y\beta$. Let $x \xrightarrow{\alpha} y$, $x \in \mathrm{dom}(\beta)$, and $y \notin \mathrm{dom}(\beta)$. Then $x\beta \notin \mathrm{dom}(\alpha)$ since otherwise we would have $x \in \mathrm{dom}(\beta\alpha) = \mathrm{dom}(\alpha\beta)$, which would imply that $y = x\alpha \in \mathrm{dom}(\beta)$. Let $x \notin \mathrm{dom}(\alpha)$ and $x \in \mathrm{dom}(\beta)$. Then $x\beta \notin \mathrm{dom}(\alpha)$ since otherwise we would have $x \in \mathrm{dom}(\beta\alpha) = \mathrm{dom}(\alpha\beta) \subseteq \mathrm{dom}(\alpha)$. Hence (1)–(3) hold.

Conversely, suppose that (1)–(3) are satisfied. Let $x \in \mathrm{dom}(\alpha\beta)$, that is, $x \in \mathrm{dom}(\alpha)$ and $y = x\alpha \in \mathrm{dom}(\beta)$. Then, by (1), $x \in \mathrm{dom}(\beta)$ and $x\beta \in \mathrm{dom}(\alpha)$, that is, $x \in \mathrm{dom}(\beta\alpha)$. Let $x \in \mathrm{dom}(\beta\alpha)$, that is, $x \in \mathrm{dom}(\beta)$ and $x\beta \in \mathrm{dom}(\alpha)$. Then $x \in \mathrm{dom}(\alpha)$ by (3), and so $y = x\alpha \in \mathrm{dom}(\beta)$ by (2). Hence $x \in \mathrm{dom}(\alpha\beta)$. We have proved that $\mathrm{dom}(\alpha\beta) = \mathrm{dom}(\beta\alpha)$. Let $x \in \mathrm{dom}(\alpha\beta)$. Then $x \xrightarrow{\alpha} x\alpha$, which implies that $x\beta \xrightarrow{\alpha} (x\alpha)\beta$ by (1). But the latter means that $(x\beta)\alpha = (x\alpha)\beta$. Thus $x(\alpha\beta) = x(\beta\alpha)$, and so $\alpha\beta = \beta\alpha$. Hence $\beta \in C(\alpha)$. □

It will be convenient to extend the concept of the chain (see Definition 2.3) by defining the chain $[x_0]$ of length 0 (where $x_0 \in X$) to be the set $\{x_0\}$ and agree that $\mathrm{span}([x_0]) = \{x_0\}$. We also agree that, for a cycle $(y_0\, y_1\, \ldots\, y_{k-1})$ and an integer i, y_i will mean y_r where $r \equiv i \pmod{k}$ and $r \in \{0, \ldots, k-1\}$.

Definition 3.2. Let $\beta \in \mathcal{I}(X)$. Let $\sigma = (x_0 \ldots x_{k-1})$ and $\sigma_1 = (y_0 \ldots y_{k-1})$ be cycles of the same length, $\eta = [x_0\, x_1 \ldots\rangle$ and $\eta_1 = [y_0\, y_1 \ldots\rangle$ be right rays, $\omega = \langle \ldots x_{-1}\, x_0\, x_1 \ldots\rangle$ and $\omega_1 = \langle \ldots y_{-1}\, y_0\, y_1 \ldots\rangle$ be double rays, $\lambda = \langle \ldots x_1\, x_0]$ and $\lambda_1 = \langle \ldots y_1\, y_0]$ be left rays, and $\tau = [x_0 \ldots x_k]$ and $\tau_1 = [y_0 \ldots y_k]$ be chains of the same length (possibly zero).

We say that β *maps* σ *onto* σ_1 if $\operatorname{span}(\sigma_1) \subseteq \operatorname{dom}(\beta)$ and, for some $j \in \{0, \ldots, k-1\}$,

$$x_0\beta = y_j,\, x_1\beta = y_{j+1}, \ldots,\; x_{k-1}\beta = y_{j+k-1};$$

β *maps* η *onto* η_1 if $\operatorname{span}(\eta) \subseteq \operatorname{dom}(\beta)$ and $x_i\beta = y_i$ for all $i \geq 0$; β *maps* ω *onto* ω_1 if $\operatorname{span}(\omega) \subseteq \operatorname{dom}(\beta)$ and, for some j, $x_i\beta = y_{j+i}$ for all i; β *maps* λ *onto* λ_1 if $\operatorname{span}(\lambda) \subseteq \operatorname{dom}(\beta)$ and $x_i\beta = y_i$ for all $i \geq 0$; and β *maps* τ *onto* τ_1 if $\operatorname{span}(\tau) \subseteq \operatorname{dom}(\beta)$ and $x_i\beta = y_i$ for all $i \in \{0, \ldots, k\}$.

Definition 3.3. Let $\eta = [x_0\, x_1 \ldots\rangle$ be a right ray, $\tau = [x_0 \ldots x_k]$ be a chain ($k \geq 0$), $\omega = \langle \ldots x_{-1}\, x_0\, x_1 \ldots\rangle$ be a double ray, and $\lambda = \langle \ldots x_1\, x_0]$ be a left ray.

Any chain $[x_0 \ldots x_i]$, where $i \geq 0$, is an *initial segment* of η; and any chain $[x_0 \ldots x_i]$, where $0 \leq i \leq k$, is an *initial segment* of τ.

Any left ray $\langle \ldots x_{i-1}\, x_i]$, where i is any integer, is an *initial segment* of ω; and any left ray $\langle \ldots x_{i+1}\, x_i]$, where $i \geq 0$, is an *initial segment* of λ.

Any chain $[x_i \ldots x_k]$, where $0 \leq i \leq k$, is a *terminal segment* of τ; and any chain $[x_i \ldots x_0]$, where $i \geq 0$, is a *terminal segment* of λ.

For $\alpha \in \mathcal{I}(X)$, let A, B, C, P, and Q be the sets that occur in the ray–cycle–chain decomposition of α (see (2.1)). By A_α, B_α, C_α, P_α, and Q_α we will mean the following sets:

$$A_\alpha = A, \quad B_\alpha = B, \quad C_\alpha = C, \quad P_\alpha = P, \quad Q_\alpha = Q \cup \{[x_0] : x_0 \notin \operatorname{span}(\alpha)\}.$$

We now have the tools to characterise the members of the centraliser $C(\alpha)$.

Theorem 3.4. *Let $\alpha, \beta \in \mathcal{I}(X)$. Then $\beta \in C(\alpha)$ if and only if for all $\eta \in A_\alpha$, $\omega \in B_\alpha$, $\sigma \in C_\alpha$, $\lambda \in P_\alpha$, and $\tau \in Q_\alpha$, the following conditions are satisfied.*

(1) *If $\operatorname{span}(\eta) \subseteq \operatorname{dom}(\beta)$, then there is $\eta_1 = [y_0\, y_1 \ldots\rangle \in A_\alpha$ or $\omega_1 = \langle \ldots y_{-1}\, y_0\, y_1 \ldots\rangle \in B_\alpha$ such that β maps η onto $[y_j\, y_{j+1} \ldots\rangle$ for some j.*
(2) *If $\operatorname{span}(\eta) \cap \operatorname{dom}(\beta) \neq \emptyset$ but $\operatorname{span}(\eta) \nsubseteq \operatorname{dom}(\beta)$, then there is an initial segment τ' of η such that $\operatorname{span}(\eta) \cap \operatorname{dom}(\beta) = \operatorname{span}(\tau')$ and β maps τ' onto a terminal segment of some $\lambda_1 \in P_\alpha$ or onto a terminal segment of some $\tau_1 \in Q_\alpha$.*
(3) *If $\operatorname{span}(\omega) \subseteq \operatorname{dom}(\beta)$, then β maps ω onto some $\omega_1 \in B_\alpha$.*
(4) *If $\operatorname{span}(\omega) \cap \operatorname{dom}(\beta) \neq \emptyset$ but $\operatorname{span}(\omega) \nsubseteq \operatorname{dom}(\beta)$, then there is an initial segment λ' of ω such that $\operatorname{span}(\omega) \cap \operatorname{dom}(\beta) = \operatorname{span}(\lambda')$ and β maps λ' onto some $\lambda_1 \in P_\alpha$.*
(5) *If $\operatorname{span}(\sigma) \cap \operatorname{dom}(\beta) \neq \emptyset$, then β maps σ onto some $\sigma_1 \in C_\alpha$.*
(6) *If $\operatorname{span}(\lambda) \cap \operatorname{dom}(\beta) \neq \emptyset$, then there is an initial segment λ' (possibly λ itself) of λ such that $\operatorname{span}(\lambda) \cap \operatorname{dom}(\beta) = \operatorname{span}(\lambda')$ and β maps λ' onto some $\lambda_1 \in P_\alpha$.*

(7) *If* $\operatorname{span}(\tau) \cap \operatorname{dom}(\beta) \neq \emptyset$*, then there is an initial segment* τ' *(possibly* τ *itself) of* τ *such that* $\operatorname{span}(\tau) \cap \operatorname{dom}(\beta) = \operatorname{span}(\tau')$ *and* β *maps* τ' *onto a terminal segment of some* $\lambda_1 \in P_\alpha$ *or onto a terminal segment of some* $\tau_1 \in Q_\alpha$*.*

PROOF. Suppose that $\beta \in C(\alpha)$. Let $\eta = [x_0\ x_1\ x_2 \ldots\rangle \in A_\alpha$. Then

$$x_0 \xrightarrow{\alpha} x_1 \xrightarrow{\alpha} x_2 \xrightarrow{\alpha} \cdots . \tag{3.1}$$

Suppose that $\operatorname{span}(\eta) \subseteq \operatorname{dom}(\beta)$. Then, by Proposition 3.1,

$$x_0\beta \xrightarrow{\alpha} x_1\beta \xrightarrow{\alpha} x_2\beta \xrightarrow{\alpha} \cdots . \tag{3.2}$$

By Proposition 2.4, there is $\eta_1 = [y_0\ y_1 \ldots\rangle \in A_\alpha$ or $\omega_1 = \langle \ldots\ y_{-1}\ y_0\ y_1 \ldots\rangle \in B_\alpha$ such that $x_0\beta = y_j$ for some j. (By Remark 2.5, $x_0\beta$ cannot be in the span of $\sigma \in A_\alpha$, $\lambda \in P_\alpha$, or $\tau \in Q_\alpha$.) Hence β maps η onto $[y_j\ y_{j+1} \ldots\rangle$ by (3.2).

