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GENERALISED ARMENDARIZ PROPERTIES OF CROSSED PRODUCT TYPE

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Abstract. Let R be a ring and M a monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. We introduce and study the concepts of CM-Armendariz and CM-quasi-Armendariz rings to generalise various Armendariz and quasi-Armendariz properties of rings by working on the context of the crossed product $R * M$ over R. The following results are proved: (1) If M is a u.p.-monoid, then any M-rigid ring R is CM-Armendariz; (2) if I is a reduced ideal of an M-compatible ring R with M a strictly totally ordered monoid, then R/I being CM-Armendariz implies that R is CM-Armendariz; (3) if M is a u.p.-monoid and R is a semiprime ring, then R is CM-quasi-Armendariz. These results generalise and unify many known results on this subject.

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1. Introduction. Throughout, unless otherwise indicated, R denotes an associative ring with identity and M is a monoid. Following the literature, a ring R is called Armendariz (resp., quasi-Armendariz) if for any $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$, $f(x)g(x) = 0$ (resp., $f(x)R[x]g(x) = 0$) implies that $a_i b_j = 0$ (resp., $a_i R b_j = 0$) for all i and j [**5**, **19**]. These rings and various generalisations such as skew Armendariz rings, M-Armendariz rings, M-quasi-Armendariz rings have been discussed in a number of publications (see, for example, [**4**, **5**, **7**, **9**, **10**, **12**, **14**] and [**21**]).

In this paper, we investigate CM-Armendariz rings and CM-quasi-Armendariz rings, which are the Armendariz-like and quasi-Armendariz-like properties defined for the monoid crossed product $R * M$. The former is a common generalisation of Armendariz rings, skew Armendariz rings and M-Armendariz rings, while the latter generalises quasi-Armendariz rings and M-quasi-Armendariz rings. The following results are proved: For a u.p.-monoid M with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$, (1) R being M-rigid implies that R is CM-Armendariz; (2) R being semiprime implies that R is CM-quasi-Armendariz; (3) R being left APP implies that $R * M$ is left APP, in case ω satisfies condition $(*)$. Moreover, if R is an M-compatible ring and M is a strictly totally ordered monoid with twisting f and action ω as above, then for any reduced ideal I of R, R/I being CM-Armendariz implies that R is CM-Armendariz. Since the monoid crossed product $R * M$ is a common generalisation of polynomial rings, skew polynomial rings, the twisted monoid ring, (skew) Laurent polynomial rings and skew monoid rings, our results generalise and unify various known results on Armendariz rings, skew Armendariz rings, quasi-Armendariz rings, M-Armendariz rings and M-quasi-Armendariz rings.

Next, we recall some of the notions and notations needed in this paper. Let $\omega : M \to Aut(R)$ be a monoid homomorphism. For $g \in M$, we denote by ω_g the automorphism $\omega(g)$. The *crossed product* $R * M$ over R consists of all finite sums $R * M = \left\{ \sum r_g g | r_g \in R, g \in M \right\}$ with addition defined componentwise and multiplication defined by the distributive law and two rules that are called *action* and *twisting* explained below: For $g, h \in M$ and $r \in R$, $gr = \omega_g(r)g$ and $gh = f(g, h)gh$, where $f : M \times M \to U(R)$ is a twisted function and $U(R)$ denotes the set of units of R. Here, the twisted function f and the action ω of M on R satisfy the following conditions: $\omega_g(\omega_h(r)) = f(g, h)\omega_{gh}(r)f(g, h)^{-1}$, $\omega_g(f(h, k))f(g, hk) = f(g, h)f(gh, k)$, $f(1, g) = f(g, 1) = 1$ for all $g, h, k \in M$. Notice that monoid crossed product is a quite general ring construction. Let $R * M$ be a monoid crossed product with twisting f and action ω. If the twisting f is trivial, that is $f(x, y) = 1$ for all $x, y \in M$, then $R * M$ is the *skew monoid ring* $R\natural M$. If the action ω is trivial, i.e., $\omega_g = i_R$ with i_R the identity map over R, then $R * M$ is the *twisted monoid ring* $R^{\tau}[M]$. If both the twisting f and the action ω are trivial, then $R * M$ is a *monoid ring*, denoted by $R[M]$ (see [**11**] and [**18**]).

An endomorphism α of a ring R is said to be *compatible* if for any $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$, and to be *rigid* if $a\alpha(a) = 0$, $a \in R$, implies $a = 0$. The ring is called α-*rigid* (resp., α-*compatible*) ring if there exists a rigid (resp., compatible) endomorphism α [**13**]. By [**3**, Lemma 2.2], R is α-rigid if and only if R is α-compatible and reduced. A monoid homomorphism $\omega : M \to Aut(R)$ is said to satisfy *condition* $(*)$ if for every $a \in R$, the left ideal $\sum_{g\in M} R\omega_g(a)$ is finitely generated. We call a ring R an M-*compatible* (resp., M-*rigid*) ring if ω_g is compatible (resp., rigid) for any $g \in M$. A monoid M is called a *u.p.-monoid* (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq M$, there exists an element $g \in M$ uniquely in the form of ab with $a \in A$ and $b \in B$. From now on, $\omega : M \to Aut(R)$ is a monoid homomorphism.

2. M-Armendariz rings of crossed product type. In this section, we consider Armendariz properties relative to a monoid crossed product $R * M$. We begin with the following definition.

DEFINITION 2.1. Let R be a ring and M a monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. The ring R is called an M-Armendariz ring of crossed product type relative to the given twisting f and action ω (or simply, a CM-Armendariz ring) if whenever $\alpha = a_1g_1 + \cdots + a_ng_n$, $\beta = b_1h_1 + \cdots + b_mh_m \in R * M$ satisfy $\alpha\beta = 0$, we have $a_i\omega_{g_i}(b_j) = 0$ for all i, j. In particular, if R is CM-Armendariz with f trivial, then we call R a skew M-Armendariz ring. If R is CM-Armendariz with ω trivial, then R is called a TM-Armendariz (i.e., twisted M-Armendariz) ring.

