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SEQUENCES BY NUMBER OF w-RISES

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An m-permutation of n, repetitions allowed, is an m-sequence

$$(1)\qquad e_1, e_2, \ldots, e_m \qquad e_i \in \{1, 2, \ldots, n\}.$$

A *w-rise* is a pair (e_i, e_{i+1}) such that $e_{i+1}-e_i \geq w > 0$. In this note we find an expression for $T_{k,w}(n, m)$, the number of m-sequences having precisely k w-rises. The case $w=1$ is given in [1] [2]. Also, when $w=1$ we give the number when each of the integers $1, 2, \ldots, r$ must appear at least once. Throughout we take $\binom{n}{k}=0$ for $n<0$ except where noted in (6).

In a sequence (1), let $P(i)$ be the property that $e_{i+1}-e_i \geq w$. There are a total of $m-1$ such properties. For each subset S of $Z_{m-1}=\{1, 2, \ldots, m-1\}$, let $A(S)$ be the number of sequences which have all of the properties $P(i)$ for $i \in S$ (and possibly others), and let

$$(2)\qquad s(r) = \sum A(S)$$

where the summation extends over all subsets of order r. Thus, by the principle of inclusion and exclusion we find that

$$(3)\qquad T_{k,w}(n, m) = \sum_{i=0}^{m-1-k} (-1)^i \binom{k+i}{k} s(k+i) \qquad (0 \leq k \leq m-1; 1 \leq w \leq n-1).$$

Hence, in order to evaluate $T_{k,w}(n, m)$, it is only necessary to evaluate $A(S)$.

Let S be a subset of Z_{m-1} of order r. We associate a composition $(b_1, \ldots, b_{m-r})$ of m with S as follows. If $i \in S$, we place i and $i+1$ in the same subset of Z_m; otherwise i is in a subset by itself. This obviously gives a set partition $B_1 \cup \cdots \cup B_{m-r}$ of Z_m and we set $b_j=|B_j|$ $(1 \leq j \leq m-r)$. For example, if $m=9$ and $S=\{2, 5, 6\}$, then we have $\{1\} \cup \{2, 3\} \cup \{4\} \cup \{5, 6, 7\} \cup \{8\} \cup \{9\}$ and $1+2+1+3+1+1=9$. Now, the sequence of elements indexed by elements of a subset B_j must form an incerasing sequence whose terms differ by at least w. Since there are [4, expression (23)]

$$\binom{n-(k-1)(w-1)}{k}$$

sequences $1 \leq x_1 < x_2 < \cdots < x_k \leq n$ which satisfy $x_{i+1}-x_i \geq w$ (such a sequence is equivalent to a sequence of k1's and $n-k$ 0's with every pair of 1's separated

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by at least $w-1$ 0's), we have shown that

$$A(S)=\prod_{j=1}^{m-r}\binom{n-(b_j-1)(w-1)}{b_j}. \tag{4}$$

Since the composition $(b_1, b_2, \ldots, b_{m-r})$ determines the set S, we get by (2) and (4),

$$s(r)=\sum_{\substack{b_1+\cdots+b_{m-r}=m\\ b_j\geq 1}}\binom{n-(w-1)(b_1-1)}{b_1} \times\binom{n-(w-1)(b_2-1)}{b_2}\cdots\binom{n-(w-1)(b_{m-r}-1)}{b_{m-r}}. \tag{5}$$

For the case $w=1$, expression (5) is simplified by using the generating function

$$\sum_m \sum_{\substack{b_1+\cdots+b_p=m\\ b_i>0}}\binom{n}{b_1}\cdots\binom{n}{b_p}x^m=[(1+x)^n-1]^p$$
$$=\sum_m\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}\binom{nj}{m}x^m.$$

Simplifying in (3) $(0\leq k\leq m-1)$,

$$\begin{aligned}T_{k,1}(n,m)&=\sum_{j=1}^{m-k}(-1)^{m-k-j}\binom{nj}{m}\sum_{i=0}^{m-k-j}\binom{k+i}{k}\binom{m-k-i}{j} \\ &=\sum_{j=1}^{m-k}(-1)^{m-k-j}\binom{nj}{m}\binom{m+1}{m-k-j} \\ &=\sum_{i=0}^{m-k-1}(-1)^i\binom{m+1}{i}\binom{n(m-k-i)}{m}\end{aligned} \tag{6}$$

$\left(\text{and using [3, identity (3.150)] and } \binom{-n}{k}=(-1)^k\binom{n+k-1}{k}\right)$

$$\begin{aligned}&=-\sum_{i=m-k}^{m+1}(-1)^i\binom{m+1}{i}\binom{n(m-k-i)}{m} \\ &=\sum_{j=0}^{k+1}(-1)^j\binom{m+1}{j}\binom{n(k+1-j)+m-1}{m},\end{aligned}$$

[1, p. 356, e_1 is counted as an initial rise] and [2, p. 1091].

We now give the number of sequences (1) with precisely k 1-rises and with each of the integers $1, 2, \ldots, r$ appearing at least once. These are the sequences counted by $T_{k,1}(n, m)$ but satisfying none of the properties, "integer i does not appear", $i=1, 2, \ldots, r$. Hence, using the sieve formula the number is

$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}T_{k,1}(n-i, m) \qquad (1\leq r\leq n)$$
$$=\sum_{j=0}^{k+1}(-1)^j\binom{m+1}{j}\sum_{i=0}^{r}(-1)^i\binom{r}{i}\binom{m-1+(n-i)(k+1-j)}{m}.$$

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In the case $m=r$, using [3, identity (3.150)], we obtain the familiar Eulerian number

$$\sum_{j=0}^{k+1}(-1)^j\binom{m+1}{j}(k+1-j)^m,$$

which counts the number of permutations of $1, 2, \ldots, m$ with precisely k rises [5, pp. 216–19].

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