Suppose that $\operatorname{span}(\eta) \cap \operatorname{dom}(\beta) \neq \emptyset$ but $\operatorname{span}(\eta) \nsubseteq \operatorname{dom}(\beta)$. Then, there is $i \geq 0$ such that $x_i \in \operatorname{dom}(\beta)$ but $x_{i+1} \notin \operatorname{dom}(\beta)$. By (3.1) and Proposition 3.1, $\operatorname{span}(\eta) \cap \operatorname{dom}(\beta) = \{x_0, \ldots, x_i\}$, $x_i\beta \notin \operatorname{dom}(\alpha)$, and

$$x_0\beta \xrightarrow{\alpha} x_1\beta \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} x_i\beta. \tag{3.3}$$

Since $x_i\beta \notin \operatorname{dom}(\alpha)$, it follows by Proposition 2.4 that there is $\lambda_1 = \langle \ldots\ y_1\ y_0] \in P_\alpha$ such that $x_i\beta = y_0$, or there is $\tau_1 = [y_0 \ldots\ y_k] \in Q_\alpha$ such that $x_i\beta = y_k$. Hence, by (3.3), for the initial segment $\tau' = [x_0 \ldots\ x_i]$ of η, β maps τ' onto the terminal segment $[y_{i-1} \ldots\ y_0]$ of λ_1 or onto the terminal segment $[y_{k-i} \ldots\ y_k]$ of τ_1. We have proved (1) and (2). The proofs of (3) and (4) are similar.

Let $\sigma = (x_0 \ldots\ x_{k-1}) \in A_\alpha$. Then

$$x_0 \xrightarrow{\alpha} x_1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} x_{k-1} \xrightarrow{\alpha} x_0.$$

Suppose that $\operatorname{span}(\sigma) \cap \operatorname{dom}(\beta) \neq \emptyset$, that is, $x_i \in \operatorname{dom}(\beta)$ for some i. Then, by Proposition 3.1, $\operatorname{span}(\sigma) \subseteq \operatorname{dom}(\beta)$ and

$$x_0\beta \xrightarrow{\alpha} x_1\beta \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} x_{k-1}\beta \xrightarrow{\alpha} x_0\beta,$$

and so β maps σ onto $\sigma_1 = (x_0\beta \ldots\ x_{k-1}\beta) \in A_\alpha$. This proves (5).

Let $\lambda = \langle \ldots\ x_2\ x_1\ x_0] \in P_\alpha$, so

$$\cdots \xrightarrow{\alpha} x_2 \xrightarrow{\alpha} x_1 \xrightarrow{\alpha} x_0. \tag{3.4}$$

Suppose that $\operatorname{span}(\lambda) \cap \operatorname{dom}(\beta) \neq \emptyset$. Let i be the smallest nonnegative integer such that $x_i \in \operatorname{dom}(\beta)$. By (3.4) and Proposition 3.1, $\operatorname{span}(\lambda) \cap \operatorname{dom}(\beta) = \{\ldots, x_{i+1}, x_i\}$, $x_i\beta \notin \operatorname{dom}(\alpha)$, and

$$\cdots \xrightarrow{\alpha} x_{i+2}\beta \xrightarrow{\alpha} x_{i+1}\beta \xrightarrow{\alpha} x_i\beta. \tag{3.5}$$

Since $x_i\beta \notin \operatorname{dom}(\alpha)$, it follows by Proposition 2.4 that there is $\lambda_1 = \langle \ldots\ y_1\ y_0] \in P_\alpha$ such that $x_i\beta = y_0$, or there is $\tau_1 = [y_0 \ldots\ y_k] \in Q_\alpha$ such that $x_i\beta = y_k$. But the latter is impossible since we would have $y_0 \notin \mathrm{dom}(\alpha)$ and $y_0 = x_{i+k}\beta \in \mathrm{dom}(\alpha)$. Hence, by (3.5), for the initial segment $\lambda' = \langle \ldots\ x_{i+1}\ x_i]$ of λ, β maps λ' onto λ_1. We have proved (6). The proof of (7) is similar.

Conversely, suppose that β satisfies (1)–(7). We will prove that (1)–(3) of Proposition 3.1 hold for β. Let $x, y \in X$. Suppose that $x \xrightarrow{\alpha} y$ and $y \in \mathrm{dom}(\beta)$. If $y \in \mathrm{span}(\eta)$ for some $\eta \in A_\alpha$, then $x \in \mathrm{dom}(\beta)$ and $x\beta \xrightarrow{\alpha} y\beta$ by (1) and (2). Similarly, $x \in \mathrm{dom}(\beta)$ and $x\beta \xrightarrow{\alpha} y\beta$ in each of the remaining possibilities: if $y \in \mathrm{span}(\omega)$ for some $\omega \in B_\alpha$ by (3) and (4); if $y \in \mathrm{span}(\sigma)$ for some $\sigma \in A_\alpha$ by (5); if $y \in \mathrm{span}(\lambda)$ for some $\lambda \in P_\alpha$ by (6); and finally, if $y \in \mathrm{span}(\tau)$ for some $\tau \in Q_\alpha$ by (7).

Suppose that $x \xrightarrow{\alpha} y$, $x \in \mathrm{dom}(\beta)$, and $y \notin \mathrm{dom}(\beta)$. This is only possible when β satisfies (2), (4), (6), or (7) with x being the terminal point of the relevant initial segment, and so $x\beta \notin \mathrm{dom}(\alpha)$. Finally, suppose that $x \notin \mathrm{dom}(\alpha)$ and $x \in \mathrm{dom}(\beta)$. This can only happen when x is the terminal point of some $\lambda \in P_\alpha$ or some $\tau \in Q_\alpha$, and so $x\beta \notin \mathrm{dom}(\alpha)$ by (6) and (7).

Hence β satisfies (1)–(3) of Proposition 3.1, and so $\beta \in C(\alpha)$. □

4. Inverse and completely regular centralisers

In this section, for an arbitrary $\alpha \in \mathcal{I}(X)$, we characterise the regular elements of $C(\alpha)$. We also determine for which $\alpha \in \mathcal{I}(X)$ the centraliser $C(\alpha)$ is an inverse semigroup, and for which $\alpha \in \mathcal{I}(X)$ it is a completely regular semigroup.

An element a of a semigroup S is called *regular* if $a = axa$ for some $x \in S$. If all elements of S are regular, we say that S is a *regular semigroup*. An element $a' \in S$ is called an *inverse* of $a \in S$ if $a = aa'a$ and $a' = a'aa'$. Since regular elements are precisely those that have inverses (if $a = axa$ then $a' = xax$ is an inverse of a), we may define a regular semigroup as a semigroup in which each element has an inverse [9, p. 51].

Two important classes of regular semigroups are inverse semigroups [26] and completely regular semigroups [27]. A semigroup S is called an *inverse semigroup* if every element of S has exactly one inverse [26, Definition II.1.1]. An alternative definition is that S is an inverse semigroup if it is a regular semigroup and its idempotents (elements $e \in S$ such that $ee = e$) commute [9, Theorem 5.1.1]. A semigroup S is called a *completely regular semigroup* if every element of S is in some subgroup of S [9, p. 103].

For $\beta \in P(X)$ and $Y \subseteq X$, we denote by $Y\beta$ the image of Y under β, that is, $Y\beta = \{x\beta : x \in Y \cap \mathrm{dom}(\beta)\}$.

Definition 4.1. Let $\alpha \in \mathcal{I}(X)$, $M_\alpha = A_\alpha \cup B_\alpha \cup C_\alpha \cup P_\alpha \cup Q_\alpha$, and $\beta \in C(\alpha)$. We define a partial transformation Ψ_β on M_α by

$$\mathrm{dom}(\Psi_\beta) = \{\varepsilon \in M_\alpha : \mathrm{span}(\varepsilon) \cap \mathrm{dom}(\beta) \neq \emptyset\},$$
$$\varepsilon\Psi_\beta = \text{the unique } \varepsilon_1 \in M_\alpha \text{ such that } (\mathrm{span}(\varepsilon))\beta \subseteq \mathrm{span}(\varepsilon_1).$$

Note that Ψ_β is well defined and injective by Theorem 3.4; that is, $\Psi_\beta \in \mathcal{I}(M_\alpha)$.

The following lemma follows immediately from Definition 4.1 and Theorem 3.4.

LEMMA 4.2. *Let* $\alpha \in \mathcal{I}(X)$. *Then for all* $\beta, \gamma \in C(\alpha)$:

(1) $\Psi_{\beta\gamma} = \Psi_\beta \Psi_\gamma$;
(2) $A_\alpha \Psi_\beta \subseteq A_\alpha \cup B_\alpha \cup P_\alpha \cup Q_\alpha$;
(3) $B_\alpha \Psi_\beta \subseteq B_\alpha \cup P_\alpha$;
(4) *if* $\sigma \in C_\alpha \cap \mathrm{dom}(\Psi_\beta)$, *then* $\sigma\Psi_\beta$ *is a cycle in* C_α *of the same length as* σ;
(5) $P_\alpha \Psi_\beta \subseteq P_\alpha$;
(6) $Q_\alpha \Psi_\beta \subseteq Q_\alpha \cup P_\alpha$.

LEMMA 4.3. *Let* $\alpha \in \mathcal{I}(X)$ *and let* $\beta, \gamma \in C(\alpha)$ *be such that* $\beta = \beta\gamma\beta$. *Then* $A_\alpha \Psi_\beta \subseteq A_\alpha$, $B_\alpha \Psi_\beta \subseteq B_\alpha$ *and* $Q_\alpha \Psi_\beta \subseteq Q_\alpha$.

