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$\mathbb{Z}^n$ and $\mathbb{R}^n$ cocycle extensions and complementary algebras

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Abstract. We present here an investigation of the degree to which mixing properties can be shown to lift to cocycle extensions of an ergodic map by $\mathbb{Z}^n$ and $\mathbb{R}^n$ weakly mixing actions. A number of other results on such extensions are also included.

Let $(T, X, \mathscr{F}, \mu)$ be an ergodic measure preserving transformation of a Lebesgue probability space. Let G be $\mathbb{Z}^n$ or $\mathbb{R}^n$, where n can be finite or countably infinite, and $S = \{S_g\}_{g \in G}$ a measurable, measure preserving free action of G on the Lebesgue space $(Y, \mathscr{G}, \nu)$. By a G-cocycle $f(x, n)$ over T we mean a measurable map $f: X \times \mathbb{Z} \to G$ satisfying

$$f(x, n_1 + n_2) = f(T^{n_1}(x), n_2) + f(x, n_1)$$

i.e.

$$f(x, n) = \sum_{i=0}^{n-1} f(T^i(x), 1).$$

We will abbreviate $f(x, 1)$ as $f(x)$, the generating function of the cocycle.

The 'S, f-extension' of T is, then

$$\hat{T}^n(x, y) = (T^n(x), S_{f(x, n)}(y)),$$

a measure preserving invertible map of $(X \times Y, \mathscr{F} \times \mathscr{G}, \mu \times \nu)$.

$$\hat{S}_g(x, y) = (x, S_g(y))$$

also gives an action of G on this space, commuting with $\hat{T}$.

This kind of cocycle construction has arisen in many contexts, most especially where G is compact, or where $G = \mathbb{Z}$. Much is known about compact cocycle extensions [**P1**], [**R1**], [**R2**]. Our motivation here is to investigate the degree to which these results extend to $\mathbb{Z}^n$ or $\mathbb{R}^n$ extensions, most especially the degree to which mixing properties of T must lift to $\hat{T}$. The two mixing properties for which we can prove results are weakly mixing and the K property, because both can be defined by disjointness from a certain class of maps [**F**] (pure point spectrum, and 0-entropy), and simultaneously by the lack of a factor map in this class. Hence our analysis turns to an investigation of how certain factor actions of $\hat{T}$ can arise.

We now define the necessary notions for our work, and state and discuss the results we will prove, postponing the proofs of the main results to the appendix.

As is standard, we say a sequence of measure preserving maps U_i converges weakly to $U(\overset{w}{\to})$ if for any set $A\in\mathscr{F}$,

$$\mu(U_i(A)\Delta U(A)) \underset{i}{\to} 0.$$

As our measure space is separable, it is enough to check the convergence on a generating collection of sets.

We say a G-action S is 'partially rigid' [**F-W**] if there is a non-trivial set A and elements $g_i \in G$, $g_i \nrightarrow \mathrm{id}$ but

$$\nu(S_{g_i}(A)\Delta A) \underset{i}{\to} 0.$$

As for our purposes G is abelian, the collection of all sets for which

$$\nu(S_{g_i}(A)\Delta A) \underset{i}{\to} 0,$$

g_i fixed, is an S-invariant σ-algebra. We call this a *rigid factor* of S.

If S is a mixing G-action, then S is its own weak closure and has no rigid factors. On the other hand if S is not weakly mixing, it must have rigid factors. There are, though, weakly mixing $\mathbb{Z}^n$ and $\mathbb{R}^n$-actions with rigid factors.

A 'complementary algebra' $\mathscr{A}\subset\mathscr{F}\times\mathscr{G}$ is a non-trivial σ-algebra, invariant under both $\hat{T}$ and $\hat{S}_g$, $g\in G$ and $\mathscr{A}\perp\mathscr{F}$. Our main results concern what the existence of such an algebra forces.