It is clear that an M-Armendariz ring is just a CM-Armendariz ring with both twisting and action trivial. If $M = (\mathbb{N} \cup \{0\}, +)$ with the trivial twisting f and ω given by $\omega(n) = \alpha^n$ for $n \in M$ where $\alpha \in Aut(R)$, then R is CM-Armendariz if and only if R is α-skew Armendariz. In particular, if both twisting f and action ω are trivial with

$M = (\mathbb{N} \cup \{0\}, +)$, then R is CM-Armendariz if and only R is Armendariz. Some other variants of Armendariz rings can be obtained when specialised to special M, f and ω.

PROPOSITION 2.2. *Let M be a u.p.-monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. If R is M-rigid, then R is CM-Armendariz.*

Proof. It is similar to the proof of [**14**, Proposition 1.1]. □

COROLLARY 2.3 ([**14**, Proposition 1.1]). *Let R be a reduced ring and M a u.p.-monoid. Then, R is M-Armendariz.*

An *ordered monoid* is a pair $(M, \preceq)$ consisting of a monoid M and an order $\preceq$ on M such that for all $g, h, k \in M$, $g \preceq h$ implies $kg \preceq kh$ and $gk \preceq hk$. An ordered monoid $(M, \preceq)$ is said to be *strictly ordered* if for all $g, h, k \in M$, $g \prec h$ implies $kg \prec kh$ and $gk \prec hk$. It is known that torsion-free nilpotent groups and free groups are ordered groups by [**17**, Lemmas 13.1.6 and 13.2.8]. Hence, any submonoid of a torsion-free nilpotent group or a free group is an ordered monoid.

COROLLARY 2.4. *Let R be an M-rigid ring and M a strictly ordered monoid with a monoid homomorphism $\omega : M \to Aut(R)$. Then R is CM-Armendariz. In particular, if M is a strictly ordered monoid and R is a reduced ring, then R is M-Armendariz.*

COROLLARY 2.5. *Let R be an M-rigid ring and M a u.p.-monoid with action $\omega :$ $M \to Aut(R)$. Then R is skew M-Armendariz.*

Let I be an ideal of R and $\omega : M \to Aut(R)$ a monoid homomorphism. If $\bar{\omega} : M \to Aut(R/I)$ is defined by $\bar{\omega}_g(r + I) = \omega_g(r) + I$, then $\bar{\omega}$ is a monoid homomorphism. Note that the twisting $f : M \times M \to U(R)$ induces a twisting $\bar{f} : M \times M \to U(R/I)$ given by $\bar{f}(x, y) = f(x, y) + I$. Moreover, for every $\alpha = \sum_{i=1}^{n} a_i g_i$ in $R * M$, we denote $\bar{\alpha} = \sum_{i=1}^{n} \bar{a}_i g_i$ in $(R/I) * M \cong (R * M)/(I * M)$, where $\bar{a}_i = a_i + I$ for $1 \le i \le n$. It can be easily checked that the map $\mu : R * M \to (R/I) * M$ defined by $\mu(\alpha) = \bar{\alpha}$ is a ring homomorphism.

In contrast to the fact that there exits a ring R, which is not Armendariz, but R/I and I are Armendariz for every nonzero proper ideal I of R (see [**12**, Example 14]), we have the following proposition.

PROPOSITION 2.6. *Let R be an M-compatible ring and $(M, \preceq)$ a strictly totally ordered monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. If I is a reduced ideal of R such that R/I is CM-Armendariz, then R is CM-Armendariz.*

Proof. The proof is a modification of that of [**14**, Proposition 1.4]. Let $\alpha = a_1 g_1 + \cdots + a_n g_n$, $\beta = b_1 h_1 + \cdots + b_m h_m$ be elements of $R * M$ such that $\alpha\beta = 0$. Without loss of generality, we can assume that $g_1 \prec g_2 \prec \cdots \prec g_n$ and $h_1 \prec h_2 \prec \cdots \prec h_m$. We use transfinite induction on the strictly totally ordered monoid $(M, \preceq)$ to show that $a_i \omega_{g_i}(b_j) = 0$ for all i and j. Since R/I is CM-Armendariz and

$$\begin{aligned}\bar{0} &= (\bar{a}_1 g_1 + \cdots + \bar{a}_m g_m)(\bar{b}_1 h_1 + \cdots + \bar{b}_n h_n)\\ &= (a_1 + I)\overline{\omega}_{g_1}(b_1 + I)\bar{f}(g_1, h_1) g_1 h_1 + \cdots + (a_m + I)\overline{\omega}_{g_m}(b_n + I)\bar{f}(g_m, h_n) g_m h_n\\ &= (a_1 \omega_{g_1}(b_1) f(g_1, h_1) + I) g_1 h_1 + \cdots + (a_m \omega_{g_m}(b_n) f(g_m, h_n) + I) g_m h_n,\end{aligned}$$

in $(R/I) * M$, so we have $a_i \omega_{g_i}(b_j) f(g_i, h_j) \in I$ for all i, j. Since M is a strictly totally ordered monoid, we have $g_1 h_1 \prec g_i h_j$ for $i \neq 1$ or $j \neq 1$. It follows from $\alpha\beta = 0$ that $a_1 \omega_{g_1}(b_1) f(g_1, h_1) = 0$, i.e., $a_1 \omega_{g_1}(b_1) = 0$. Now, assume that $a_i \omega_{g_i}(b_j) = 0$ for all

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$1 \leq i \leq n$ and $1 \leq j \leq m$ with $g_ih_j \prec \xi$, and we shall show that $a_i\omega_{g_i}(b_j) = 0$ for all i and j with $g_ih_j = \xi$.