PROOF. First, notice that $\Psi_\beta = \Psi_{\beta\gamma\beta}$ (since $\beta = \beta\gamma\beta$), and so $\Psi_\beta = \Psi_\beta\Psi_\gamma\Psi_\beta$ (by Lemma 4.2). Let $\eta \in A_\alpha \cap \mathrm{dom}(\Psi_\beta)$. Then, by Lemma 4.2, $\eta\Psi_\beta \in A_\alpha \cup B_\alpha \cup P_\alpha \cup Q_\alpha$. Suppose that $\eta\Psi_\beta \in B_\alpha$ and let $\omega = \eta\Psi_\beta$. Then

$$\eta\Psi_\beta = \eta(\Psi_\beta\Psi_\gamma\Psi_\beta) = ((\eta\Psi_\beta)\Psi_\gamma)\Psi_\beta = (\omega\Psi_\gamma)\Psi_\beta.$$

But then $\omega\Psi_\gamma = \eta$ (since Ψ_β is injective), which contradicts Lemma 4.2 (since $\omega \in B_\alpha$ and $\eta \in A_\alpha$). Hence $\eta\Psi_\beta \notin B_\alpha$. By similar arguments, $\eta\Psi_\beta$ cannot belong to P_α or Q_α, and so $\eta\Psi_\beta \in A_\alpha$. We have proved that $A_\alpha\Psi_\beta \subseteq A_\alpha$. The proofs that the remaining two inclusions hold are similar. □

LEMMA 4.4. *Let* $\alpha \in \mathcal{I}(X)$ *and let* $\beta, \gamma \in C(\alpha)$ *be such that* $\beta = \beta\gamma\beta$. *Then:*

(1) *if* $\eta = [x_0\, x_1\, \ldots\rangle \in A_\alpha \cap \mathrm{dom}(\Psi_\beta)$ *and* $\eta\Psi_\beta = [y_0\, y_1\, \ldots\rangle \in A_\alpha$, *then* $x_0\beta = y_0$;
(2) *if* $\lambda = \langle \ldots\, x_1\, x_0] \in P_\alpha \cap \mathrm{dom}(\Psi_\beta)$ *and* $\lambda\Psi_\beta = \langle \ldots\, y_1\, y_0] \in P_\alpha$, *then* $x_0 \in \mathrm{dom}(\beta)$ *and* $x_0\beta = y_0$;
(3) *if* $\tau = [x_0 \ldots x_k] \in Q_\alpha \cap \mathrm{dom}(\Psi_\beta)$ *and* $\tau\Psi_\beta = [y_0 \ldots y_m] \in Q_\alpha$, *then* $k = m$, $x_0\beta = y_0$, $x_k \in \mathrm{dom}(\beta)$, *and* $x_k\beta = y_k$.

PROOF. Suppose that $\eta = [x_0\, x_1\, \ldots\rangle \in A_\alpha \cap \mathrm{dom}(\Psi_\beta)$ and $\eta\Psi_\beta = \eta_1 = [y_0\, y_1\, \ldots\rangle \in A_\alpha$. Then, by Theorem 3.4, $\mathrm{span}(\eta) \subseteq \mathrm{dom}(\beta)$ and β maps η onto $[y_j\, y_{j+1}\, \ldots\rangle$ for some j. Since $\beta = \beta\gamma\beta$, we have $x_0\beta = ((x_0\beta)\gamma)\beta = (y_j\gamma)\beta$ and so $y_j\gamma = x_0$ (since β is injective). Thus, by Theorem 3.4 again, γ maps η_1 onto $[x_i\, x_{i+1}\, \ldots\rangle$ for some $i \geq 0$. But since $y_j\gamma = x_0$, this is only possible when $i = j = 0$. Hence $x_0\beta = y_j = y_0$. We have proved (1).

Suppose that $\lambda = \langle \ldots\, x_1\, x_0] \in P_\alpha \cap \mathrm{dom}(\Psi_\beta)$ and $\lambda\Psi_\beta = \lambda_1 = \langle \ldots\, y_1\, y_0] \in P_\alpha$. Then, by Theorem 3.4, β maps some initial segment of λ, say $\langle \ldots\, x_{i+1}\, x_i]$, onto λ_1. Since $\beta = \beta\gamma\beta$, we have $x_i\beta = ((x_i\beta)\gamma)\beta = (y_0\gamma)\beta$ and so $y_0\gamma = x_i$. Thus, by Theorem 3.4 again, γ maps η_1 onto η. Thus $x_i = y_0\gamma = x_0$, so $x_0 = x_i \in \mathrm{dom}(\beta)$ and $x_0\beta = x_i\beta = y_0$. We have proved (2).

Suppose that $\tau = [x_0 \ldots x_k] \in Q_\alpha \cap \mathrm{dom}(\Psi_\beta)$ and $\tau\Psi_\beta = \tau_1 = [y_0 \ldots y_m] \in Q_\alpha$. Then, by Theorem 3.4, β maps some initial segment of τ, say $[x_0 \ldots x_i]$, onto some terminal segment of τ_1, say $[y_j \ldots y_m]$. Then $x_0\beta = ((x_0\beta)\gamma)\beta = (y_j\gamma)\beta$, and so $y_j\gamma = x_0$. But then, by Theorem 3.4, γ maps some initial segment on τ_1, say $[y_0 \ldots y_p]$, onto some terminal segment of τ, say $[x_t \ldots\ x_k]$. Thus $x_0 = y_j\gamma = x_{t+j}$, which implies that $j = t = 0$. Hence β maps $[x_0 \ldots\ x_i]$ onto $[y_0 \ldots\ y_m]$, and γ maps $[y_0 \ldots\ y_p]$ onto $[x_0 \ldots\ x_k]$. It follows that $i = m$ and $p = k$, so $m = i \leq k = p \leq m$. Hence $k = m$ and β maps τ onto τ_1, so $x_0\beta = y_0$, $x_k \in \operatorname{dom}(\beta)$, and $x_k\beta = y_k$. We have proved (3). □

We can now characterise the regular elements of $C(\alpha)$.

Theorem 4.5. *Let $\alpha \in \mathcal{I}(X)$ and $\beta \in C(\alpha)$. Then β is a regular element of $C(\alpha)$ if and only if, for every $\varepsilon \in M_\alpha$:*
(1) *if* $\operatorname{span}(\varepsilon) \cap \operatorname{dom}(\beta) \neq \emptyset$ *then* $\operatorname{span}(\varepsilon) \subseteq \operatorname{dom}(\beta)$*; and*
(2) *if* $\operatorname{span}(\varepsilon) \cap \operatorname{im}(\beta) \neq \emptyset$ *then* $\operatorname{span}(\varepsilon) \subseteq \operatorname{im}(\beta)$.

Proof. Suppose that β is a regular element of $C(\alpha)$, that is, $\beta = \beta\gamma\beta$ for some $\gamma \in C(\alpha)$. Let $\varepsilon \in M_\alpha = A_\alpha \cup B_\alpha \cup C_\alpha \cup P_\alpha \cup Q_\alpha$.

Suppose that $\varepsilon = [x_0\, x_1 \ldots\rangle \in A_\alpha$ and $\operatorname{span}(\varepsilon) \cap \operatorname{dom}(\beta) \neq \emptyset$. Then $\varepsilon\Psi_\beta \in A_\alpha$ by Lemma 4.3, and so $\operatorname{span}(\varepsilon) \subseteq \operatorname{dom}(\beta)$ by Theorem 3.4. Suppose that $\varepsilon = \langle \ldots\ x_1\, x_0] \in P_\alpha$ and $\operatorname{span}(\varepsilon) \cap \operatorname{dom}(\beta) \neq \emptyset$. Then $\varepsilon\Psi_\beta \in P_\alpha$ by Lemma 4.3. Let $\varepsilon_1 = \varepsilon\Psi_\beta = \langle \ldots\ y_1\, y_0]$. By Lemma 4.4, $x_0 \in \operatorname{dom}(\beta)$ and $x_0\beta = y_0$. Thus β maps ε onto ε_1, and so $\operatorname{span}(\varepsilon) \subseteq \operatorname{dom}(\beta)$. If $\varepsilon \in B_\alpha \cup C_\alpha \cup Q_\alpha$, then (1) follows by similar arguments.

Suppose that $\varepsilon = [y_0\, y_1 \ldots\rangle \in A_\alpha$ and $\operatorname{span}(\varepsilon) \cap \operatorname{im}(\beta) \neq \emptyset$. Then $\varepsilon \in \operatorname{im}(\Psi_\beta)$, that is, $\varepsilon = \varepsilon_1\Psi_\beta$ for some $\varepsilon_1 \in M_\alpha$. By Lemmas 4.2 and 4.3, $\varepsilon_1 \in A_\alpha$. Let $\varepsilon_1 = [x_0\, x_1 \ldots\rangle$. By Lemma 4.4, $x_0\beta = y_0$. Hence β maps ε_1 onto ε, and so $\operatorname{span}(\varepsilon) \subseteq \operatorname{im}(\beta)$. Suppose that $\varepsilon = [y_0 \ldots\ y_m] \in Q_\alpha$ and $\operatorname{span}(\varepsilon) \cap \operatorname{dom}(\beta) \neq \emptyset$. Then $\varepsilon \in \operatorname{im}(\Psi_\beta)$, that is, $\varepsilon = \varepsilon_1\Psi_\beta$ for some $\varepsilon_1 \in M_\alpha$. By Lemmas 4.2 and 4.3, $\varepsilon_1 \in Q_\alpha$. Let $\varepsilon_1 = [x_0 \ldots\ x_k]$. By Lemma 4.4, $k = m$, $x_0\beta = y_0$, $x_k \in \operatorname{dom}(\beta)$, and $x_k\beta = y_k$. Hence β maps ε_1 onto ε, and so $\operatorname{span}(\varepsilon) \subseteq \operatorname{im}(\beta)$. If $\varepsilon \in B_\alpha \cup C_\alpha \cup P_\alpha$, then (2) follows by similar arguments.

Conversely, suppose that (1) and (2) hold for every $\varepsilon \in M_\alpha$. We will define $\gamma \in C(\alpha)$ such that $\beta = \beta\gamma\beta$. Set $\operatorname{dom}(\gamma) = \bigcup\{\operatorname{span}(\varepsilon_1) : \varepsilon_1 \in \operatorname{im}(\Psi_\beta)\}$ and note that $\operatorname{dom}(\gamma) = \operatorname{im}(\beta)$. Let $\varepsilon_1 = \lambda_1 \in \operatorname{im}(\Psi_\beta) \cap P_\alpha$. Then $\lambda_1 = \varepsilon\Psi_\beta$ for some $\varepsilon \in M_\alpha$.

Suppose that $\varepsilon \in A_\alpha$. Then, by Theorem 3.4, β maps some initial segment τ' of ε onto a terminal segment of λ_1, and $\operatorname{span}(\varepsilon) \cap \operatorname{dom}(\beta) = \operatorname{span}(\tau')$. But this is impossible since $\operatorname{span}(\varepsilon) \subseteq \operatorname{dom}(\beta)$ by (1). Suppose that $\varepsilon \in B_\alpha$. Then, by Theorem 3.4, β maps some initial segment λ' of ε onto λ, and $\operatorname{span}(\varepsilon) \cap \operatorname{dom}(\beta) = \operatorname{span}(\lambda')$. Again, this contradicts (1). Suppose that $\varepsilon \in Q_\alpha$. Then, by Theorem 3.4, β maps some initial segment τ' of ε onto some terminal segment τ_1 of λ_1. But then $\operatorname{span}(\lambda_1) \cap \operatorname{im}(\beta) = \operatorname{span}(\tau_1)$, which contradicts (2).