THEOREM 1. *Assume T is weakly mixing, S a weakly mixing G-action, f a measurable cocycle into G and $\hat{T}$ the corresponding S,f-extension. If there is a complementary algebra $\mathscr{A}$, then*

$$T|_{\mathscr{F}\times\mathscr{A}} \cong T\times\tilde{S},$$

where the isomorphism map, defined $\mu\times\nu$ a.e., is of the form

$$\psi(x,\bar{y}) = (x, U_x(\bar{y})),$$

$\tilde{S}$ and U_x are in the weak closure of $\hat{S}|_{\mathscr{A}}$, and $U_x(\bar{y})$ is bimeasurable.

The results that interest us we prove as corollaries of this.

COROLLARY 2. *Under the assumptions of theorem 1, if S also has no rigid factors, then there is a measurable map ϕ; $X\to G$ and constant $c\in G$ with*

$$f(x)-c=\phi(T(x))-\phi(x),$$

i.e. f differs from a constant by a coboundary.

Proof. As S is its own weak closure, on all factors,

$$\tilde{S}(\bar{y}) = \hat{S}_c(\bar{y})$$

for some $c\in G$, and

$$U_x(\bar{y}) = \hat{S}_{\phi(x)}(\bar{y}),$$

ϕ measurable. As $S|_{\mathscr{A}}$ is free,

$$f(x)-c=\phi(T(x))-\phi(x). \qquad \square$$

We can give the constant c in this result an alternate description as follows. If $f \in L^1(\mu)$ we know

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(x))$$

converges in n to $\int f\,d\mu$.

We define a weaker notion of $L^1(\mu)^\cdot$ ('almost $L^1(\mu)$') as those functions $f;\ X \to G$ so that for a.e. x there is a sequence $n_i(x) \xrightarrow{i}$ of full density and

$$\frac{1}{n_i(x)}\sum_{j=0}^{n_i(x)-1} f(T^j(x))$$

converges to some finite value.

It is easy to argue that the limit is measurable, is independent of which sequence of full density is chosen hence is T-invariant and so constant a.e.

If $f \in L^1(\mu)^\cdot$ we write this limit as $\int f\,d\mu^\cdot$.

LEMMA 3. *If* $f \in L^1(\mu)^\cdot$ *and* $g(x) - f(x) = \phi(T(x)) - \phi(x)$ *then* $g \in L^1(\mu)^\cdot$ *and*

$$\int f\,d\mu^\cdot = \int g\,d\mu^\cdot.$$

Proof. Fix $\varepsilon < 0$ and choose a set A and bound b with $\mu(A) > 1-\varepsilon$ and for $x \in A$, $|\phi(x)| < b$. Now let $n_i^\varepsilon(x)$ be those values that are both an $n_j(x)$ and have $T^{n_i^\varepsilon(x)} \in A$. It follows that

$$\frac{1}{n_i^\varepsilon(x)}\sum_{j=0}^{n_i^\varepsilon(x)} g(T^j(x))$$

converges in i to $\int f\,d\mu^\cdot$. Diagonalizing over a sequence of ε's $\to 0$ completes the result. □

Using the techniques of [**R3**], (**dJ-R**] one can in fact show that $f \in L^1(\mu)^\cdot$ iff there is a bounded g and measurable ϕ satisfying

$$f(x) - g(x) = \phi(T(x)) - \phi(x).$$

We leave this argument to the interested reader.

COROLLARY 4. *Under the assumptions of corollary* 1, *we must have* $f \in L^1(\mu)^\cdot$ *and* $c = \int f\,d\mu^\cdot$.

COROLLARY 5. *If T is weakly mixing and S has no rigid factors and $\hat{T}$ is ergodic, then $\hat{T}$ must be weakly mixing.*

Proof. Assuming $\hat{T}$ is not weakly mixing, let $\mathscr{a}$ be the σ-algebra generated by the point spectrum of $\hat{T}$. We must have $\hat{T}(\mathscr{a}) = \mathscr{a}$ and $\hat{S}_g(\mathscr{a}) = \mathscr{a}$ as $\hat{S}_g$ commutes with $\hat{T}$. Further, as $T|_{\mathscr{a}}$ has pure point spectrum and $\hat{T}|_{\mathscr{F}}$ is weakly mixing, $\mathscr{a} \perp \mathscr{F}$. Hence $\mathscr{a}$ is a complementary algebra. We conclude

$$f(x) - c = \phi(T(x)) - \phi(x).$$

Defining

$$\psi(x, y) = (x, S_{\phi(x)}(y)),$$
$$\psi^{-1}(\hat{T})\psi = T \times S_c.$$

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If S_c is ergodic, as S is weakly mixing, $\hat{T}$ is also. If S_c is non-ergodic, $\hat{T}$ is non-ergodic, and we are done. □