In fact, $X := \{(g_i, h_j)|g_ih_j = \xi\}$ is a finite set and hence we can rewrite $X = \{(g_{i_r}, h_{j_r})|1 \leq r \leq t\}$ such that $g_{i_1} \prec g_{i_2} \prec \cdots \prec g_{i_t}$. If $g_{i_c} = g_{i_d}$ for $c \neq d$, then the equalities $g_{i_c}h_{j_c} = \xi = g_{i_d}h_{j_d}$ implies $h_{j_c} = h_{j_d}$ (since M is cancellative), a contradiction. Moreover, since $(M, \preceq)$ is a strictly totally ordered monoid, it follows that for any c and d with $c \prec d$, $g_{i_c}h_{j_c} = g_{i_d}h_{j_d} = \xi$ implies $h_{j_d} \prec h_{j_c}$. Thus, we have $h_{j_t} \prec h_{j_{t-1}} \prec \cdots \prec h_{j_1}$, and hence

$$\sum_{g_ih_j=\xi} a_i\omega_{g_i}(b_j)f(g_i, h_j) = \sum_{(g_i,h_j)\in X} a_i\omega_{g_i}(b_j)f(g_i, h_j) = \sum_{r=1}^{t} a_{i_r}\omega_{g_{i_r}}(b_{j_r})f(g_{i_r}, h_{j_r}) = 0.$$

Note that for any $r \geq 2$, $g_{i_1}h_{j_r} \prec g_{i_r}h_{j_r} = \xi$, and so $a_{i_1}\omega_{g_{i_1}}(b_{j_r})f(g_{i_1}, h_{j_r}) = 0$ by induction hypothesis. Thus $a_{i_1}\omega_{g_{i_1}}(b_{j_r}) = 0$. It follows that $a_{i_1}b_{j_r} = 0$ and hence $a_{i_1}\omega_{g_{i_r}}(b_{j_r}) = 0$, because R is M-compatible. Since I is reduced and $\omega_{g_{i_r}}(b_{j_r})Ia_{i_1} \subseteq I$, we have $(\omega_{g_{i_r}}(b_{j_r})Ia_{i_1})^2 = \omega_{g_{i_r}}(b_{j_r})I(a_{i_1}\omega_{g_{i_r}}(b_{j_r}))Ia_{i_1} = 0$. This implies that $\omega_{g_{i_r}}(b_{j_r})Ia_{i_1} = 0$ and hence $\omega_{g_{i_r}}(b_{j_r})f(g_{i_r}, h_{j_r})Ia_{i_1} \subseteq \omega_{g_{i_r}}(b_{j_r})Ia_{i_1} = 0$ since I is an ideal of R. For any $r \geq 2$, we obtain

$$\begin{aligned} 0 &= (a_{i_r}\omega_{g_{i_r}}(b_{j_r})f(g_{i_r}, h_{j_r}))(a_{i_1}\omega_{g_{i_1}}(b_{j_1})f(g_{i_1}, h_{j_1}))^2 \\ &= (a_{i_r}\omega_{g_{i_r}}(b_{j_r})f(g_{i_r}, h_{j_r}))(a_{i_1}\omega_{g_{i_1}}(b_{j_1})f(g_{i_1}, h_{j_1}))(a_{i_1}\omega_{g_{i_1}}(b_{j_1})f(g_{i_1}, h_{j_1})) \\ &\in (a_{i_r}\omega_{g_{i_r}}(b_{j_r})f(g_{i_r}, h_{j_r}))I(a_{i_1}\omega_{g_{i_1}}(b_{j_1})f(g_{i_1}, h_{j_1})) \\ &= a_{i_r}(\omega_{g_{i_r}}(b_{j_r})f(g_{i_r}, h_{j_r})Ia_{i_1})\omega_{g_{i_1}}(b_{j_1})f(g_{i_1}, h_{j_1}). \end{aligned}$$

Multiplying $\sum_{r=1}^{t} a_{i_r}\omega_{g_{i_r}}(b_{j_r})f(g_{i_r}, h_{j_r}) = 0$ on the right side by $(a_{i_1}\omega_{g_{i_1}}(b_{j_1})f(g_{i_1}, h_{j_1}))^2$, we have

$$\begin{aligned} 0 &= \left(\sum_{r=1}^{t} a_{i_r}\omega_{g_{i_r}}(b_{j_r})f(g_{i_r}, h_{j_r})\right)(a_{i_1}\omega_{g_{i_1}}(b_{j_1})f(g_{i_1}, h_{j_1}))^2 \\ &= (a_{i_1}\omega_{g_{i_1}}(b_{j_1})f(g_{i_1}, h_{j_1}))(a_{i_1}\omega_{g_{i_1}}(b_{j_1})f(g_{i_1}, h_{j_1}))^2 \\ &= (a_{i_1}\omega_{g_{i_1}}(b_{j_1})f(g_{i_1}, h_{j_1}))^3. \end{aligned}$$

Since $a_{i_1}\omega_{g_{i_1}}(b_{j_1})f(g_{i_1}, h_{j_1}) \in I$ and I is reduced, we have $a_{i_1}\omega_{g_{i_1}}(b_{j_1})f(g_{i_1}, h_{j_1}) = 0$. Thus $a_{i_1}\omega_{g_{i_1}}(b_{j_1}) = 0$. This implies that

$$\sum_{r=2}^{t} a_{i_r}\omega_{g_{i_r}}(b_{j_r})f(g_{i_r}, h_{j_r}) = 0. \tag{$\natural$}$$

Multiplying equation ($\natural$) from the right-hand side by $(a_{i_2}\omega_{g_{i_2}}(b_{j_2})f(g_{i_2}, h_{j_2}))^2$, we obtain $a_{i_2}\omega_{g_{i_2}}(b_{j_2}) = 0$ by the similar method as above. Continuing this procedure, we can prove that $a_{i_t}\omega_{g_{i_r}}(b_{j_r}) = 0$ for all r. This shows that $a_i\omega_{g_i}(b_j) = 0$ for all i, j with $g_ih_j = \omega$. Therefore, by transfinite induction, we have proved $a_i\omega_{g_i}(b_j) = 0$ for all i, j. □

For any $\alpha \in R * M$, we denote by C_α the set of all coefficients of α.

LEMMA 2.7. *Let R be a ring and M a monoid with $\omega : M \to Aut(R)$ a monoid homomorphism and twisting $f : M \times M \to U(R)$. Suppose that R is an M-rigid CM-Armendariz ring. If $\alpha_1\alpha_2 \cdots \alpha_n = 0$ with each $\alpha_i \in R * M$, then $a_1a_2 \cdots a_n = 0$ for all $a_i \in C_{\alpha_i}$ and all $1 \leq i \leq n$.*

For a ring R and $n \geq 2$, let $S_n(R)$ be the ring of all $n \times n$ upper triangular matrices over R that are constant on the diagonal. Let $\omega : M \to Aut(R)$ be a monoid

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homomorphism. For each $g \in M$, ω can be extended to a monoid homomorphism $\bar{\omega}$ from M to $Aut(S_n(R))$ defined by $\bar{\omega}_g((a_{ij})) = (\omega_g(a_{ij}))$.