Thus $\varepsilon = \lambda \in P_\alpha$ and β maps an initial segment of λ onto λ_1. By (1), that initial segment must be λ. We have proved that for every $\lambda_1 \in \operatorname{im}(\Psi_\beta) \cap P_\alpha$, there is a (necessarily unique) $\lambda \in P_\alpha$ such that β maps λ onto λ_1. By similar arguments, for every $\eta_1 \in \operatorname{im}(\Psi_\beta) \cap A_\alpha$ ($\omega_1 \in \operatorname{im}(\Psi_\beta) \cap B_\alpha$, $\tau_1 \in \operatorname{im}(\Psi_\beta) \cap Q_\alpha$) there is a unique $\eta \in A_\alpha$ ($\omega \in B_\alpha$, $\tau \in Q_\alpha$) such that β maps η onto η_1 (ω onto ω_1, τ onto τ_1).

Let $\eta_1 \in \operatorname{im}(\Psi_\beta) \cap A_\alpha$. Define γ on $\operatorname{span}(\eta_1)$ in such a way that γ maps η_1 onto η (where η is as in the preceding paragraph). Let $\omega_1, \lambda_1, \tau_1 \in \operatorname{im}(\Psi_\beta)$ with $\omega_1 \in B_\alpha$, $\lambda_1 \in P_\alpha$, and $\tau_1 \in Q_\alpha$. We define γ on $\operatorname{span}(\omega_1)$, on $\operatorname{span}(\lambda_1)$, and on $\operatorname{span}(\tau_1)$

in a similar way with the following restriction: if $\omega_1 = \langle \ldots\ y_{-1}\, y_0\, y_1 \ldots \rangle$ and $\omega = \langle \ldots\ x_{-1}\, x_0\, x_1 \ldots \rangle$ with $x_0\beta = y_p$, then $y_i\gamma = x_{i-p}$ for every i.

By the definition of γ and Theorem 3.4, $\gamma \in \mathcal{I}(X)$, $\gamma \in C(\alpha)$, and $\beta = \beta\gamma\beta$. Hence β is a regular element of $C(\alpha)$. □

The class of regular semigroups is larger than the class of inverse semigroups. For example, the semigroups $P(X)$ and $T(X)$ of partial and full transformations on a set X are regular semigroups but not inverse semigroups (unless $|X| = 1$). However, for every subsemigroup S of $\mathcal{I}(X)$, S is a regular semigroup if and only if S is an inverse semigroup. This is because $\mathcal{I}(X)$ is an inverse semigroup, and so its idempotents commute (see the beginning of this section).

Theorem 4.6. *Let $\alpha \in \mathcal{I}(X)$. Then $C(\alpha)$ is an inverse semigroup if and only if $\alpha = \emptyset$ or α is a permutation on its domain.*

Proof. First note that a nonzero $\alpha \in \mathcal{I}(X)$ is a permutation on its domain if and only if it is a join of double rays and cycles; that is, if and only if $A_\alpha = P_\alpha = \emptyset$ and $Q_\alpha = \{[x_0] : x_0 \notin \text{span}(\alpha)\}$.

Suppose that $C(\alpha)$ is inverse and $\alpha \neq \emptyset$. Then, since $\alpha \in C(\alpha)$, there exists $\beta \in C(\alpha)$ with $\alpha = \alpha\beta\alpha = \alpha(\alpha\beta)$ (since $\beta\alpha = \alpha\beta$) and it follows that $\text{im}(\alpha) \subseteq \text{dom}(\alpha\beta) \subseteq \text{dom}(\alpha)$. Also, $\alpha\beta$ is idempotent, so $\alpha\beta = \beta\alpha = \text{id}_Y$ for some Y containing $\text{dom}(\alpha)$ (since $\alpha = \alpha\beta\alpha = \text{id}_Y\alpha$). It follows that $\text{dom}(\alpha) \subseteq \text{im}(\alpha)$ (since if $x \in \text{dom}(\alpha)$, then $x \in Y$, and so $x = x\,\text{id}_Y = x(\beta\alpha) \in \text{im}(\alpha)$). Therefore, $\text{dom}(\alpha) = \text{im}(\alpha)$, and so, since α is injective, it is a permutation on its domain.

Conversely, if $\alpha = \emptyset$ then $C(\alpha) = \mathcal{I}(X)$ is an inverse semigroup. Suppose that $\alpha \neq \emptyset$ and α is a permutation on its domain. Let $\beta \in C(\alpha)$. We will prove that β is regular. Let $\varepsilon \in B_\alpha \cup C_\alpha \cup Q_\alpha$ (recall that $A_\alpha = P_\alpha = \emptyset$). We claim that if $\text{span}(\varepsilon) \cap \text{dom}(\beta) \neq \emptyset$ ($\text{span}(\varepsilon) \cap \text{im}(\beta) \neq \emptyset$), then $\text{span}(\varepsilon) \subseteq \text{dom}(\beta)$ ($\text{span}(\varepsilon) \subseteq \text{im}(\beta)$). Let $\varepsilon = \omega \in B_\alpha$. Suppose that $\text{span}(\omega) \cap \text{dom}(\beta) \neq \emptyset$. Then $\text{span}(\omega) \subseteq \text{dom}(\beta)$ by Theorem 3.4 (since $P_\alpha = \emptyset$). Suppose that $\text{span}(\omega) \cap \text{im}(\beta) \neq \emptyset$. Then, by Theorem 3.4 again, β maps some $\omega_1 \in B_\alpha$ onto ω (since $A_\alpha = \emptyset$), and so $\text{span}(\omega) \subseteq \text{im}(\beta)$. The claim is true for $\varepsilon \in C_\alpha$ by a similar argument, and it is certainly true for $\varepsilon = [x_0] \in Q_\alpha$. (Recall that α does not have any chain of length greater than 0.) Thus β is regular by Theorem 4.5. Hence $C(\alpha)$ is a regular semigroup, and so an inverse semigroup (since the idempotents in $C(\alpha)$ commute). □

Let $\alpha \in \mathcal{I}(X)$. If $C(\alpha)$ is a completely regular semigroup, then it is an inverse semigroup. As the next result shows, the class of completely regular centralisers in $\mathcal{I}(X)$ is much smaller than the class of inverse centralisers. For $n \geq 1$, we denote by C_α^n the subset of C_α consisting of all cycles in C_α of length n.

Theorem 4.7. *Let $\alpha \in \mathcal{I}(X)$. Then $C(\alpha)$ is a completely regular semigroup if and only if:*

(1) *$\alpha = \emptyset$ or α is a permutation on its domain; and*
(2) *$|B_\alpha| \leq 1$, $|Q_\alpha| \leq 1$, and $|C_\alpha^n| \leq 1$ for every $n \geq 1$.*

Proof. Suppose that $C(\alpha)$ is a completely regular semigroup. Then (1) holds by Theorem 4.6. Suppose that $\omega = \langle \dots\ x_{-1}\ x_0\ x_1 \dots \rangle$ and $\omega_1 = \langle \dots\ y_{-1}\ y_0\ y_1 \dots \rangle$ are two distinct double rays in B_α. Define $\beta \in \mathcal{I}(X)$ by $\mathrm{dom}(\beta) = \mathrm{span}(\omega)$ and $x_i\beta = y_i$ for every i. Then $\beta \in C(\alpha)$ by Theorem 3.4, and $\beta^2 = \emptyset$. Thus β is not in a subgroup of $C(\alpha)$ since there is no group with at least two elements and a zero. Hence $|B_\alpha| \le 1$. By similar arguments, $|Q_\alpha| \le 1$ and $|C_\alpha^n| \le 1$ for every $n \ge 1$. Thus (2) holds.

Conversely, suppose that (1) and (2) are satisfied. If $\alpha = \emptyset$, then $X = \{x_0\}$ by (2), and so $C(\alpha) = \mathcal{I}(X) = \{0, \mathrm{id}_X\}$ is a completely regular semigroup. Suppose that $\alpha \ne \emptyset$ and let $\beta \in C(\alpha)$. If $\beta = \emptyset$, then β is an element of a subgroup of $C(\alpha)$, namely $\{0\}$. Suppose that $\beta \ne \emptyset$ and let $Z = \mathrm{dom}(\beta)$. By (1) and Theorem 4.6, β is regular. Hence, by (2) and Theorem 4.5,

$$Z = \mathrm{dom}(\beta) = \mathrm{im}(\beta) = \bigcup \{\mathrm{span}(\varepsilon) : \varepsilon \in \mathrm{dom}(\Psi_\beta)\}. \tag{4.1}$$

Hence, the idempotent $\varepsilon_Z \in \mathcal{I}(X)$ with $\mathrm{dom}(\varepsilon_Z) = Z$ is an element of $C(\alpha)$. We will define $\gamma \in C(\alpha)$ with $\mathrm{dom}(\gamma) = Z$ such that $\beta\gamma = \gamma\beta = \varepsilon_Z$. Let $\omega = \langle \dots\ x_{-1}\ x_0\ x_1 \dots \rangle \in B_\alpha \cap \mathrm{dom}(\Psi_\beta)$. Since $|B_\alpha| \le 1$, β must map ω onto itself, that is, there is p such that $x_i\beta = x_{i+p}$ for every i. We define γ on $\mathrm{span}(\omega)$ by $x_i\gamma = x_{i-p}$ for every i. Let $\sigma = (x_0 \dots\ x_{n-1}) \in C_\alpha \cap \mathrm{dom}(\Psi_\beta)$. Since $|C_\alpha^n| \le 1$, β must map σ onto itself, that is, there is $p \in \{0, \dots, n-1\}$ such that $x_i\beta = x_{i+p}$ for every $i \in \{0, \dots, n-1\}$. We define γ on $\mathrm{span}(\sigma)$ by $x_i\gamma = x_{i-p}$ for every $i \in \{0, \dots, n-1\}$. Let $[x_0] \in Q_\alpha \cap \mathrm{dom}(\Psi_\beta)$. Since $|Q_\alpha| \le 1$, β must map $[x_0]$ onto itself, that is, $x_0\beta = x_0$. We define $x_0\gamma = x_0$.

By the definition of γ, Theorem 3.4, and (4.1), we have $\gamma \in C(\alpha)$, $\mathrm{dom}(\gamma) = \mathrm{im}(\gamma) = Z$, and $\beta\gamma = \gamma\beta = \varepsilon_Z$. Hence the subsemigroup $\langle \beta, \gamma \rangle$ of $C(\alpha)$ generated by β and γ is a group. It follows that $C(\alpha)$ is a completely regular semigroup. □

5. Green's relations

In this section we determine Green's relations in $C(\alpha)$, including the partial orders of $\mathcal{L}$-, $\mathcal{R}$-, and $\mathcal{J}$-classes, for an arbitrary $\alpha \in \mathcal{I}(X)$ such that $\mathrm{dom}(\alpha) = X$.