COROLLARY 6. *If T is a K-automorphism and S has no rigid factors then $\hat{T}$ is either isomorphic to $T \times S_c$ for some $c \in G$ or $\hat{T}$ is K.*

Proof. Same as for corollary 5 but assuming $\hat{T}$ is not K, let $\mathcal{a}$ be the Pinsker algebra of $\hat{T}$. □

Here are some special cases of this result.

COROLLARY 7. *If both T and S are K-systems and $\hat{T}$ is ergodic, then $\hat{T}$ is K.*

Proof. By corollary 6 if $\hat{T}$ is not K, then $\hat{T} \cong T \times S_c$. If this is not K, $c = \vec{0}$ and $\hat{T}$ is non-ergodic. □

COROLLARY 8. *If T is a K-automorphism S has no rigid factors, $f \in L^1(\mu)^{\cdot}$ and $\int f\, d\mu^{\cdot} = 0$, then if $\hat{T}$ is ergodic, it must be K.*

Proof. If $\hat{T}$ is not K, we know $\hat{T} \cong T \times S_c$ where $c = \int f\, d\mu^{\cdot} = \vec{0}$, so $\hat{T}$ is non-ergodic. □

Either of corollaries 7 or 8 provides insight into the K-ness of the now famous T, T^{-1} map **[Ka1]** and the work of Meilijson **[M]**, as it says K-ness in these cases is not so delicate a property, following simply from ergodicity.

COROLLARY 9. *If both T and S are K-systems and either $f \notin L^1(\mu)^{\cdot}$ or $\int f\, d\mu^{\cdot} \neq \vec{0}$, then $\hat{T}$ must be K.*

The T, T^{-1} map shows that the Bernoulli property cannot replace the K-property in corollaries 7 and 8. If $G = \mathbb{Z}$, S is Bernoulli, $f \in L^1(\mu)^{\cdot}$ but $\int f\, d\mu^{\cdot} \neq 0$ it is a nice exercise to show $\hat{T}$ is Bernoulli. What happens when $f \notin L^1(\mu)^{\cdot}$? Another interesting question is to what degree the strong mixing property can replace the K-property in these results. Complementary algebra arguments, of course, don't apply.

We now investigate the role of partial rigidity in these results.

THEOREM 10. *For any aperiodic T and partially rigid free $\mathbb{Z}^n$ or $\mathbb{R}^n$ action S there is an $f \in L^1(\mu)$ so that the S, f-extension $\hat{T}$ is non-ergodic but*

$$f(x) - \int f\, d\mu = \phi(T(x)) - \phi(x)$$

has no measurable solution ϕ a.e.

This is a fairly simple construction. It is still true that $\mathcal{a} =$ (algebra of $\hat{T}$ invariant sets) is a complementary algebra and $\hat{T}|_{\mathcal{F} \times \mathcal{a}} \cong T \times \mathrm{id}$, so $\tilde{S} = \mathrm{id}$ is in S but the U_x are not, but only in the weak closure of S. Using the cocycle generated by $f_c(x) = f(x) + c$ easily gives an example where $\tilde{S} = S_c$. This leads to the interesting possibility that perhaps only assuming S weakly mixing forces $\tilde{S} \in S$. It is also easily seen from the construction that $\int f_c\, d\mu = c$. It is not difficult, under the conditions of theorem 10 to construct a cocycle $f \geq 0$ with the S, f-extension $\hat{T}$ once more non-ergodic. Our next result says, though, that this forces $f \notin L^1(\mu)^{\cdot}$.