PROPOSITION 2.8. *Let R be an M-rigid ring and M a monoid with action $\omega : M \to Aut(R)$, where $|M| \geq 2$. Then R is skew M-Armendariz if and only if $S_3(R)$ is skew M-Armendariz.*

Proof. Let R be a skew M-Armendariz ring and $\alpha = A_1g_1 + A_2g_2 + \cdots + A_ng_n$, $\beta = B_1h_1 + B_2h_2 + \cdots + B_mh_m$ be nonzero elements of $S_3(R)\sharp M$ with $\alpha\beta = 0$, where

$$A_i = \begin{pmatrix} a^{(i)} & {a_{12}}^{(i)} & {a_{13}}^{(i)} \\ 0 & a^{(i)} & {a_{23}}^{(i)} \\ 0 & 0 & a^{(i)} \end{pmatrix}, \quad B_j = \begin{pmatrix} b^{(j)} & {b_{12}}^{(j)} & {b_{13}}^{(j)} \\ 0 & b^{(j)} & {b_{23}}^{(j)} \\ 0 & 0 & b^{(j)} \end{pmatrix}.$$

We note that there is an obvious isomorphism $S_3(R)\sharp M \cong S_3(R\sharp M)$. Therefore, we can rewrite α and β as

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_1 & \alpha_{23} \\ 0 & 0 & \alpha_1 \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 & \beta_{12} & \beta_{13} \\ 0 & \beta_1 & \beta_{23} \\ 0 & 0 & \beta_1 \end{pmatrix}.$$

So we have the following equations: $\alpha_1\beta_1 = 0$, $\alpha_1\beta_{12} + \alpha_{12}\beta_1 = 0$, $\alpha_1\beta_{13} + \alpha_{12}\beta_{23} + \alpha_{13}\beta_1 = 0$, $\alpha_1\beta_{23} + \alpha_{23}\beta_1 = 0$. Since R is skew M-Armendariz, it follows that $a^{(i)}\omega_{g_i}(b^{(j)}) = 0$ for all i and j. Therefore, $a^{(i)}b^{(j)} = 0$ and thus $b^{(j)}a^{(i)} = 0$ since R is M-rigid. This implies that $b^{(j)}\omega_{h_j}(a^{(i)}) = 0$ and we obtain $\beta_1\alpha_1 = 0$. Multiplying $\alpha_1\beta_{12} + \alpha_{12}\beta_1 = 0$ on the left side by β_1, we have $\beta_1\alpha_{12}\beta_1 = 0$. Since R is skew M-Armendariz, we have $b^{(j)}{a_{12}}^{(i)}b^{(j)} = 0$ by Lemma 2.7. This shows that ${a_{12}}^{(i)}\omega_{g_i}(b^{(j)}) = 0$ since R is M-rigid, and hence $\alpha_{12}\beta_1 = 0$. Thus, we deduce $\alpha_1\beta_{12} = 0$ and so $a^{(i)}\omega_{g_i}({b_{12}}^{(j)}) = 0$. Similarly, if we multiply $\alpha_1\beta_{23} + \alpha_{23}\beta_1 = 0$ on the left side by β_1, then we have $\beta_1\alpha_{23}\beta_1 = 0$, so $b^{(j)}{a_{23}}^{(i)}b^{(j)} = 0$ by Lemma 2.7. Thus ${a_{23}}^{(i)}\omega_{g_i}(b^{(j)}) = 0$ for all i, j since R is M-rigid. Hence, $\alpha_1\beta_{23} = 0$ and so ${b_{23}}^{(j)}\omega_{h_j}(a^{(i)}) = 0$ for all i and j. Moreover, if we multiply $\alpha_1\beta_{13} + \alpha_{12}\beta_{23} + \alpha_{13}\beta_1 = 0$ on the left side by β_1, then $\beta_1\alpha_{13}\beta_1 = 0$. Similarly, we have ${a_{13}}^{(i)}\omega_{g_i}(b^{(j)}) = 0$ and so $\alpha_{13}\beta_1 = 0$. Thus, the third equation above becomes $\alpha_1\beta_{13} + \alpha_{12}\beta_{23} = 0$. If we multiply $\alpha_1\beta_{13} + \alpha_{12}\beta_{23} = 0$ on the right side by α_1, then we have $\alpha_1\beta_{13}\alpha_1 = 0$ since ${b_{23}}^{(j)}\omega_{h_j}(a^{(i)}) = 0$ (hence $\beta_{23}\alpha_1 = 0$). A similar argument shows that $a^{(i)}\omega_{g_i}({b_{13}}^{(j)}) = 0$. Therefore, we obtain $\alpha_{12}\beta_{23} = 0$ and so ${a_{12}}^{(i)}\omega_{g_i}({b_{23}}^{(j)}) = 0$. Now, it is straightforward to see that $A_i\omega_{g_i}(B_j) = 0$ for all i, j.

Conversely, assume that $S_3(R)$ is skew M-Armendariz. Let $\mu = a_1g_1 + a_2g_2 + \cdots + a_ng_n$ and $\nu = b_1h_1 + b_2h_2 + \cdots + b_mh_m$ be nonzero polynomials in $R\sharp M$ with $\mu\nu = 0$. Then

$$\begin{pmatrix} \mu & 0 & \cdots & 0 \\ 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu \end{pmatrix} \begin{pmatrix} \nu & 0 & \cdots & 0 \\ 0 & \nu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nu \end{pmatrix} = 0,$$

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in $S_3(R)\sharp M$. Since $S_3(R)$ is skew M-Armendariz, a routine verification shows that R is skew M-Armendariz. □

COROLLARY 2.9. *Let R be a ring and M a monoid with $|M| \geq 2$. Then R is M-Armendariz if and only if $S_3(R)$ is M-Armendariz.*

Note that any endomorphism α of a ring R can be extended to an endomorphism $\bar{\alpha}$ of $S_3(R)$ defined by $\bar{\alpha}(a_{ij}) = (\alpha(a_{ij}))$.

COROLLARY 2.10 ([**7**, Proposition 17]). *Let R be a ring and α an endomorphism of R. If R is α-rigid, then $S_3(R)$ is $\bar{\alpha}$-skew Armendariz.*

COROLLARY 2.11 ([**12**, Proposition 2]). *If R is a reduced ring, then $S_3(R)$ is Armendariz.*

COROLLARY 2.12. *Let R be an M-rigid ring and M a monoid with action $\omega : M \to Aut(R)$, where $|M| \geq 2$. If R is skew M-Armendariz, then $S_2(R)$ is skew M-Armendariz.*

In view of Proposition 2.8, one may suspect that $S_n(R)$ is skew M-Armendariz if R is skew M-Armendariz for $n \geq 4$. But the following example eliminates the possibility.