Denote by $\Gamma(X)$ the subsemigroup of $\mathcal{I}(X)$ consisting of all $\alpha \in \mathcal{I}(X)$ such that $\mathrm{dom}(\alpha) = X$. Green's relations of the centraliser of $\alpha \in \Gamma(X)$ relative to $\Gamma(X)$ have been determined in [18]. However, except for the relation $\mathcal{L}$, the results for the centraliser of $\alpha \in \Gamma(X)$ relative to $\mathcal{I}(X)$ are quite different.

If S is a semigroup and $a, b \in S$, we say that $a \,\mathcal{L}\, b$ if $S^1a = S^1b$, $a \,\mathcal{R}\, b$ if $aS^1 = bS^1$, and $a \,\mathcal{J}\, b$ if $S^1aS^1 = S^1bS^1$, where S^1 is the semigroup S with an identity adjoined. We define $\mathcal{H}$ as the intersection of $\mathcal{L}$ and $\mathcal{R}$, and $\mathcal{D}$ as the join of $\mathcal{L}$ and $\mathcal{R}$, that is, the smallest equivalence relation on S containing both $\mathcal{L}$ and $\mathcal{R}$. These five equivalence relations are known as *Green's relations* [9, p. 45]. The relations $\mathcal{L}$ and $\mathcal{R}$ commute [9, Proposition 2.1.3], and consequently $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. Green's relations are one of the most important tools in studying semigroups.

If $\mathcal{G}$ is one of Green's relations and $a \in S$, we denote the equivalence class of a with respect to $\mathcal{G}$ by G_a. Since $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ are defined in terms of principal ideals in S, which are partially ordered by inclusion, we have the induced partial orders in the sets of the equivalence classes of $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$: $L_a \le L_b$ if $S^1a \subseteq S^1b$, $R_a \le R_b$ if $aS^1 \subseteq bS^1$, and $J_a \le J_b$ if $S^1aS^1 \subseteq S^1bS^1$.

Green's relations in the symmetric inverse semigroup are well known [9, Exercise 5.11.2]. For all $\alpha, \beta \in \mathcal{I}(X)$:
(a) $\alpha \mathcal{L} \beta$ if and only if $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$;
(b) $\alpha \mathcal{R} \beta$ if and only if $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$;
(c) $\alpha \mathcal{J} \beta$ if and only if $|\mathrm{dom}(\alpha)| = |\mathrm{dom}(\beta)|$;
(d) $\mathcal{D} = \mathcal{J}$.

Let S be a semigroup and let $\mathcal{G}$ be one of Green's relation in S. For a subsemigroup U of S, denote by $\mathcal{G}^U$ the corresponding Green's relation in U. We always have

$$\mathcal{G}^U \subseteq \mathcal{G} \cap (U \times U)$$

[9, p. 56]. We will say that $\mathcal{G}^U$ is S*-inheritable* if

$$\mathcal{G}^U = \mathcal{G} \cap (U \times U).$$

For example, if U is a regular subsemigroup of S, then $\mathcal{L}^U$, $\mathcal{R}^U$, and $\mathcal{H}^U$ are S-inheritable [9, Proposition 2.4.2]. If $\mathcal{G}^U$ is S-inheritable, then a description of $\mathcal{G}$ carries over to $\mathcal{G}^U$. We will see that $\mathcal{L}$ is the only $\mathcal{I}(X)$-inheritable Green's relation in $C(\alpha)$, where $\mathrm{dom}(\alpha) = X$.

Let $\alpha \in \mathcal{I}(X)$. Then $\mathrm{dom}(\alpha) = X$ if and only if $P_\alpha = Q_\alpha = \emptyset$. Therefore, the following corollary follows immediately from Theorem 3.4 and Definition 4.1.

Corollary 5.1. *Let $\alpha, \beta \in \mathcal{I}(X)$ with $\mathrm{dom}(\alpha) = X$. Then $\beta \in C(\alpha)$ if and only if for all $\eta \in A_\alpha$, $\omega \in B_\alpha$, and $\sigma \in C_\alpha$ such that $\eta, \omega, \sigma \in \mathrm{dom}(\Psi_\beta)$, the following conditions are satisfied.*
(1) *There is $\eta_1 = [y_0\, y_1\, \ldots\rangle \in A_\alpha$ or $\omega_1 = \langle \ldots\, y_{-1}\, y_0\, y_1\, \ldots\rangle \in B_\alpha$ such that β maps η onto $[y_j\, y_{j+1}\, \ldots\rangle$ for some j.*
(2) *β maps ω onto some $\omega_1 \in B_\alpha$.*
(3) *β maps σ onto some $\sigma_1 \in C_\alpha$.*

We will use Corollary 5.1 frequently, not always referring to it explicitly.

Theorem 5.2. *Let $\alpha \in \mathcal{I}(X)$ with $\mathrm{dom}(\alpha) = X$, and let $\beta, \gamma \in C(\alpha)$. Then $L_\beta \le L_\gamma$ if and only if $\mathrm{im}(\beta) \subseteq \mathrm{im}(\gamma)$. Consequently, $\beta \mathcal{L} \gamma$ if and only if $\mathrm{im}(\beta) = \mathrm{im}(\gamma)$.*

Proof. Suppose that $L_\beta \le L_\gamma$. Then $\beta = \delta\gamma$ for some $\delta \in C(\alpha)$, and so $\mathrm{im}(\beta) = \mathrm{im}(\delta\gamma) \subseteq \mathrm{im}(\gamma)$. Conversely, suppose that $\mathrm{im}(\beta) \subseteq \mathrm{im}(\gamma)$. Then $\beta = \delta\gamma$ for some $\gamma \in \mathcal{I}(X)$. We may assume that $\mathrm{dom}(\delta) = \mathrm{dom}(\beta)$. It now suffices to show that $\delta \in C(\alpha)$. Since $\mathrm{dom}(\alpha) = X$, $\beta \in C(\alpha)$, and $\mathrm{dom}(\beta) = \mathrm{dom}(\delta)$, it follows by Proposition 3.1 that for every $x \in X$,

$$x \in \mathrm{dom}(\delta) \Leftrightarrow x\alpha \in \mathrm{dom}(\delta). \tag{5.1}$$

We claim that $\mathrm{dom}(\alpha\delta) = \mathrm{dom}(\delta\alpha)$. Indeed, it follows from (5.1) and $\mathrm{dom}(\alpha) = X$ that for every $x \in X$,

$$x \in \mathrm{dom}(\alpha\delta) \Leftrightarrow x\alpha \in \mathrm{dom}(\delta) \Leftrightarrow x \in \mathrm{dom}(\delta) \Leftrightarrow x \in \mathrm{dom}(\delta\alpha).$$

We have $(\alpha\delta)\gamma = \alpha\beta = \beta\alpha = (\delta\gamma)\alpha = (\delta\alpha)\gamma$ and $\operatorname{im}(\delta) \subseteq \operatorname{dom}(\gamma)$ (since $\beta = \delta\gamma$ and $\operatorname{dom}(\beta) = \operatorname{dom}(\gamma)$). Let x be an element of the common domain of $\alpha\delta$ and $\delta\alpha$. Then $x(\alpha\delta) \in \operatorname{im}(\delta)$, and so $x(\alpha\delta) \in \operatorname{dom}(\gamma)$. Thus $(x(\alpha\delta))\gamma = (x(\delta\alpha))\gamma$ (since $(\alpha\delta)\gamma = (\delta\alpha)\gamma$), and so $x(\alpha\delta) = x(\delta\alpha)$ (since γ is injective). Hence $\alpha\delta = \delta\alpha$, which concludes the proof. □

As we have already mentioned, other Green's relations in $C(\alpha)$ are not $\mathcal{I}(X)$-inheritable. For their characterisation, we will need the following notation.

Notation 5.3. Let $\alpha, \beta \in \mathcal{I}(X)$ with $\beta \in C(\alpha)$. Suppose that $\eta = [x_0\, x_1 \ldots\rangle \in A_\alpha \cap \operatorname{dom}(\Psi_\beta)$ and $\eta\Psi_\beta = [y_0\, y_1 \ldots\rangle \in A_\alpha$. Then β maps η onto $[y_i\, y_{i+1} \ldots\rangle$ for some $i \geq 0$. We denote the integer i by $(\eta\Psi_\beta)_0$. In other words, $i = (\eta\Psi_\beta)_0$ if and only if $y_i = x_0\beta$.

It may happen that $\eta_1 = \eta\Psi_\beta = \eta\Psi_\gamma$ for some $\gamma \in C(\alpha)$ with $\gamma \neq \beta$. Then the notation $(\eta_1)_0$ would be ambiguous. However, we will always write such an η_1 in the form $\eta\Psi_\beta$ (or $\eta\Psi_\gamma$) so that the ambiguity will never arise.

Proposition 5.4. *Let $\alpha \in \mathcal{I}(X)$ with $\operatorname{dom}(\alpha) = X$, and let $\beta, \gamma \in C(\alpha)$. Then $R_\beta \leq R_\gamma$ if and only if:*

(1) $\operatorname{dom}(\Psi_\beta) \subseteq \operatorname{dom}(\Psi_\gamma)$*; and*

(2) *for every $\eta \in A_\alpha \cap \operatorname{dom}(\Psi_\beta)$, if $\eta\Psi_\beta \in A_\alpha$, then $\eta\Psi_\gamma \in A_\alpha$ and $(\eta\Psi_\gamma)_0 \leq (\eta\Psi_\beta)_0$.*

Proof. Suppose that $R_\beta \leq R_\gamma$, that is, $\beta = \gamma\delta$ for some $\delta \in C(\alpha)$. Then, by Lemma 4.2, $\Psi_\beta = \Psi_{\gamma\delta} = \Psi_\gamma\Psi_\delta$, and so $\operatorname{dom}(\Psi_\beta) \subseteq \operatorname{dom}(\Psi_\gamma)$. Let $\eta = [x_0\, x_1 \ldots\rangle \in A_\alpha \cap \operatorname{dom}(\Psi_\beta)$ and suppose that $\eta\Psi_\beta = [y_0\, y_1 \ldots\rangle \in A_\alpha$. Then $(\eta\Psi_\gamma)\Psi_\delta = \eta(\Psi_\gamma\Psi_\delta) = \eta\Psi_\beta \in A_\alpha$, and so $\eta\Psi_\gamma = [z_0\, z_1 \ldots\rangle \in A_\alpha$ (since $\omega\Psi_\delta \in B_\alpha$ for every $\omega \in B_\alpha$). Let $i = (\eta\Psi_\beta)_0$ and $j = (\eta\Psi_\gamma)_0$, that is, $x_0\beta = y_i$ and $x_0\gamma = z_j$. We have $[z_0\, z_1 \ldots\rangle\Psi_\delta = [y_0\, y_1 \ldots\rangle$, so δ maps $[z_0\, z_1 \ldots\rangle$ onto $[y_p\, y_{p+1} \ldots\rangle$ for some $p \geq 0$. Then $y_i = x_0\beta = (x_0\gamma)\delta = z_j\delta = y_{p+j}$. Thus $i = p + j$, and so $(\eta\Psi_\gamma)_0 = j \leq i = (\eta\Psi_\beta)_0$.

