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CONNECTION PROPERTIES IN NEARNESS SPACES

BY

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ABSTRACT. We prove that a topological space X has a locally connected regular T_1 extension if and only if X is the underlying topological space of a nearness space Y which is concrete, regular and uniformly locally uniformly connected.

Introduction. In [1] the first author proved that a topological space X has a locally connected compactification iff it possesses a compatible uniformity with respect to which it is uniformly locally uniformly connected. This result provided the motivation to characterize those topological spaces which have a locally connected regular extension, not necessarily compact. The concomitant generality requires that our investigation be carried out in the category of nearness spaces instead of the more restricted category of uniform spaces. As in [1], the concepts of uniform connectedness and uniform local uniform connectedness are used in this characterization. In the first section of the paper the following result is obtained: A topological space X has a locally connected regular T_1 extension iff X is the underlying topological space of a nearness space Y which is concrete, regular and uniformly locally uniformly connected. The remainder of the paper is devoted to a study of uniform local connectedness in the setting of nearness spaces. In particular, we show that local connectedness, uniform local connectedness and uniform local uniform connectedness are equivalent for a regular topological nearness space.

Preliminaries. While we shall assume that the reader is familiar with the material in Herrlich's papers [4] and [6] we present here the basic definitions for the sake of completeness. A nearness space X consists of an underlying set of points and a structure on that underlying set consisting of a distinguished set of covers of X called "uniform covers of X". These covers satisfy the following axioms:

(1) If $\mathcal{A}$ is a uniform cover of X and if $\mathcal{A}$ refines $\mathcal{B}$ then $\mathcal{B}$ is a uniform cover of X.

(2) $\{X\}$ is a uniform cover of X and ϕ is not a uniform cover of X.

(3) If $\mathcal{A}$ and $\mathcal{B}$ are uniform covers of X then $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}$ and $B \in \mathcal{B}\}$ is also a uniform cover of X.

(4) If $\mathcal{A}$ is a uniform cover of X then so is $\text{int}_X\mathcal{A} = \{\text{int}_X A \mid A \in \mathcal{A}\}$ where $\text{int}_X A = [x \in X \mid \{A, X - \{x\}\}$ is a uniform cover of $X]$.

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For the purpose of this paper we shall assume that every nearness space X satisfies the T_1 axiom: If $x, y \in X$ and $x \neq y$ then $\{X - \{x\}, X - \{y\}\}$ is a uniform cover of X. A function $f: X \to Y$ between nearness spaces is called a *uniformly continuous map* if for each uniform cover $\mathcal{A}$ of Y, $f^{-1}(\mathcal{A}) = \{f^{-1}(A) | A \in \mathcal{A}\}$ is a uniform cover of X. A *nearness subspace* A of X has as uniform covers the traces on A of uniform covers of X. We shall denote the concrete category of nearness spaces and uniformly continuous maps by *Near*. A nearness space X is called *topological* iff it satisfies the condition: If $X = U[\mathrm{int}_X A | A \in \mathcal{A}]$ then $\mathcal{A}$ is a uniform cover of X.

The full subcategory *Top* of *Near* whose objects are topological nearness spaces is bicoreflective in *Near* and is isomorphic to the cateogry of symmetric topological spaces (in the usual sense). (A symmetric topological space is one which satisfies the axiom: $x \in \mathrm{cl}\,\{y\}$ if $y \in \mathrm{cl}\,\{x\}$, where cl is the closure operator on X).

The topological coreflection of a nearness space X is denoted by TX and is called the underlying topological space of X. TX is defined by the condition: $\mathcal{A}$ is a uniform cover of TX iff $X = U\{\mathrm{int}_X A | A \in \mathcal{A}\}$. Note that if A is a nearness subspace of a nearness space X then TA is the subspace, in the topological sense, of TX.

If A and B are subsets of a nearness space X then $A < B$ means that $\{X - A, B\}$ is a uniform cover of X. A nearness space X is called *regular* iff whenever $\mathcal{B}$ is a uniform cover of X then the collection $\{A \subset X | A < B$ for some $B \in \mathcal{B}\}$ is also a uniform cover of X.

Section 1.

DEFINITION 1.1. *A nearness space X is said to be uniformly connected iff every uniformly continuous function on X to a discrete nearness space is a constant.*

DEFINITION 1.2. *A nearness space X is said to be uniformly locally uniformly connected iff each uniform cover $\mathcal{U}$ of X is refined by a uniform cover $\mathcal{V}$ of X each member of which is uniformly connected as a subspace of X.*

The following proposition is obvious. In fact the first part is implicit in [3] on page 229.

PROPOSITION 1.1.