EXAMPLE 2.13. Let R and M be given as in Proposition 2.8. Since R is M-rigid, we note that $\omega_g(e) = e$ for every $e^2 = e \in R$ by [**8**, Proposition 5]. Let $\alpha = e_{12}e + (e_{12} - e_{13})g$ and $\beta = e_{34}e + (e_{24} + e_{34})g \in S_4(R) * M$ with $e \neq g \in M$, where the e_{ij}'s are the matrix units in $S_4(R)$. Then we have $\alpha\beta = 0$, but $(e_{12} - e_{13})\omega_g(e_{34}) \neq 0$. This shows that $S_4(R)$ is not skew M-Armendariz. Similarly, $S_n(R)$ is not skew M-Armendariz for all $n \geq 5$.

If N is an ideal of the monoid M with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$, then the restrictions $f|_{N\times N} : N \times N \to U(R)$ and $\omega|_N : N \to Aut(R)$ are induced twisting and action.

PROPOSITION 2.14. *Let R be an M-compatible ring and M a cancellative monoid. If R is CN-Armendariz for an ideal N of M, then R is CM-Armendariz.*

Proof. The proof is similar to that of [**14**, Proposition 1.10]. □

COROLLARY 2.15. *Let M be a cancellative monoid and N an ideal of M. If R is N-Armendariz, then R is M-Armendariz.*

Let $\triangle$ be a multiplicative monoid consisting of central regular elements of R. Then $\triangle^{-1}R := \{u^{-1}a |\ u \in \triangle, a \in R\}$ is a ring. Let $\omega : M \to Aut(R)$ be a monoid homomorphism. If $\omega_g(\triangle) \subseteq \triangle$ for every $g \in M$, then ω can be extended to $\bar{\omega} : M \to Aut(\triangle^{-1}R)$ defined by $\bar{\omega}_g(u^{-1}a) = \omega_g(u)^{-1}\omega_g(a)$. If $f : M \times M \to U(R)$ is a twisted function, then f can be viewed as a twisted function from $M \times M$ to $U(\triangle^{-1}R)$ as $U(R) \subseteq U(\triangle^{-1}R)$.

PROPOSITION 2.16. *Let R be an M-compatible ring and M a cancellative monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. Then R is CM-Armendariz if and only if $\triangle^{-1}R$ is CM-Armendariz.*

Proof. It suffices to show the necessity. Assume that R is CM-Armendariz. Let $\alpha = \sum_{i=0}^{m} u_i^{-1}a_ig_i$, $\beta = \sum_{j=0}^{n} v_j^{-1}b_jh_j$ be elements in $\triangle^{-1}R * M$ with $\alpha\beta = 0$. Then $\bar{\alpha} = (u_mu_{m-1}\cdots u_0)\alpha$, $\bar{\beta} = (v_nv_{n-1}\cdots v_0)\beta$ are in $R * M$. Since R is CM-Armendariz and $\bar{\alpha}\bar{\beta} = 0$, we have $(u_mu_{m-1}\cdots u_0u_i^{-1}a_i)\omega_{g_i}(v_nv_{n-1}\cdots v_0v_j^{-1}b_j) = 0$ for all i, j. It follows

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that $a_i\omega_{g_i}(b_j) = 0$, because $\triangle$ is a multiplicative monoid consisting of central regular elements of R and all $u_i, v_j \in \triangle$. Hence $(u_i^{-1}a_i)\omega_{g_i}(v_j^{-1}b_j) = a_i\omega_{g_i}(b_j)(\omega_{g_i}(v_j)u_i)^{-1} = 0$ for all i, j. This shows that $\triangle^{-1}R$ is CM-Armendariz. $\square$

Corollary 2.17. *Let R be an M-compatible ring and M a cancellative monoid with monoid homomorphism $\omega : M \to Aut(R)$. Then R is skew M-Armendariz if and only if $\triangle^{-1}R$ is skew M-Armendariz.*

Corollary 2.18. *Let R be an M-compatible ring and M a monoid. If R is M-Armendariz, then $\triangle^{-1}R$ is M-Armendariz.*

The ring of Laurent polynomials over a ring R in one variable x is denoted by $R[x; x^{-1}]$. Each endomorphism α of R can be extended to an endomorphism $\bar{\alpha}$ of $R[x; x^{-1}]$, where $\bar{\alpha}$ is given by $\bar{\alpha}(\sum_{i=k}^{n} a_i x^i) = \sum_{i=k}^{n} \alpha(a_i)x^i$ for $\sum_{i=k}^{n} a_i x^i \in R[x; x^{-1}]$.

Corollary 2.19 ([**14**, Proposition 2.5]). *Let R be a reduced ring and M a monoid. If R is M-Armendariz, then $R[x; x^{-1}]$ is M-Armendariz.*

Corollary 2.20. *Let R be a reduced ring and M a monoid. Then $R[x]$ is M-Armendariz if and only if $R[x; x^{-1}]$ is M-Armendariz.*

A ring R is *left p.q.-Baer* if the left annihilator of any principal left ideal of R is generated as a left ideal by an idempotent [**2**]. As a generalisation of left p.q.-Baer rings, Liu and Zhao in [**15**] introduced left APP-rings. A ring R is a *left APP-ring* if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any $a \in R$. Here an ideal I of R is said to be *right s-unital* if, for each $a \in I$ there exists $x \in I$ such that $ax = a$. Note that an ideal I is right s-unital if and only if R/I is flat as a left R-module if and only if I is pure as a left ideal of R (see [**20**, Proposition 11.3.13]). In [**5**, Theorem 3.9], it was shown that a ring R is left APP if and only if $R[x]$ is left APP.

For the crossed product $R * M$, we have the following.