Conversely, suppose that (1) and (2) are satisfied. We will define $\delta \in C(\alpha)$ such that $\beta = \gamma\delta$. Set $\operatorname{dom}(\delta) = \bigcup\{\operatorname{span}(\varepsilon\Psi_\gamma) : \varepsilon \in \operatorname{dom}(\Psi_\beta)\}$. Note that this definition makes sense since $\operatorname{dom}(\Psi_\beta) \subseteq \operatorname{dom}(\Psi_\gamma)$. Let $\eta = [x_0\, x_1 \ldots\rangle \in A_\alpha \cap \operatorname{dom}(\Psi_\beta)$ and suppose that $\eta\Psi_\beta = [y_0\, y_1 \ldots\rangle \in A_\alpha$. Then $\eta\Psi_\gamma = [z_0\, z_1 \ldots\rangle \in A_\alpha$ by (2). Let $y_i = x_0\beta$ and $z_j = x_0\gamma$, and note that $j \leq i$ by (2). We define δ on $\operatorname{span}(\eta\Psi_\gamma)$ in such a way that δ maps $[z_0\, z_1 \ldots\rangle$ onto $[y_{i-j}\, y_{i-j+1} \ldots\rangle$. Note that $x_0(\gamma\delta) = z_j\delta = y_{i-j+j} = y_i = x_0\beta$.

Let $\eta = [x_0\, x_1 \ldots\rangle \in A_\alpha \cap \operatorname{dom}(\Psi_\beta)$ and suppose that $\eta\Psi_\beta = \langle \ldots\, y_{-1}\, y_0\, y_1 \ldots\rangle \in B_\alpha$. By Lemma 4.2, $\eta\Psi_\gamma = [z_0\, z_1 \ldots\rangle \in A_\alpha$ or $\eta\Psi_\gamma = \langle \ldots\, z_{-1}\, z_0\, z_1 \ldots\rangle \in B_\alpha$. In either case, let $y_i = x_0\beta$ and $z_j = x_0\gamma$. If $\eta\Psi_\gamma = [z_0\, z_1 \ldots\rangle$, we define δ on $\operatorname{span}(\eta\Psi_\gamma)$ in such a way that δ maps $[z_0\, z_1 \ldots\rangle$ onto $[y_{i-j}\, y_{i-j+1} \ldots\rangle$. If $\eta\Psi_\gamma = \langle \ldots\, z_{-1}\, z_0\, z_1 \ldots\rangle \in B_\alpha$, we define δ on $\operatorname{span}(\eta\Psi_\gamma)$ in such a way that δ maps $\langle \ldots\, z_{-1}\, z_0\, z_1 \ldots\rangle$ onto $\langle \ldots\, y_{-1}\, y_0\, y_1 \ldots\rangle$ and $z_j\delta = y_i$. Note that in both cases $x_0(\gamma\delta) = y_i = x_0\beta$.

Let $\omega = \langle \ldots\, x_{-1}\, x_0\, x_1 \ldots\rangle \in B_\alpha \cap \operatorname{dom}(\Psi_\beta)$. By Lemma 4.2, $\omega\Psi_\beta = \langle \ldots\, y_{-1}\, y_0\, y_1 \ldots\rangle \in B_\alpha$ and $\omega\Psi_\gamma = \langle \ldots\, z_{-1}\, z_0\, z_1 \ldots\rangle \in B_\alpha$. Let $y_i = x_0\beta$ and $z_j = x_0\gamma$. We define δ on $\operatorname{span}(\omega\Psi_\gamma)$ in such a way that δ maps $\langle \ldots\, z_{-1}\, z_0\, z_1 \ldots\rangle$ onto $\langle \ldots\, y_{-1}\, y_0\, y_1 \ldots\rangle$ and $z_j\delta = y_i$.

Finally, let $\sigma = (x_0 \dots x_{n-1}) \in C_\alpha \cap \mathrm{dom}(\Psi_\beta)$. By Lemma 4.2, $\sigma\Psi_\beta = (y_0 \dots y_{n-1}) \in C_\alpha$ and $\sigma\Psi_\gamma = (z_0 \dots z_{n-1}) \in C_\alpha$. Let $y_i = x_0\beta$ and $z_j = x_0\gamma$. We define δ on $\mathrm{span}(\sigma\Psi_\gamma)$ in such a way that δ maps $(z_0 \dots z_{n-1})$ onto $(y_0 \dots y_{n-1})$ and $z_j\delta = y_i$.

By the definition of δ and Corollary 5.1, we have $\delta \in \mathcal{I}(X)$, $\delta \in C(\alpha)$, and $\beta = \gamma\delta$. Hence $R_\beta \leq R_\gamma$, which concludes the proof. □

Proposition 5.4 immediately gives us a characterisation of the relation $\mathcal{R}$ in $C(\alpha)$.

Theorem 5.5. *Let $\alpha \in \mathcal{I}(X)$ with $\mathrm{dom}(\alpha) = X$, and let $\beta, \gamma \in C(\alpha)$. Then $\beta \,\mathcal{R}\, \gamma$ if and only if $\mathrm{dom}(\Psi_\beta) = \mathrm{dom}(\Psi_\gamma)$ and for all $\eta \in A_\alpha \cap \mathrm{dom}(\Psi_\beta)$ and $k \geq 0$,*

$$\eta\Psi_\beta \in A_\alpha \quad \text{and} \quad (\eta\Psi_\beta)_0 = k \Leftrightarrow \eta\Psi_\gamma \in A_\alpha \quad \text{and} \quad (\eta\Psi_\gamma)_0 = k.$$

For semigroups S and T, we write $S \leq T$ to mean that S is a subsemigroup of T. Recall that $\Gamma(X) = \{\alpha \in \mathcal{I}(X) : \mathrm{dom}(\alpha) = X\}$. For $\alpha \in \Gamma(X)$, denote by $C'(\alpha)$ the centraliser of α in $\Gamma(X)$, and by $C(\alpha)$ the centraliser of α in $\mathcal{I}(X)$. Then clearly $C'(\alpha) \leq C(\alpha)$.

We note that the relation $\mathcal{R}$ in $C'(\alpha)$ is not $C(\alpha)$-inheritable. Indeed, let $X = \{x_0^1, x_1^1, x_2^1, \dots\} \cup \{x_0^2, x_1^2, x_2^2, \dots\} \cup \dots$, and consider

$$\alpha = [x_0^1\, x_1^1\, x_2^1 \dots\rangle \sqcup [x_0^2\, x_1^2\, x_2^2 \dots\rangle \sqcup \dots \in \Gamma(X).$$

Define $\beta, \gamma \in \Gamma(X)$ by $x_i^n\beta = x_i^{n+1}$ and $x_i^n\gamma = x_i^{2n}$. Then $(\beta, \gamma) \in \mathcal{R}$ in $C(\alpha)$ by Theorem 5.5. However, $|A_\alpha \setminus A_\alpha\Psi_\beta| = 1$ and $|A_\alpha \setminus A_\alpha\Psi_\gamma| = \aleph_0$, and so $(\beta, \gamma) \notin \mathcal{R}$ in $C'(\alpha)$ by [18, Theorem 4.7].

Recall that for $\alpha \in \mathcal{I}(X)$ and $n \geq 1$, $C_\alpha^n = \{\sigma \in C_\alpha : \sigma \text{ has length } n\}$.

Theorem 5.6. *Let $\alpha \in \mathcal{I}(X)$ with $\mathrm{dom}(\alpha) = X$, and let $\beta, \gamma \in C(\alpha)$. Then $\beta \,\mathcal{D}\, \gamma$ if and only if there is a bijection $f : \mathrm{dom}(\Psi_\beta) \to \mathrm{dom}(\Psi_\gamma)$ such that for all $\varepsilon \in \mathrm{dom}(\Psi_\beta)$, $n \geq 1$, and $k \geq 0$:*

(1) *if $\varepsilon \in A_\alpha$ ($\varepsilon \in B_\alpha$, $\varepsilon \in C_\alpha^n$), then $\varepsilon f \in A_\alpha$ ($\varepsilon f \in B_\alpha$, $\varepsilon f \in C_\alpha^n$);*

(2) *$\varepsilon\Psi_\beta \in A_\alpha$ and $(\varepsilon\Psi_\beta)_0 = k \Leftrightarrow (\varepsilon f)\Psi_\gamma \in A_\alpha$ and $((\varepsilon f)\Psi_\gamma)_0 = k$.*

Proof. Suppose that $\beta \,\mathcal{D}\, \gamma$. Then, since $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, there is $\delta \in C(\alpha)$ such that $\beta \,\mathcal{L}\, \delta$ and $\delta \,\mathcal{R}\, \gamma$. Let $\varepsilon \in \mathrm{dom}(\Psi_\beta)$. Then, by Theorem 5.2 and Definition 4.1, there is a unique $\varepsilon_1 \in \mathrm{dom}(\Psi_\delta)$ such that $\varepsilon\Psi_\beta = \varepsilon_1\Psi_\delta$. Define $f : \mathrm{dom}(\Psi_\beta) \to \mathrm{dom}(\Psi_\gamma)$ by $\varepsilon f = \varepsilon_1$. Note that f indeed maps $\mathrm{dom}(\Psi_\beta)$ to $\mathrm{dom}(\Psi_\gamma)$ since $\mathrm{dom}(\Psi_\gamma) = \mathrm{dom}(\Psi_\delta)$ by Theorem 5.5.

Suppose that $\varepsilon_1 = \varepsilon f = \varepsilon' f = \varepsilon'_1$, where $\varepsilon, \varepsilon' \in \mathrm{dom}(\Psi_\beta)$. Then $\varepsilon\Psi_\beta = \varepsilon_1\Psi_\delta = \varepsilon'_1\Psi_\delta = \varepsilon'\Psi_\beta$, and so $\varepsilon = \varepsilon'$ since Ψ_β is injective. Let $\varepsilon_1 \in \mathrm{dom}(\Psi_\gamma)$. Then $\varepsilon_1 \in \mathrm{dom}(\Psi_\delta)$, and so $\varepsilon_1\Psi_\delta \in \mathrm{im}(\Psi_\delta)$. Since $\mathrm{im}(\Psi_\delta) = \mathrm{im}(\Psi_\beta)$, there is $\varepsilon \in \mathrm{dom}(\Psi_\beta)$ such that $\varepsilon\Psi_\beta = \varepsilon_1\Psi_\delta$, so $\varepsilon f = \varepsilon_1$. We have proved that f is a bijection.