(*a*) *A nearness space X is uniformly connected iff for every $x, y \in X$ and every uniform cover $\mathcal{U}$ of X there exists a finite sequence of points $x = x_1, x_2, \ldots, x_n = y$ such that for each $i = 1, \ldots, n - 1$ there is $U \in \mathcal{U}$ with $\{x_i, x_{i+1}\} \subset U$.*

(*b*) *If X is a topological nearness space then X is uniformly connected iff X is connected (as a topological space).*

(*c*) *If X is a nearness space such that its underlying topological space TX is connected then X is uniformly connected.*

REMARK 1.1. Let R_u denote the real line with its usual uniform structure and let Q_u denote the subspace of R_u determined by the set of all rational numbers. Then Q_u is uniformly connected but TQ_u is not connected. Thus the converse of proposition 1.1 (*c*) does not hold.

LEMMA 1.1 *Let A be a dense subspace (dense in the topological sense and subspace in the nearness sense) of a nearness space X. Then X is uniformly connected iff A is uniformly connected.*

PROOF: Assume that A is uniformly connected. Let $f: X \to \{0, 1\}$ be a uniformly continuous map into the two point discrete space. Then $f|A : A \to \{0, 1\}$ is constant, say $f|A = 0$. Since $f: TX \to T\{0, 1\}$ is continuous on the underlying topological spaces,

$$fX = f\,\mathrm{cl}_X A \subset \mathrm{cl}(fA) = \mathrm{cl}\,\{0\} = \{0\}$$

Conversely assume that X is uniformly connected and let $f: A \to \{0, 1\}$ be uniformly continuous. Since $\{0, 1\}$ is a regular T_1 nearness space f has a unique uniformly continuous extension $g: X \to \{0, 1\}$ (see [5]). The map g is constant because X is uniformly connected. Therefore f is constant.

PROPOSITION 1.2. *The union of a family of uniformly connected subspaces of a nearness space whose intersection is non-empty is uniformly connected.*

PROOF. Follows from the relevant definitions.

REMARK 1.2(a) Because of the above proposition, each point x of a nearness space X is contained in a maximal uniformly connected subspace, called the uniform connected component of x in X.

(b) Lemma 1.1 and proposition 1.2 above appear in a slightly more general context in the paper [7] by G. Preuss.

PROPOSITION 1.3. *Let X be a uniformly locally uniformly connected nearness space and let G be a subspace of X which is an open subset of TX. Then each uniformly connected component of G is open in X.*

PROOF. Let B be a uniformly connected component of G and let $x \in B$. Then $\{G, X - \{x\}\}$ is a uniform cover of X. There exists a uniform cover $\mathcal{U}$ of X which refines $\{G, X - \{x\}\}$ and such that each member of $\mathcal{U}$ is uniformly connected. Thus there exists $V \in \mathcal{U}$ with $x \in \mathrm{int}_X V$. Therefore $V \subset G$ and since V is uniformly connected, $V \subset B$. Hence $x \in \mathrm{int}_X\, B$.

PROPOSITION 1.4. *A nearness space X is uniformly locally uniformly connected iff each uniform cover $\mathcal{U}$ of X has a refinement $\mathcal{V}$ which is an open uniform cover of X with each member of $\mathcal{V}$ being uniformly connected.*

PROOF. Assume that X is uniformly locally uniformly connected and let $\mathcal{U}$ be a uniform cover of X. Then $\mathrm{int}_X \mathcal{U}$ is a uniform cover of X and so it is refined by a uniform cover $\mathcal{B}$ (of X) each member of which is uniformly connected. For each $B \in \mathcal{B}$, choose $U_B \in \mathcal{U}$ with $B \subset \mathrm{int}_X\, U_B$. Let C_B be the uniformly connected component of $\mathrm{int}_X\, U_B$ containing B (if $B = \emptyset$, $C_B = \emptyset$). Then $\mathcal{V} = \{C_B | B \in \mathcal{B}\}$ is the desired uniform cover of X.

The proof of the following proposition is analogous to Corollary 3.4 of [1].

PROPOSITION 1.5. *Let A be a dense subspace of a nearness space X and let G be an open uniformly connected subspace of X. Then $G \cap A$ is uniformly connected.*

PROOF. Follows from the fact that $\mathrm{cl}_X G = \mathrm{cl}_X(G \cap A)$.