Proposition 2.21. *Let M be a u.p.-monoid with twisting $f : M \times M \to U(R)$ and with action $\omega : M \to Aut(R)$ satisfying condition $(*)$. If R is left APP, then $R * M$ is left APP.*

Proof. The proof is a modification of that of [**15**, Theorem 3.10]. Suppose $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m \in R * M$ such that $\alpha \in l_{R*M}((R * M)\beta)$. Then we have $\alpha(R * M)\beta = 0$, and so $\alpha(rc)\beta = 0$ for every $r \in R$ and every $c \in M$. In the following, we freely use the fact that $\omega_{g_i}(R)f(g_i, h_j) = Rf(g_i, h_j) = R$ for any $g_i, h_j \in M$. Since ω is a map from M to $Aut(R)$, there exist $c_1, c_2, \ldots, c_n \in R$ such that $a_i = \omega_{g_i}(c_i)$ for $i = 1, 2, \ldots, n$. We shall prove by induction on m that $c_i \in l_R(R\omega_c(b_j))$ for every $c \in M$ and for $1 \leq i \leq n$ and $1 \leq j \leq m$. Note that M is a cancellative monoid by [**1**, Lemma 1.1].

If $m = 1$, then $\beta = b_1h_1$ and $(a_1g_1 + a_2g_2 + \cdots + a_ng_n)(rc)(b_1h_1) = 0$. A routine calculation shows that $c_i \in l_R(R\omega_c(b_1))$.

If $m \geq 2$, there exist $1 \leq s \leq n$ and $1 \leq t \leq m$ such that g_sch_t is uniquely presented by considering the two subsets $\{g_1, g_2, \ldots, g_n\}$ and $\{ch_1, ch_2, \ldots, ch_m\}$ of M. It follows from $\alpha(rc)\beta = 0$ that $a_s\omega_{g_s}(R)f(g_s, c)\omega_{g_sc}(b_t)f(g_sc, h_t)g_sch_t = 0$, and hence $a_s\omega_{g_s}(R)f(g_s, c)\omega_{g_sc}(b_t)f(g_sc, h_t) = 0$. Then $a_s\omega_{g_s}(R)\omega_{g_sc}(b_t) = 0$ since $\omega_{g_s}(R)f(g_s, c) = R$. This shows that $a_s\omega_{g_s}(R\omega_c(b_t)) = 0$ and so $\omega_{g_s}(c_sR\omega_c(b_t)) = 0$, which implies that $c_sR\omega_c(b_t) = 0$ since ω_{g_s} is a ring automorphism. Hence $c_s \in l_R(R\omega_c(b_t))$. Since R is a left

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APP-ring, $l_R(R\omega_c(b_t))$ is pure as a left ideal of R and hence there exists $e_t \in l_R(R\omega_c(b_t))$ such that $c_s = c_s e_t$. For every $r \in R$, we have

$$\begin{aligned}0 &= \alpha(e_t rc)\beta = (a_1g_1 + a_2g_2 + \cdots + a_ng_n)(e_t rc)(b_1h_1 + b_2h_2 + \cdots + b_mh_m)\\ &= (a_1g_1 + a_2g_2 + \cdots + a_ng_n)(e_t rc)(b_1h_1 + b_2h_2 + \cdots + b_{t-1}h_{t-1} + b_{t+1}h_{t+1}\\ &\quad + \cdots + b_mh_m) + (a_1g_1 + a_2g_2 + \cdots + a_ng_n)((e_t r\omega_c(b_t)f(c,h_t))(ch_t))\\ &= (a_1\omega_{g_1}(e_t)g_1 + a_2\omega_{g_2}(e_t)g_2 + \cdots + a_n\omega_{g_n}(e_t)g_n)(rc)\\ &\quad \cdot(b_1h_1 + b_2h_2 + \cdots + b_{t-1}h_{t-1} + b_{t+1}h_{t+1} + \cdots + b_mh_m).\end{aligned}$$

Moreover, since $a_i\omega_{g_i}(e_t) = \omega_{g_i}(c_ie_t)$, by induction we obtain $c_ie_t \in l_R(R\omega_c(b_j))$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, t-1, t+1, \ldots, m$. Thus $c_s = c_se_t \in \bigcap_{j=1}^{m} l_R(R\omega_c(b_j))$. Now, we have $a_s\omega_{g_s}(R\omega_c(b_j)) = \omega_{g_s}(c_sR\omega_c(b_j)) = 0$ for $j = 1, 2, \ldots, m$. For every $g_i \in M$, since ω_{g_i} is an automorphism of R and $\omega_{g_s}(R)f(g_s, c) = Rf(g_s, c) = R$, we obtain $a_s\omega_{g_s}(R)f(g_s, c)\omega_{g_sc}(b_j)f(g_sc, h_j) = 0$. It follows from $\alpha(rc)\beta = 0$ that $(a_1g_1 + a_2g_2 + \cdots + a_{s-1}g_{s-1} + a_{s+1}g_{s+1} + \cdots + a_ng_n)(rc)(b_1h_1 + b_2h_2 + \cdots + b_mh_m) = 0$.

Similarly, there exists $\gamma \in \{1, 2, \ldots, s-1, s+1, \ldots, n\}$ such that $c_\gamma \in \bigcap_{j=1}^{m} l_R(R\omega_c(b_j))$. This implies that $a_\gamma\omega_{g_\gamma}(R\omega_c(b_j)) = \omega_{g_\gamma}(c_\gamma R\omega_c(b_j)) = 0$ for $j = 1, 2, \ldots, m$. Then we have $(a_1g_1 + a_2g_2 + \cdots + a_{s-1}g_{s-1} + a_{s+1}g_{s+1} + \cdots + a_{\gamma-1}g_{\gamma-1} + a_{\gamma+1}g_{\gamma+1} + \cdots + a_ng_n)(rc)(b_1h_1 + b_2h_2 + \cdots + b_mh_m) = 0$. Continuing this procedure yields $c_1, c_2, \ldots, c_n \in \bigcap_{j=1}^{m} l_R(R\omega_c(b_j))$ for every $c \in M$. Now let $I = \sum_{j=1}^{m}\sum_{c\in M} R\omega_c(b_j)$. It is easy to see that $c_1, c_2, \ldots, c_n \in l_R(I)$ and I is finitely generated since ω satisfies condition $(*)$.