THEOREM 11. *Under the conditions of theorem* 1, *if* $\tilde{S}=S_c\in S$ *and* $f\in L^1(\mu)^{\bullet}$ *then*

$$\int f\,d\mu^{\bullet}=c.$$

Furthermore, if S *and* T *are such that whenever*:

(i) $\int f\,d\mu^{\bullet}=\vec{0}$; *and*

(ii) *a complementary algebra exists*; *then*

$$\tilde{S}=\mathrm{id}.$$

It follows that whenever:

(i) $f\in L_1(\mu)^{\bullet}$; *and*

(ii) *a complementary algebra exists*; *then*

$$\tilde{S}=S_c,\qquad c=\int f\,d\mu^{\bullet}.$$

We can refine our question about whether $\tilde{S}\in S$, then as follows. First, can there be an $f\in L^1(\mu)^{\bullet}$ so that $\hat{T}$ has a complementary algebra, $\tilde{S}$ is ergodic but $\int f\,d\mu^{\bullet}=0$? If not, then whenever $f\in L_1(\mu)^{\bullet}$, $\tilde{S}\in S$. Second, allowing f to be outside $L_1(\mu)^{\bullet}$ can we obtain a complementary algebra with $\tilde{S}\notin S$?

Our interest in investigating these issues arose from a study of the deep construction of Katok [**A-Kat**], [**Kat**] of a smooth K-automorphism that is not Bernoulli and a desire to understand the degree to which his proof of K-ness was true for purely measure algebra reasons. What is clear from our work is that in fact the smooth theory is essential to his proof, as his choice for S is necessarily rigid. Our last result will demonstrate this even more strongly, by showing that when X is a compact metric space continuity of the cocycle can be assumed in the construction in theorem 2.

THEOREM 12. *Let* $\mathcal{A}$ *be an algebra of real valued functions satisfying*:

(i) $\mathcal{A}$ *is dense in* $L^1(\mu)$;

(ii) $\mathcal{A}$ *is closed under uniform convergence*; *and*

(iii) *if* $|g|<\alpha$, *then for any* $\varepsilon>0$ *there is an* $f\in\mathcal{A}$, $\|f-g\|_1<\varepsilon$ *and* $|f|<\alpha$.

We conclude that for any $g\in L^1(\mu)$, *there is an* $f\in\mathcal{A}$ *and* $\phi:X\to\mathbb{R}$ *satisfying*

$$g(x)-f(x)=\phi(T(x))-\phi(x).$$

Of course the algebra $\mathcal{A}$ that is most natural is the algebra of continuous functions where X is a compact metric space and μ a Borel measure.

Thus in theorem 1 if we chose T to be a hyperbolic map of a compact metric space, S a smooth rigid flow as constructed in [**A-Kat**] then the cocycle constructed in theorem 2 can, by theorem 2, be assumed continuous. It cannot be made C^1 though, as Katok, [**Kat**] has shown that in this case the coboundary equation

$$f(x)-\int f\,d\mu=\phi(T(x))-\phi(x)$$

must have a solution.

Height functions in the representation of flows under a function are also cocycles with the added requirement that they be positive. Parry [**P2**] and Kočergin [**Ko**]

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have obtained results concerning continuous time changes of flows on compact metric spaces. This yields a natural question: does a version of theorem 12 exist where the cocycles are required to be positive? Such a theorem can probably be obtained via the method we have used. The author has been informed that Kočergin has proved such a result, but has been unable to find a reference.

Appendix

Proof of theorem 1. As $\mathscr{F} \perp \mathscr{a}$, for any $A \in \mathscr{a}$,

$$E(A|\mathscr{F}) = \mu \times \nu(A)$$

almost everywhere. For a set $S \in X \times Y$, let

$$S_x = \{y|(x, y) \in S\}.$$

Select a countable dense algebra of sets $\mathscr{a}_0 \subset \mathscr{a}$, invariant under a countable dense subset of $\hat{S}$, and then a T-invariant subset X_0 of full measure so that for all $A \in \mathscr{a}$ and $x \in X_0$,

$$E(A|\mathscr{F})(x) = \nu(A_x) = \mu \times \nu(A).$$

For each $x \in X_0$, let $\mathscr{a}_x$ be the completion in $(Y, \mathscr{F}, \nu)$ of the algebra $\{A_x\}_{A \in \mathscr{a}_0}$, an S-invariant sub-σ-algebra of $\mathscr{F}$.