PROPOSITION 1.6. *Let A be a dense subspace of a uniformly locally uniformly connected nearness space X. Then A is uniformly locally uniformly connected.*

PROOF. Assume that X is uniformly locally uniformly connected. Let $\mathcal{U}$ be a uniform cover of A. Then there exists a uniform cover $\mathcal{C}$ of X such that $\mathcal{U} = \{A\} \wedge \mathcal{C}$. Also there exists an open uniform cover $\mathcal{G}$ (of X) which refines $\mathcal{C}$ each member of $\mathcal{G}$ being uniformly connected. Then $\{A\} \wedge \mathcal{G}$ is a uniform cover of A which refines $\mathcal{U}$ and which (by proposition 1.5) consists of uniformly connected sets.

LEMMA 1.2. *Let X be a nearness space and let X^* be its completion. Then for an open subspace A of X, A is uniformly connected iff* op A *is. (We are using the notation* op $A = X^* - \mathrm{cl}_{X^*}(X - A)$.)

PROOF. $A = X \cap \mathrm{op}\, A$ and so by Proposition 1.5, if op A is uniformly connected then so is A. Conversely assume that A is uniformly connected. Since $A \subset \mathrm{op}\, A \subset \mathrm{cl}_{X^*} A$, A is dense in op A. Therefore op A is uniformly connected.

PROPOSITION 1.7. *A nearness space X is uniformly locally uniformly connected iff its completion X^* is.*

PROOF. Assume that X is uniformly locally uniformly connected. Let $\mathcal{U}$ be a uniform cover of X^*. Then there exists an open uniform cover $\mathcal{G}$ of X such that op $\mathcal{G}$ refines $\mathcal{U}$. (Here we are using the strictness of the completion map c.f [2] page 28). For some open uniform cover $\mathcal{V}$ of X, $\mathcal{V}$ refines $\mathcal{G}$ and each member of $\mathcal{V}$ is uniformly connected. Therefore op $\mathcal{V}$ refines $\mathcal{U}$, op $\mathcal{V}$ is a uniform cover of X^*, and each member of op $\mathcal{V}$ is uniformly connected. The converse follows from Proposition 1.6.

PROPOSITION 1.8. *If a topological nearness space is locally connected then it is uniformly locally uniformly connected.*

PROOF. Assume that X is a topological nearness space which is locally connected. Let $\mathcal{U}$ be an open cover of X. If $x \in U \in \mathcal{U}$ then there exists a connected neighborhood V of x with $V \subset U$. This means that such a V is connected as a subspace of X in the topological sense. Taking V as a nearness subspace of X, we are saying that TV is connected. By Proposition 1.1 (c), V is uniformly connected. Thus it is clear that there exists a uniform cover $\mathcal{V}$ of X which refines $\mathcal{U}$ and such that every $V \in \mathcal{V}$ is uniformly connected.

We now come to the main result of this paper.

PROPOSITION 1.9. *The following conditions are equivalent for a topological space X:*

(a) *X has a locally connected, regular T_1 extension.*

(b) *X is the underlying topological space of a nearness space Y which is concrete, regular and uniformly locally uniformly connected.*

PROOF. $(a) \Rightarrow (b)$: This follows from Propositions 1.8 and 1.6.

$(b) \Rightarrow (a)$: We shall show that Y^*, the completion of Y, is the desired locally connected, regular T_1 extension. It suffices to show that Y^* (which is topological) is locally connected. Let $\mathcal{G}$ be an open cover of Y^*. Since Y^* is regular, there exists an open cover $\mathfrak{D}$ of Y^* such that $\mathrm{Cl}_{Y^*}\mathfrak{D} = \{\mathrm{Cl}_{Y^*}D \mid D \in \mathfrak{D}\}$ refines $\mathcal{G}$. Since $\mathfrak{D}$ is a uniform cover of Y^* there exists a uniform cover $\mathcal{U}$ of Y such that op $\mathcal{U}$ refines $\mathfrak{D}$. Since Y is uniformly locally uniformly connected there exists a uniform cover $\mathcal{V}$ of Y which refines $\mathcal{U}$ and with each member of $\mathcal{V}$ being uniformly connected. Then $\mathrm{Cl}_{Y^*}\mathcal{V}$ is a uniform cover of Y^* (it is refined by op $\mathcal{V}$) and $\mathrm{Cl}_{Y^*}\mathcal{V}$ refines $\mathcal{G}$. For each $V \in \mathcal{V}$, $\mathrm{Cl}_{Y^*}V$ is uniformly connected (because V is) and since $\mathrm{Cl}_{Y^*}V$ is topological (because Y^* is) then $\mathrm{Cl}_{Y^*}V$ is connected. This completes the proof.

Section 2. The concept of uniform local connectedness was introduced in the theory of uniform spaces by A. M. Gleason. In this section we present a general theory of uniform local connectedness.