Let $\varepsilon \in \mathrm{dom}(\Psi_\beta)$. To prove (1), suppose that $\varepsilon \in A_\alpha$ and $\varepsilon_1 = \varepsilon f$. If $\varepsilon\Psi_\beta \in A_\alpha$ then $\varepsilon_1\Psi_\delta = \varepsilon\Psi_\beta \in A_\alpha$, and so $\varepsilon_1 \in A_\alpha$ by Lemma 4.2. Suppose that $\varepsilon\Psi_\beta = \langle \dots y_{-1}\, y_0\, y_1 \dots\rangle \in B_\alpha$. Then, since $\varepsilon \in A_\alpha$, β maps ε onto $[y_i\, y_{i+1} \dots\rangle$ for some i. We have $\varepsilon_1\Psi_\delta = \varepsilon\Psi_\beta$, so $\varepsilon_1 \in A_\alpha$ or $\varepsilon_1 \in B_\alpha$. The latter is impossible, however, since δ would map ε_1 onto $\varepsilon\Psi_\beta$, which would imply that $\mathrm{span}(\varepsilon\Psi_\beta) \subseteq \mathrm{im}(\delta)$ and contradict the fact that $\operatorname{im}(\beta) = \operatorname{im}(\delta)$. We have proved that if $\varepsilon \in A_\alpha$ then $\varepsilon f \in A_\alpha$. The proofs of (1) in the two remaining cases, when $\varepsilon \in B_\alpha$ and when $\varepsilon \in C^n_\alpha$, are similar.

To prove (2), suppose that $\varepsilon\Psi_\beta \in A_\alpha$ and $\varepsilon_1 = \varepsilon f$. Then $\varepsilon_1\Psi_\delta = \varepsilon\Psi_\beta \in A_\alpha$, and so $\varepsilon_1 \in A_\alpha$ by Lemma 4.2. By Theorem 5.5, $\varepsilon_1 \in \operatorname{dom}(\Psi_\gamma)$, $\varepsilon_1\Psi_\gamma \in A_\alpha$, and $(\varepsilon_1\Psi_\delta)_0 = (\varepsilon_1\Psi_\gamma)_0$. But $\operatorname{im}(\beta) = \operatorname{im}(\delta)$ implies that $(\varepsilon_1\Psi_\beta)_0 = (\varepsilon_1\Psi_\delta)_0$, so $(\varepsilon_1\Psi_\beta)_0 = (\varepsilon_1\Psi_\gamma)_0$. The proof of the converse of (2) is similar.

Conversely, suppose that there exists a bijection $f : \operatorname{dom}(\Psi_\beta) \to \operatorname{dom}(\Psi_\gamma)$ such that (1) and (2) are satisfied for all $\varepsilon \in \operatorname{dom}(\Psi_\beta)$, $n \geq 1$, and $k \geq 0$. We will construct $\delta \in C(\alpha)$ such that $\beta \mathcal{L} \delta$ and $\delta \mathcal{R} \gamma$. We set $\operatorname{dom}(\delta) = \bigcup\{\operatorname{span}(\varepsilon_1) : \varepsilon_1 \in \operatorname{dom}(\Psi_\gamma\}$ (which is equal to $\operatorname{dom}(\gamma)$). Let $\varepsilon_1 = \varepsilon f \in \operatorname{dom}(\Psi_\gamma)$.

Let $\varepsilon_1 \in A_\alpha$. Then $\varepsilon \in A_\alpha$ by (1). Suppose that $\varepsilon\Psi_\beta = [y_0\, y_1 \ldots\rangle \in A_\alpha$ with $i = (\varepsilon\Psi_\beta)_0$. By (2), $\varepsilon_1\Psi_\gamma \in A_\alpha$ and $(\varepsilon_1\Psi_\gamma)_0 = i$. We define δ on $\operatorname{span}(\varepsilon_1)$ in such a way that δ maps ε_1 onto $[y_i\, y_{i+1} \ldots\rangle$. Suppose that $\varepsilon\Psi_\beta = \langle \ldots\, y_{-1}\, y_0\, y_1 \ldots\rangle \in B_\alpha$. Then β maps ε onto $[y_i\, y_{i+1} \ldots\rangle$ for some i. By (2), $\varepsilon_1\Psi_\gamma \notin A_\alpha$, so $\varepsilon_1\Psi_\gamma \in B_\alpha$. We define δ on $\operatorname{span}(\varepsilon_1)$ in such a way that δ maps ε_1 onto $[y_i\, y_{i+1} \ldots\rangle$.

Let $\varepsilon_1 \in B_\alpha$. Then $\varepsilon \in B_\alpha$ by (1), and $\varepsilon\Psi_\beta, \varepsilon_1\Psi_\gamma \in B_\alpha$ by Lemma 4.2. We define δ on $\operatorname{span}(\varepsilon_1)$ in such a way that δ maps ε_1 onto $\varepsilon\Psi_\beta$. Finally, let $\varepsilon_1 \in C^n_\alpha$, where $n \geq 1$. Then $\varepsilon \in C^n_\alpha$ by (1), and $\varepsilon_1\Psi_\gamma \in C^n_\alpha$ by Lemma 4.2. We define δ on $\operatorname{span}(\varepsilon_1)$ in such a way that δ maps ε_1 onto $\varepsilon\Psi_\beta$.

By the definition of δ, Corollary 5.1, Theorems 5.2 and 5.5, we have $\delta \in \mathcal{I}(X)$, $\delta \in C(\alpha)$, $\beta \mathcal{L} \delta$, and $\delta \mathcal{R} \gamma$. Hence $\beta \mathcal{D} \gamma$, which concludes the proof. □

In the semigroup $\mathcal{I}(X)$, we have $\mathcal{J} = \mathcal{D}$. We will see that, in general, this is not true in $C(\alpha)$. The following theorem describes the partial order of the $\mathcal{J}$-classes in $C(\alpha)$.

THEOREM 5.7. *Let $\alpha \in \mathcal{I}(X)$ with $\operatorname{dom}(\alpha) = X$, and let $\beta, \gamma \in C(\alpha)$. Then $J_\beta \leq J_\gamma$ if and only if there is an injection $g : \operatorname{dom}(\Psi_\beta) \to \operatorname{dom}(\Psi_\gamma)$ such that, for all $\varepsilon \in \operatorname{dom}(\Psi_\beta)$ and $n \geq 1$, the following conditions are satisfied.*

(1) *If $\varepsilon \in A_\alpha$, then $\varepsilon g \in A_\alpha \cup B_\alpha$.*

(2) *If $\varepsilon \in B_\alpha$ ($\varepsilon \in C^n_\alpha$), then $\varepsilon g \in B_\alpha$ ($\varepsilon g \in C^n_\alpha$).*

(3) *If $\varepsilon\Psi_\beta \in A_\alpha$, then $(\varepsilon g)\Psi_\gamma \in A_\alpha$ and $((\varepsilon g)\Psi_\gamma)_0 \leq (\varepsilon\Psi_\beta)_0$.*

PROOF. Suppose that $J_\beta \leq J_\gamma$, that is, $\beta = \delta\gamma\kappa$ for some $\delta, \kappa \in C(\alpha)$. Then, by Lemma 4.2, $\Psi_\beta = \Psi_{\delta\gamma\kappa} = \Psi_\delta\Psi_\gamma\Psi_\kappa$, and so if $\varepsilon \in \operatorname{dom}(\Psi_\beta)$, then $\varepsilon \in \operatorname{dom}(\Psi_\delta)$ and $\varepsilon\Psi_\delta \in \operatorname{dom}(\Psi_\gamma)$. Define $g : \operatorname{dom}(\Psi_\beta) \to \operatorname{dom}(\Psi_\gamma)$ by $\varepsilon g = \varepsilon\Psi_\delta$. Then g is injective since Ψ_δ is injective.

Let $\varepsilon \in \operatorname{dom}(\Psi_\beta)$ and $n \geq 1$. Then g satisfies (1) and (2) by Lemma 4.2. Suppose that $\varepsilon\Psi_\beta = [y_0\, y_1 \ldots\rangle \in A_\alpha$. Then $\varepsilon = [x_0\, x_1 \ldots\rangle \in A_\alpha$ by Lemma 4.2, and $((\varepsilon g)\Psi_\gamma)\Psi_\kappa = \varepsilon(\Psi_\delta\Psi_\gamma\Psi_\kappa) = \varepsilon\Psi_\beta \in A_\alpha$. Thus $(\varepsilon g)\Psi_\gamma = [z_0\, z_1 \ldots\rangle \in A_\alpha$ (since $\omega\Psi_\kappa \in B_\alpha$ for every $\omega \in B_\alpha$) and $[z_0\, z_1 \ldots\rangle\Psi_\kappa = [y_0\, y_1 \ldots\rangle$. Let $\varepsilon g = \varepsilon\Psi_\delta = [v_0\, v_1 \ldots\rangle$ and note that $[v_0\, v_1 \ldots\rangle\Psi_\gamma = [z_0\, z_1 \ldots\rangle$. Let $x_0\beta = y_i$, $x_0\delta = v_p$, $v_0\gamma = z_j$, and $z_0\kappa = y_q$ (so $i = (\varepsilon\Psi_\beta)_0$ and $j = ((\varepsilon g)\Psi_\gamma)_0$). Then $y_i = x_0\beta = (x_0\delta)(\gamma\kappa) = (v_p\gamma)\kappa = z_{p+j}\kappa = y_{p+j+q}$. Thus $i = p + j + q$, and so $((\varepsilon g)\Psi_\gamma)_0 = j = i - p - q \leq i = (\varepsilon\Psi_\beta)_0$. This proves (3).

Conversely, suppose that there exists an injection $g : \mathrm{dom}(\Psi_\beta) \to \mathrm{dom}(\Psi_\gamma)$ such that (1)–(3) are satisfied for all $\varepsilon \in \mathrm{dom}(\Psi_\beta)$ and $n \geq 1$. We will construct $\delta, \kappa \in C(\alpha)$ such that $\beta = \delta\gamma\kappa$. Set

$$\mathrm{dom}(\delta) = \bigcup\{\mathrm{span}(\varepsilon) : \varepsilon \in \mathrm{dom}(\Psi_\beta)\},$$
$$\mathrm{dom}(\kappa) = \bigcup\{\mathrm{span}(\varepsilon_1) : \varepsilon_1 = (\varepsilon g)\Psi_\gamma \text{ for some } \varepsilon \in \mathrm{dom}(\Psi_\beta)\}.$$

(Note that $\mathrm{dom}(\delta) = \mathrm{dom}(\beta)$.) Suppose that $\varepsilon \in \mathrm{dom}(\Psi_\beta)$.

Let $\varepsilon = \eta = [x_0\ x_1 \ldots\rangle \in A_\alpha$.