If $A_x \in \mathscr{a}_x$, and $A^i \in \mathscr{a}_0$ are a sequence of sets with

$$\nu(A^i_x \triangle A_x) \underset{i}{\to} 0,$$

then for all $x' \in X_0$, $A^i_{x'}$ must converge *uniformly* to some $A_{x'}$. Hence defining $\mathscr{a}_1 \in \mathscr{a}$ as the algebra of sets uniformly approximable on X_0 by sets in $\mathscr{a}_0$,

$$\mathscr{a}_x = \{A_x | A \in \mathscr{a}_1\}.$$

Now $\mathscr{a}_1$ is not a complete algebra, but any set in $\mathscr{a}$ is within measure zero of one in $\mathscr{a}_1$, and $\mathscr{a}_1$ is clearly $\hat{T}$ and $\hat{S}_g$ invariant.

For $x \in X_0$, let $\pi_x : \mathscr{a}_x \to \mathscr{a}_1$ be the set map taking $A_x \to A$, clearly measure preserving and invertible mod 0 sets, and $\pi_x S_g = \hat{S}_g \pi_x$ as $\mathscr{a}$ is $\hat{S}$ invariant.

As X_0 is a Lebesgue space, for a.e. x there are sets $C_i(x)$ in a countable algebra with $\bigcap_i C_i(x) = x$ and if $x_i \in C_i(x)$ then $\pi_{x_i}^{-1} \xrightarrow{w} \pi_x^{-1}$ on $\mathscr{a}$.

Let x_0 be such a point, $C_i = C_i(x_0)$, and let

$$\tilde{S} = \pi_{x_0}^{-1} \hat{T} \pi_{x_0}|_{\mathscr{a}_{x_0}}$$

be a set map $\mathscr{a}_{x_0} \to \mathscr{a}_{x_0}$. For any $A \in \mathscr{a}_1$,

$$A = \bigcup_{x \in X_0} (x, \pi_x^{-1} \pi x_0(A_{x_0}))$$

so

$$\hat{T}(A) = \bigcup_{x \in X} (T(x), \pi_{T(x)}^{-1} \pi_{x_0} (\tilde{S}(A_{x_0}))).$$

Letting $\psi(B \times A) = B \times A_{x_0}$, a set map from $\mathscr{F} \times \mathscr{a}_1 \to \mathscr{F} \times \mathscr{a}_{x_0}$,

$$\psi(x, A) = (x, \pi_{x_0}^{-1} \pi_x(A))$$

and first note

$$\hat{T}^n(A) = \bigcup_{x\in X} (T^n(x), \pi^{-1}_{T^n(x)}\pi_{x_0}(\tilde{S}^n(A_{x_0})))$$
$$= \bigcup_{x\in X} (T^n(x), S_{f(x,n)}(\pi_x^{-1}\pi_{x_0}(A_{x_0})).$$

Select $x_1(i)$ and $x_2(i)\in C_i$ and n_i so that

$$T^{n_i}(x_1(i)),\ T^{n_i+1}(x_2(i))\in C_i.$$

This is possible as T is weakly mixing. Thus

$$\pi^{-1}_{T^{n_i}(x_1(i))}\pi_{x_0}(S^{n_i}(A_{x_0})) = \tilde{S}_{f(x_1(i),n_i)}(\pi^{-1}_{x_1(i)}\pi_{x_0}(A_{x_0}))$$

and

$$\pi^{-1}_{T^{n_i+1}(x_2(i))}\pi_{x_0}(\tilde{S}^{n_i+1}(A_{x_0})) = S_{f(x_2(i),n_i)}(\pi^{-1}_{x_2(i)}\pi_{x_0}(A_{x_0})).$$
$$\tilde{S}(A_{x_0}) = S_{f(x_2(i),n_i+1)-f(x_1(i),n_i)}\cdot \pi^{-1}_{x_0}\pi_{x_1(i)}\pi^{-1}_{T^{n_i}(x_1(i))}\pi_{T^{n_i+1}(x_2(i))}\pi^{-1}_{x_2(i)}\pi_{x_0}(A_{x_0})$$

and

$$S_{f(x_2(i),n_i+1-f(x_1(i),n_i)} \xrightarrow{w} \tilde{S} \qquad \text{on } a_x.$$