Since R is left APP, $l_R(I)$ is pure as a left ideal of R, so there exists $\xi \in l_R(I)$ such that $c_i = c_i\xi$ with $i = 1, 2, \ldots, n$. Note that for every $r \in R$ and every $c \in M$, we have $r\omega_c(b_j) \in I$, and thus

$$\begin{aligned}(\xi e)(rc)\beta &= (\xi e)(rc)(\Sigma_{j=1}^{m} b_jh_j) = \xi\omega_e(r)f(e, c)(ec)(\Sigma_{j=1}^{m} b_jh_j)\\ &= \Sigma_{j=1}^{m}\xi rf(e, c)\omega_{ec}(b_j)f(ec, h_j)(ch_j)\\ &= \Sigma_{j=1}^{m}\xi rf(e, c)\omega_c(b_j)f(c, h_j)(ch_j) = 0.\end{aligned}$$

This shows that $(\xi e) \in l_{R*M}((R*M)\beta)$. Hence, $\alpha(\xi e) = \sum_{i=1}^{n} a_ig_i(\xi e) = \sum_{i=1}^{n} a_i\omega_{g_i}(\xi e)g_i = \sum_{i=1}^{n} a_i\omega_{g_i}(\xi)g_i = \sum_{i=1}^{n}\omega_{g_i}(c_i\xi)g_i = \sum_{i=1}^{n}\omega_{g_i}(c_i)g_i = \sum_{i=1}^{n} a_ig_i = \alpha$. It follows that $R*M$ is left APP. □

Corollary 2.22. *Let M be a u.p.-monoid with $\omega : M \to Aut(R)$ satisfying condition $(*)$. If R is left APP, then the skew monoid ring $R\sharp M$ (i.e., the crossed product $R*M$ with the trivial twisted function) is left APP.*

3. CM-quasi-Armendariz rings.

Given a ring R and a monoid M with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$, we define the notion of a CM-quasi-Armendariz ring, which unifies several quasi-Armendariz properties of rings.

Definition 3.1. A ring R is called M-quasi-Armendariz of crossed product type relative to the given twisting f and action ω (or simply, CM-quasi-Armendariz) if whenever $\alpha = a_0g_0 + a_1g_1 + \cdots + a_ng_n$, $\beta = b_0h_0 + b_1h_1 + \cdots + b_mh_m \in R*M$ with $g_i, h_j \in M$ satisfy $\alpha(R*M)\beta = 0$, we have $a_iR\omega_{g_ig}(b_j) = 0$ for any i, j and $g \in M$.

Remark 3.2. If R is CM-quasi-Armendariz with f trivial, then we call M a skew M-quasi-Armendariz ring. If R is CM-quasi-Armendariz with ω trivial, then we call R a TM-quasi-Armendariz (i.e., twisted M-quasi-Armendariz) ring. It is easy to see that if both twisting f and action ω are trivial, then R is M-quasi-Armendariz. In particular, if both twisting f and action ω are trivial with $M = (\mathbb{N} \cup \{0\}, +)$, then R is CM-quasi-Armendariz if and only R is quasi-Armendariz.

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Proposition 3.3. *If R is a left p.q.-Baer ring and M is a strictly totally ordered monoid, then R is TM-quasi-Armendariz.*

Proof. The proof is a modification of that of [**6**, Lemma 1]. Let $\alpha = a_0g_0 + a_1g_1 + \cdots + a_ng_n$, $\beta = b_0h_0 + b_1h_1 + \cdots + b_mh_m \in R^\tau[M]$ such that $\alpha(R^\tau[M])\beta = 0$. Since M is a strictly totally ordered monoid, we can assume that $g_i \prec g_j$ and $h_i \prec h_j$ whenever $i < j$. Now, we claim $a_iRb_j = 0$ for all i, j. Let r be an element of R. Then, we have $\alpha(re)\beta = 0$ since $\alpha(R^\tau[M])\beta = 0$, and so

$$\begin{aligned} 0 = \alpha(re)\beta = {} & a_0rf(g_0, e)b_0f(g_0, h_0)g_0h_0 + \cdots + \{a_nrf(g_n, e)b_{m-2}f(g_n, h_{m-2})g_nh_{m-2} \\ & + a_{n-1}rf(g_{n-1}, e)b_{m-1}f(g_{n-1}, h_{m-1})g_{n-1}h_{m-1} + a_{n-2}rf(g_{n-2}, e)b_mf(g_{n-2}, h_m)g_{n-2}h_m\} \\ & + \{a_nrf(g_n, e)b_{m-1}f(g_n, h_{m-1})g_nh_{m-1} + a_{n-1}rf(g_{n-1}, e)b_mf(g_{n-1}, h_m)g_{n-1}h_m\} \\ & + a_nrf(g_n, e)b_mf(g_n, h_m)g_nh_m. \qquad (\dagger) \end{aligned}$$

It follows that $a_nrf(g_n, e)b_mf(g_n, h_m) = 0$ since g_nh_m is of highest order in the g_ih_j's. Hence $a_nrf(g_n, e)b_m = 0$. This shows that $a_n \in l_R(Rf(g_n, e)b_m) = l_R(Rb_m)$. Hence, $l_R(Rb_m) = Re_m$ for some idempotent e_m by hypothesis. Replacing r by re_m in equation $(\dagger)$, we obtain

$$\begin{aligned} 0 = {} & a_0re_mf(g_0, e)b_0f(g_0, h_0)g_0h_0 + \cdots + \{a_nre_mf(g_n, e)b_{m-2}f(g_n, h_{m-2})g_nh_{m-2} \\ & + a_{n-1}re_mf(g_{n-1}, e)b_{m-1}f(g_{n-1}, h_{m-1})g_{n-1}h_{m-1}\} \\ & + a_nre_mf(g_n, e)b_{m-1}f(g_n, h_{m-1})g_nh_{m-1}. \qquad (\ddagger) \end{aligned}$$

So $a_nre_mf(g_n, e)b_{m-1}f(g_n, h_{m-1}) = 0$, because g_nh_{m-1} is of highest order in $\{g_ih_j | 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\} \backslash \{g_{n-1}h_m, g_nh_m\}$. Hence $a_nre_mf(g_n, e)b_{m-1} = 0$. Since Re_m is an ideal of R and $e_m \in Re_m$, we have $e_mr \in Re_m$ and thus $e_mr = e_mre_m$ for all $r \in R$. On the other hand, we also have $a_n = a_ne_m$ since $a_n \in l_R(Rb_m) = Re_m$. Hence $a_nrf(g_n, e)b_{m-1} = a_ne_mrf(g_n, e)b_{m-1} = a_ne_mre_mf(g_n, e)b_{m-1} = a_nre_mf(g_n, e)b_{m-1} = 0$. This implies that $a_n \in l_R(Rb_m + Rb_{m-1})$, and hence $l_R(Rb_m + Rb_{m-1}) = Re_{m-1}$ for some idempotent $e_{m-1} \in R$ since R is a left p.q.-Baer ring.