DEFINITION 2.1. *A nearness space X is called uniformly locally connected if each uniform cover $\mathcal{U}$ of X is refined by a uniform cover $\mathcal{V}$ of X such that for each $V \in \mathcal{V}$, TV, the underlying topological space of V, is connected.*

The proof of the following Proposition follows from the relevant definitions.

PROPOSITION 2.1.

(*a*) *Every uniformly locally connected nearness space is uniformly locally uniformly connected.*

(*b*) *If a nearness space X is uniformly locally connected then its underlying topological space TX is locally connected.*

REMARK 2.1. (*a*) Let Q_u be as in Remark 1.1. Then Q_u is uniformly locally uniformly connected but not uniformly locally connected. Thus the converse of Proposition 2.1 (*a*) does not hold.

(*b*) Let $X = \{(x, \sin 1/x) \mid 0 < x \leq 1\}$ as a subspace of the Euclidean plane with its usual (metric induced) uniformity. Then the completion of X is

$$X^* = XU\{(0, y) \mid -1 \leq y \leq 1\},$$

also as a subspace of the Euclidean plane with its usual uniformity. X^* is topological and is not locally connected. Therefore X^* is not uniformly locally uniformly connected. It follows that X is not uniformly locally uniformly connected and so, X is not uniformly locally connected. Thus the converse of Proposition 2.1 (*b*) does not hold.

PROPOSITION 2.2. *If a regular topological nearness space is uniformly locally uniformly connected then it is uniformly locally connected.*

PROOF. Assume that X is a regular topological nearness space which is uniformly locally uniformly connected. Let $\mathcal{U}$ be a uniform cover of X. By regularity, there exists a uniform cover $\mathcal{H}$ of X such that $\mathrm{cl}_X\mathcal{H}$ refines $\mathcal{U}$. There exists a uniform cover $\mathcal{V}$ of

X which refines $\mathcal{H}$ and each member of which is uniformly connected. Then $\mathrm{cl}_X\mathcal{V}$ is a uniform cover of X which refines $\mathcal{U}$ and for each $V \in \mathcal{V}$, $\mathrm{cl}_X V$ is uniformly connected (because V is). Since X is topological, so is $\mathrm{cl}_X V$. Therefore $\mathrm{cl}_X V$ is connected.

COROLLARY 2.3. *The following conditions are equivalent for a regular topological nearness space X:*

(a) X is uniformly locally uniformly connected.

(b) X is uniformly locally connected.

(c) X is locally connected.

PROPOSITION 2.4. *A nearness space is uniformly locally connected iff each uniform cover $\mathcal{U}$ of X is refined by an open uniform cover $\mathcal{V}$ of X each member of which is connected as a subspace (in the topological sense) of the underlying topological space TX of X.*

PROOF. Assume that X is uniformly locally connected and let $\mathcal{U}$ be a uniform cover of X. $\mathrm{int}_X\mathcal{U} = \{\mathrm{int}_X U \mid U \in \mathcal{U}\}$ is a uniform cover of X and is refined by a uniform cover $\mathcal{A}$ of X such that TA is connected for each $A \in \mathcal{A}$. For each $A \in \mathcal{A}$ choose $U_A \in \mathcal{U}$ such that $A \subset \mathrm{int}_X U_A$ and let V_A denote the component ($= \phi$ if $A = \phi$) of $\mathrm{int}_X U_A$ which contains A. Since TX is locally connected and $\mathrm{int}_X U_A$ is open in TX, V_A is also open in TX. $\mathcal{V} = \{V_A \mid A \in \mathcal{A}\}$ is the required open uniform cover of X since it is refined by $\mathcal{A}$.

REFERENCES

1. D. Baboolal, *On some uniform connection properties related to local connectedness*, Quaestiones Mathematicae, to appear.

2. H. L. Bentley, *Normal nearness spaces*, Quaestiones Mathematicae **2** (1977), pp. 23–24.

3. H. L. Bentley and H. Herrlich, *The reals and the reals*, General Topology Appl. **9** (1978), pp. 221–232.

4. H. Herrlich, *A concept of nearness*, General Topology Appl. **4** (1974), pp. 191–212.

5. H. Herrlich, *On the extendibility of continuous functions*, General Topology Appl. **5** (1974), pp. 213–215.

6. H. Herrlich, *Topological Structures*, Math. Centre Tract (Amsterdam) **52** (1974), pp. 59–122.

7. G. Preuss, *Connectedness and disconnectedness in S-Near*, Lecture notes in Math., (Springer-Verlag) **915** (1982), pp. 275–292.

8. A. M. Gleason, *Universal locally connected refinements*, Illinois J. Math. **7** (1963), pp. 521–531.

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