For a.e. x we can select n_i with $T^{n_i}(x)\in C_i$ and now

$$\pi^{-1}_{T^{n_i}(x)}\pi_{x_0}(\tilde{S}^{n_i}(A_{x_0})) = S_{f(x,n_i)}(\pi_x^{-1}\pi_{x_0}(A_{x_0}))$$

or

$$\pi^{-1}_{x_0}\pi_{T^{n_i}(x)}(\pi_x^{-1}\pi_{x_0}(A_{x_0})) = S_{f(x,n_i)}\tilde{S}(A_{x_0}).$$

We conclude $\pi^{-1}_{x_0}\pi_{T^{n_i}(x)}\pi_x^{-1}\pi_{x_0}$ is in the weak closure of S. But as

$$\pi^{-1}_{x_0}\pi_{T^{n_i}(x)} \xrightarrow{w} \text{id} \qquad \text{on } a_x,$$

$\pi_x^{-1}\pi_{x_0}$ is in the weak closure of S and now as the weak closure of S is abelian, all $a_x = a_{x_0}$.

Restrict S to the algebra $a_{x_0} = \bar{a}$, and define $\psi(x,\bar{y}) = (x, U_x(\bar{y}))$, $U_x = \pi_x^{-1}\pi_{x_0}$

$$\hat{T}\psi(x,y) = \hat{T}(x, \pi_x^{-1}\pi_{x_0}(y))$$
$$= (T(x), S_{f(x)}\pi_x^{-1}\pi_{x_0}(y))$$
$$= (T(x), \pi^{-1}_{T(x)}\pi_{x_0}\tilde{S}(y)) = \psi(T\times\tilde{S})$$

and we are done. That $U_x(y)$ is bimeasurable follows from the weak convergence of $\pi^{-1}_{x_i}$ to π_x on $C_i(x)$. □

Proof of theorem 10. First let c_i be a sequence of values, $c_{i+1} > 2\sum_{j=0}^{i} c_i$, and A a set of measure $\frac{1}{2}$ with

$$\mu(S_{c_i}(A)\cap A) > \mu(A)\left(1 - \frac{1}{100\times 2^i}\right).$$

Select n_i so large that $\sum_{i=1}^{\infty} c_i/n_i < \infty$, $n_i/n_{i+1} < 1/(100\times 2^{i+1})$. Using the Rochlin lemma for T, find sets B_i with $B_i, T(B_i), \ldots, T^{2n_i-1}(B_i)$ all disjoint

$$\mu\left(\bigcup_{j=0}^{2n_i-1} T^j(B_i)\right) > 1 - \frac{1}{100\times 2^i}.$$

Let $A_i^+ = B_i$, $A_i^- = T^{n_i-1}(B_i)$. As $\mu(A_i^+) = \mu(A_i^-) < 1/(100\times 2^i)$ a point x is, almost surely, in only finitely many A_i^+ and A_i^-. Hence

$$I^+(x) = \{i \mid x\in A_i^+(x)\}$$

and

$$I^-(x)=\{i \mid x\in A_i^-(x)\}$$

are finite sets. Define

$$f(x)=\sum_{i\in I^+(x)} c_i-\sum_{i\in I^-(x)} c_i.$$

Thinking inductively, what we do is to take the 0th level of the ith tower and to however f has already been defined, we add a further c_i, and at level n_i-1, halfway up the tower, we subtract this value off again.

As

$$\int |f|\,d\mu=\sum_{i=1}^{\infty}\mu(A_i^+\cup A_i^-)C_i$$
$$\leq \sum_{i=1}^{\infty} C_i/n_i<\infty,$$

$f\in L^1(\mu)$. It is clear $\int f\,d\mu=0$. From our discussion it is also clear that

$$f(x,n)=\sum_{i\in I^+(x,n)} c_i-\sum_{i=I^-(x,n)} c_i,$$

where

$$I^+(x,n)=\left\{i \,\middle|\, x\in\left(\bigcup_{j=0}^{n_i-1}T^j(B_i)\right)^c,\ T^n(x)\in\bigcup_{j=0}^{n_i-1}T^j(B_i)\right\},$$
$$I^-(x,n)=\left\{i \,\middle|\, x\in\bigcup_{j=0}^{n_i-1}T^j(B_i),\ T^n(x)\in\left(\bigcup_{j=0}^{n_i-1}T^j(B_i)\right)^c\right\}.$$