Replacing r by re_{m-1} in equation $(\dagger)$, we obtain $a_nre_{m-1}f(g_n, e)b_{m-2}f(g_n, h_{m-2}) = 0$ in the same way as above. This shows that $a_n \in l_R(Rb_m + Rb_{m-1} + Rb_{m-2})$. Continuing this process we obtain $a_nRb_t = 0$ for all $t = 0, 1, \ldots, m$. So, we have $(a_0g_0 + a_1g_1 + \cdots + a_{n-1}g_{n-1})(R^\tau[M])(b_0h_0 + b_1h_1 + \cdots + b_mh_m) = 0$. Using induction on $m + n$, we obtain $a_iRb_j = 0$ for all i, j. Therefore, R is a TM-quasi-Armendariz ring. □

Corollary 3.4. *If R is a left p.q.-Baer ring and M is an ordered monoid, then R is an M-quasi-Armendariz ring.*

Proposition 3.5. *Let M be a u.p.-monoid with twisting $f : M \times M \to U(R)$ and action $\omega : M \to Aut(R)$. If R is a semiprime ring, then R is CM-quasi-Armendariz.*

Proof. The proof is a modification of that of [**9**, Theorem 1.1]. Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$ and $\beta = b_1h_1 + b_2h_2 + \cdots + b_mh_m$ be elements in $R * M$ with $\alpha(R * M)\beta = 0$. Then for any $r \in R$ and $g \in M$, we have

$$(a_1g_1 + a_2g_2 + \cdots + a_ng_n)gr(b_1h_1 + b_2h_2 + \cdots + b_mh_m) = 0. \qquad (\sharp)$$

We shall prove, by induction on n, that $a_iR\omega_{g_i}(\omega_g(b_j)) = 0$ for every $g \in M$ and for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

If $n = 1$, then $(a_1g_1)gr(b_1h_1 + b_2h_2 + \cdots + b_mh_m) = 0$. A routine calculation shows that $a_1R\omega_{g_1}(\omega_g(b_j)) = 0$ for each $1 \leq j \leq m$.

If $n \geq 2$, since M is a u.p.-monoid, there exist s, t with $1 \leq s \leq n$ and $1 \leq t \leq m$ such that g_sgh_t is uniquely presented by considering two subsets $S = \{g_1g, \ldots, g_ng\}$ and $T = \{h_1, \ldots, h_m\}$ of M. We may assume, without loss of generality, that $s = 1$, $t = 1$. It

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follows from (♯) that $a_1\omega_{g_1}(\omega_g(Rb_1))f(g_1g, h_1)g_1gh_1 = 0$, and hence $a_1\omega_{g_1}(\omega_g(Rb_1)) = 0$. Since ω_g, ω_{g_1} are automorphisms of R, we get $a_1R\omega_{g_1}(\omega_g(b_1)) = 0$. Hence, for every $z \in R$, $a_1R\omega_{g_1}(\omega_g(b_1zb_1)f(g_1g, h_1) = 0$, and so

$$\begin{aligned} 0 &= (a_1g_1 + a_2g_2 + \cdots + a_ng_n)grb_1z(b_1h_1 + b_2h_2 + \cdots + b_mh_m) \\ &= (a_2g_2 + \cdots + a_ng_n)gr(b_1zb_1h_1 + b_1zb_2h_2 + \cdots + b_1zb_mh_m). \end{aligned}$$

By induction hypothesis, we have $a_i\omega_{g_i}(\omega_g(rb_1zb_j)) = 0$ for all $2 \leq i \leq n$ and $1 \leq j \leq m$, and so $0 = a_i\omega_{g_i}(\omega_g(rb_1zb_1)) = a_i\omega_{g_i}(\omega_g(r))\omega_{g_i}(\omega_g(b_1))\omega_{g_i}(\omega_g(z))\omega_{g_i}(\omega_g(b_1))$. Since ω_{g_i} and ω_g are automorphisms for every $2 \leq i \leq n$, we have $a_iR\omega_{g_i}(\omega_g(b_1))R\omega_{g_i}(\omega_g(b_1)) = 0$. It follows that $a_iR\omega_{g_i}(\omega_g(b_1)) = 0$ for all $2 \leq i \leq n$ since R is a semiprime ring. Therefore, we have $a_iR\omega_{g_i}(\omega_g(b_1)) = 0$ for all $1 \leq i \leq n$. Thus, the equation (♯) becomes $(a_1g_1 + a_2g_2 + \cdots + a_ng_n)gr(b_2h_2 + \cdots + b_mh_m) = 0$. Continuing this process, we obtain $a_i\omega_{g_i}(\omega_g(rb_j)) = 0$ for all $g \in M$ and all i, j. This shows that $a_iR\omega_{g_i}(\omega_g(b_j)) = 0$ for all $g \in M$, $1 \leq i \leq n$ and $1 \leq j \leq m$. The proof is complete. □

Let α be an endomorphism of a ring R. According to [**9**], a ring R is called *α-skew quasi-Armendariz* if for $f(x) = \Sigma_{i=0}^{m} a_ix^i$ and $g(x) = \Sigma_{j=0}^{n} b_jx^j$ in $R[x;\alpha]$, $f(x)R[x;\alpha]g(x) = 0$ implies $a_iR\alpha^i(b_j) = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. The following corollary shows that CM-quasi-Armendariz rings generalise both α-skew quasi-Armendariz rings and semiprime rings with α an epimorphism.

Corollary 3.6. *Let R be a semiprime ring with an epimorphism α. If $f(x) = \Sigma_{i=0}^{m} a_ix^i$ and $g(x) = \Sigma_{j=0}^{n} b_jx^j \in R[x;\alpha]$ such that $f(x)R[x;\alpha]g(x) = 0$, then $a_iR\alpha^{i+k}(b_j) = 0$ for all $k \geq 0$, $0 \leq i \leq m$ and $0 \leq j \leq n$.*

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