Suppose that $\eta\Psi_\beta = [y_0\ y_1 \ldots\rangle \in A_\alpha$. Then $(\eta g)\Psi_\gamma = [z_0\ z_1 \ldots\rangle \in A_\alpha$ by (3), and so $\eta g = [v_0\ v_1 \ldots\rangle \in A_\alpha$ by Lemma 4.2. Let $x_0\beta = y_i$ and $v_0\gamma = z_j$. Then $j \leq i$ by (3). We define δ on $\mathrm{span}(\eta)$ in such a way that δ maps $[x_0\ x_1 \ldots\rangle$ onto $[v_0\ v_1 \ldots\rangle$; and κ on $\mathrm{span}((\eta g)\Psi_\gamma)$ in such a way that κ maps $[z_0\ z_1 \ldots\rangle$ onto $[y_{i-j}\ y_{i-j+1} \ldots\rangle$. Note that $x_0(\delta\gamma\kappa) = v_0(\gamma\kappa) = z_j\kappa = y_{i-j+j} = y_i = x_0\beta$.

Suppose that $\eta\Psi_\beta = \langle \ldots\ y_{-1}\ y_0\ y_1 \ldots\rangle \in B_\alpha$. By (1) and Lemma 4.2, there are three possible cases to consider.

Case 1. $\eta g = [v_0\ v_1 \ldots\rangle \in A_\alpha$ and $(\eta g)\Psi_\gamma = [z_0\ z_1 \ldots\rangle \in A_\alpha$.

Let $x_0\beta = y_i$ and $v_0\gamma = z_j$. We define δ on $\mathrm{span}(\eta)$ in such a way that δ maps $[x_0\ x_1 \ldots\rangle$ onto $[v_0\ v_1 \ldots\rangle$; and κ on $\mathrm{span}((\eta g)\Psi_\gamma)$ in such a way that κ maps $[z_0\ z_1 \ldots\rangle$ onto $[y_{i-j}\ y_{i-j+1} \ldots\rangle$.

Case 2. $\eta g = [v_0\ v_1 \ldots\rangle \in A_\alpha$ and $(\eta g)\Psi_\gamma = \langle \ldots\ z_{-1}\ z_0\ z_1 \ldots\rangle \in B_\alpha$.

Let $x_0\beta = y_i$ and $v_0\gamma = z_j$. We define δ on $\mathrm{span}(\eta)$ in such a way that δ maps $[x_0\ x_1 \ldots\rangle$ onto $[v_0\ v_1 \ldots\rangle$; and κ on $\mathrm{span}((\eta g)\Psi_\gamma)$ in such a way that κ maps $\langle \ldots\ z_{-1}\ z_0\ z_1 \ldots\rangle$ onto $\langle \ldots\ y_{-1}\ y_0\ y_1 \ldots\rangle$ and $z_j\kappa = y_i$.

Case 3. $\eta g = \langle \ldots\ v_{-1}\ v_0\ v_1 \ldots\rangle \in B_\alpha$ and $(\eta g)\Psi_\gamma = \langle \ldots\ z_{-1}\ z_0\ z_1 \ldots\rangle \in B_\alpha$.

In this case, we define δ and κ exactly as in Case 2.

Let $\varepsilon = \omega = \langle \ldots\ x_{-1}\ x_0\ x_1 \ldots\rangle \in B_\alpha$. Then $\omega\Psi_\beta = \langle \ldots\ y_{-1}\ y_0\ y_1 \ldots\rangle \in B_\alpha$, $\omega g = \langle \ldots\ v_{-1}\ v_0\ v_1 \ldots\rangle \in B_\alpha$ (by (2)), and $(\eta g)\Psi_\gamma = \langle \ldots\ z_{-1}\ z_0\ z_1 \ldots\rangle \in B_\alpha$. Let $x_0\beta = y_i$ and $v_0\gamma = z_j$. We define δ on $\mathrm{span}(\omega)$ in such a way that δ maps $\langle \ldots\ x_{-1}\ x_0\ x_1 \ldots\rangle$ onto $\langle \ldots\ v_{-1}\ v_0\ v_1 \ldots\rangle$ and $x_0\delta = v_0$; and κ on $\mathrm{span}((\eta g)\Psi_\gamma)$ in such a way that κ maps the double chain $\langle \ldots\ z_{-1}\ z_0\ z_1 \ldots\rangle$ onto $\langle \ldots\ y_{-1}\ y_0\ y_1 \ldots\rangle$ and $z_j\kappa = y_i$.

Finally, let $\varepsilon = \sigma = (x_0 \ldots\ x_{n-1}) \in C^n_\alpha$, where $n \geq 1$. Then $\sigma\Psi_\beta = (y_0 \ldots\ y_{n-1}) \in C^n_\alpha$, $\sigma g = (v_0 \ldots\ v_{n-1}) \in C^n_\alpha$ (by (2)), and $(\sigma g)\Psi_\gamma = (z_0 \ldots\ z_{n-1}) \in C^n_\alpha$. Let $x_0\beta = y_i$ and $v_0\gamma = z_j$. We define δ on $\mathrm{span}(\omega)$ in such a way that δ maps $(x_0 \ldots\ x_{n-1})$ onto $(v_0 \ldots\ v_{n-1})$ and $x_0\delta = v_0$; and κ on $\mathrm{span}((\eta g)\Psi_\gamma)$ in such a way that κ maps $(z_0 \ldots\ z_{n-1})$ onto $(y_0 \ldots\ y_{n-1})$ and $z_j\kappa = y_i$.

By the definitions of δ and κ and Corollary 5.1, we have $\delta, \kappa \in \mathcal{I}(X)$, $\delta, \kappa \in C(\alpha)$, and $\beta = \delta\gamma\kappa$. Hence $J_\beta \leq J_\gamma$. □

Theorem 5.7 gives us a characterisation of the relation $\mathcal{J}$ in $C(\alpha)$.

Theorem 5.8. *Let $\alpha \in \mathcal{I}(X)$ with $\mathrm{dom}(\alpha) = X$, and let $\beta, \gamma \in C(\alpha)$. Then $\beta \,\mathcal{J}\, \gamma$ if and only if there are injections $g_1 : \mathrm{dom}(\Psi_\beta) \to \mathrm{dom}(\Psi_\gamma)$ and $g_2 : \mathrm{dom}(\Psi_\gamma) \to \mathrm{dom}(\Psi_\beta)$ such that for all $\varepsilon_1 \in \mathrm{dom}(\Psi_\beta)$, $\varepsilon_2 \in \mathrm{dom}(\Psi_\gamma)$, $n \geq 1$, and $i \in \{1, 2\}$, the following conditions are satisfied.*

(1) *If $\varepsilon_i \in A_\alpha$, then $\varepsilon_i g_i \in A_\alpha \cup B_\alpha$.*
(2) *If $\varepsilon_i \in B_\alpha$ ($\varepsilon_i \in C^n_\alpha$), then $\varepsilon_i g_i \in B_\alpha$ ($\varepsilon_i g_i \in C^n_\alpha$).*
(3) *If $\varepsilon_1 \Psi_\beta \in A_\alpha$, then $(\varepsilon_1 g_1)\Psi_\gamma \in A_\alpha$ and $((\varepsilon_1 g_1)\Psi_\gamma)_0 \leq (\varepsilon_1 \Psi_\beta)_0$.*
(4) *If $\varepsilon_2 \Psi_\gamma \in A_\alpha$, then $(\varepsilon_2 g_2)\Psi_\beta \in A_\alpha$ and $((\varepsilon_2 g_2)\Psi_\beta)_0 \leq (\varepsilon_2 \Psi_\gamma)_0$.*

The injections g_1 and g_2 from Theorem 5.8 cannot be replaced by a bijection $g : \mathrm{dom}(\Psi_\beta) \to \mathrm{dom}(\Psi_\gamma)$. Indeed, let

$$X = \{x^1_0, x^1_1, x^1_2, \ldots\} \cup \{x^2_0, x^2_1, x^2_2, \ldots\} \cup \cdots \cup \{y^1_0, y^1_1, y^1_2, \ldots\} \cup \{y^2_0, y^2_1, y^2_2, \ldots\} \cup \ldots,$$

and consider

$$\alpha = [x^1_0\, x^1_1\, x^1_2 \ldots\rangle \sqcup [x^2_0\, x^2_1\, x^2_2 \ldots\rangle \sqcup \cdots \sqcup [y^1_0\, y^1_1\, y^1_2 \ldots\rangle \sqcup [y^2_0\, y^2_1\, y^2_2 \ldots\rangle \sqcup \cdots \in \Gamma(X).$$

Define $\beta, \gamma \in \mathcal{I}(X)$ by $\mathrm{dom}(\beta) = \{x^{2n}_i : n \geq 1, i \geq 0\}$, $x^{2n}_i \beta = y^{2n}_i$, $\mathrm{dom}(\gamma) = \{x^{2n-1}_i : n \geq 1, i \geq 0\}$, $x^1_i \gamma = y^1_{i+1}$ and $x^{2n-1}_i \gamma = y^{2n-1}_i$ for $n \geq 2$. Then (1)–(4) of Theorem 5.8 are satisfied with $[x^{2n}_0\, x^{2n}_1\, x^{2n}_2 \ldots\rangle g_1 = [x^{2n+1}_0\, x^{2n+1}_1\, x^{2n+1}_2 \ldots\rangle$ and $[x^{2n-1}_0\, x^{2n-1}_1\, x^{2n-1}_2 \ldots\rangle g_2 = [x^{2n}_0\, x^{2n}_1\, x^{2n}_2 \ldots\rangle$ $(n \geq 1)$, so $\beta \,\mathcal{J}\, \gamma$.

However, no bijection $g : \mathrm{dom}(\Psi_\beta) \to \mathrm{dom}(\Psi_\gamma)$ can satisfy (3) of Theorem 5.8. Suppose that such a bijection exists. Then $\varepsilon_1 g = [x^1_0\, x^1_1\, x^1_2 \ldots\rangle$ for some $\varepsilon_1 \in \mathrm{dom}(\Psi_\beta)$ (since g is onto). But then $((\varepsilon_1 g)\Psi_\gamma)_0 = 1$ (since $x^1_0 \gamma = y^1_1$) and $(\varepsilon_1 \Psi_\beta)_0 = 0$ (since $x^{2n}_0 \beta = y^{2n}_0$ for every $n \geq 1$), and so (3) is violated.

By the foregoing argument, there is no bijection $f : \mathrm{dom}(\Psi_\beta) \to \mathrm{dom}(\Psi_\gamma)$ such that (2) of Theorem 5.6 is satisfied. Hence $(\beta, \gamma) \notin \mathcal{D}$ in $C(\alpha)$.

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