What matters is that each c_i occurs at most once in each sum, but we also note that the index sets $I^+(x,n)$ and $I^-(x,n)$ are disjoint. Thus

$$\mu(S_{f(x,n)}(A)\bigtriangleup A)\leq\sum_{i=1}^{\infty}\mu(S_{c_i}(A)\bigtriangleup A)\leq\mu(A)/100.$$

Let $\bar{A}=X\times A$ and now

$$\mu\times\nu(\hat{T}^n(\bar{A})\bigtriangleup\bar{A})=\int_X (S_{f(x,n)}(A)\bigtriangleup A)\,d\mu\leq\mu(A)/100.$$

So computing

$$\int_{\bar{A}}\frac{1}{n}\sum_{j=0}^{n-1}\chi_{\bar{A}}(T^{-j}(x,y)\,d\mu\times\nu$$
$$=\frac{1}{n}\sum_{j=0}^{n-1}\mu\times\nu(T^j(\bar{A})\cap\bar{A})$$
$$=\frac{1}{n}\sum_{j=0}^{n}\int_X (S_{f(x,j)}(A)\cap A)\,d\mu$$
$$\geq\frac{1}{n}\sum_{j=0}^{n}\int_X \left(\tfrac{99}{100}\mu(A)\right)d\mu=0.99\,\mu(A).$$

But if $\hat{T}$ were ergodic, then by the mean ergodic theorem,

$$\lim_{n\to\infty}\int_A\frac{1}{n}\sum_{j=0}^{n-1}\chi_{\bar{A}}(T^{-j}(x,y))\,d\mu\times\nu=\mu(\bar{A})^2,$$

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and we conclude $\hat{T}$ is not ergodic. Let $\mathcal{A}$ be the algebra of $\hat{T}$-invariant sets. This is a complementary algebra and using theorem 1, $\tilde{S} = \mathrm{id}$.

It remains to see that $f(x) = \phi(T(x)) - \phi(x)$ has no measurable solution. Assume that it does. Pick a bound b so large that for $x \in B$, $\mu(B) > 0.9$, $|\phi(x)| < b$. Let

$$D_i^+ = \bigcup_{j=0}^{n_i-1} T^j(\{x \mid i \in I^+(x) \text{ but } i' \notin I^\pm(T^k(x)) \text{ if } 0 \le k < n_i \text{ and } i' > i\}).$$

This is the set of points in the lower half of the ith tower which never have any further c_i's added on to the value of f.

$$\mu(D_i^+) \ge \frac{1}{2}\left(1 - \frac{1}{100 \times 2^i}\right) - n_i \sum_{i'=i+1}^{\infty} \frac{1}{n'_i} \ge \frac{1}{2}\left(1 - \frac{1}{50 \times 2^i}\right) > 0.9(\tfrac{1}{2}).$$

Let

$$D_i^- = \bigcup_{j=n_i}^{2n_i-1} T^j(\{x \mid i \in I^-(x) \text{ but } i' \notin I^\pm(T^k(x)) \text{ if } n_i \le k < 2n_i \text{ and } i' > i\}).$$

This is the corresponding set of points in the upper half of the ith tower,

$$\mu(D_i^-) \ge 0.9(\tfrac{1}{2})$$

Now choose i so large that $c_i > 4b$. We know

$$\mu(B^+) = \mu(B \cap D_i^+) > 0.35$$

and

$$\mu(B^-) = \mu(B \cap D_i^-) > 0.35.$$

Thus

$$\mu(T^{n_i}(B^+) \cap B^-) > 0.2,$$

and there is an $x \in B^+$, $T^{n_i}(x) \in B^-$. But now

$$\begin{aligned} |f(x, n_i)| &= \left| \sum_{j \in I^+(x,n)} c_j - \sum_{j \in I^-(x,n)} c_j \right| \\ &\ge c_i - \sum_{j<i} c_j > c_i/2 > 2b. \end{aligned}$$

But

$$f(x, n_i) = \phi(T^{n_i}(x)) - \phi(x) < 2b,$$

which is a conflict, and we are done. □

Proof of theorem 11. Suppose $\tilde{S} = S_c$, $c \in G$ and w.l.o.g. we can assume $\bar{\mathcal{A}} = \mathcal{F}$. Let $\tilde{f}(x) = f(x) - c$ and define a new map

$$\hat{\tilde{T}}(x, y) = (T(x), S_{f(x)-c}(y)) = \hat{S}_{-c} \circ \hat{T}(x, y).$$

As $\hat{S}(\mathcal{A}) = \mathcal{A}$, $\mathcal{A}$ is a complementary algebra for $\hat{\tilde{T}}$ and in fact

$$\begin{aligned} \psi \tilde{I} \psi^{-1} &= \psi(\hat{S}_{-c}\hat{T})\psi^{-1} = \hat{S}_{-c}(T \times S_c) \\ &= (\mathrm{id} \times S_{-c}) \circ (T \times S_c) = T \times \mathrm{id}, \end{aligned}$$

and $\tilde{T}$ is non-ergodic with $\mathcal{A}$ its algebra of invariant sets. By theorem 1, $\psi(x, y) = (x, U_x(y))$, where U_x is in the weak closure of S. Select a set $A \in \mathcal{A}$, $\mu(A) = \frac{1}{2}$ and for each x, $c(x) \in G$ with

$$\mu(U_x(A) \triangle S_{c(x)}(A)) < \tilde{\varepsilon},$$

$\bar{\varepsilon}$ to be chosen later. Define

$$\bar{f}(x, n) = \tilde{f}(x, n) - c(x) + c(T^n(x)),$$
$$\hat{\bar{T}}(x, y) = (T(x), S_{\bar{f}(x)}(y))$$

Letting $\bar{\psi}(x, y) = (x, S_{c(x)}(y))$, $\hat{\bar{T}}\bar{\psi} = \bar{\psi}\hat{\bar{T}}$ and so $\hat{\bar{T}}$ is also non-ergodic with complementary algebra $\bar{\psi}(\mathscr{a})$. In fact $\bar{\psi}(\mathscr{a})$ is weakly close to $X \times \mathscr{G}$ as we compute

$$\begin{aligned}\nu(S_{\bar{f}(x,n)}(A) \triangle A) &= \nu(S_{c(T^n(x))} \circ S_{f(x)} \circ S_{-c(x)}(A) \triangle U_{T^n(x)}S_{f(x)}U_x^{-1}(A)) \\ &= \nu(S_{-c(x)}U_x(A) \triangle S_{-c(T^n(x))}U_{T^n(x)}(A)) < 2\bar{\varepsilon},\end{aligned}$$

for a.e. x.

Now suppose $\int \bar{f}\, d\mu^{\bullet} \neq 0$, which w.l.o.g. we assume is >0. Hence $\bar{f}(x, n_i(x)) = \sum_{j=0}^{n_i(x)-1} f(T^j(x))$ is asymptotically $n_i(x)\int f\, d\mu^{\bullet}$, and furthermore

$$\nu(S_{\bar{f}(x,n_i(x))}(A) \triangle A) < 2\bar{\varepsilon}.$$

As S is weakly mixing, select $\bar{\varepsilon}$ so small that

$$I = \{n \mid \mu(S_n(A) \triangle A) < 4\bar{\varepsilon}\}$$

has zero density in G, (the set of n for which $\mu(S_n(A) \triangle A) = 2\mu(A)(1-\mu(A)) \pm \varepsilon$ has full density in G). Next select $\bar{\alpha}$ so small that if $|c| < \bar{\alpha}$ then

$$\mu(S_c(A) \triangle A) < \bar{\varepsilon}.$$

Select a value N, $N\displaystyle\int f\, d\mu^{\bullet} > 10\bar{\alpha}$, so that for a set B, $\mu(B) > 0.9$, if $x \in B$ and $n > N$ then

$$\# \frac{\{n_i(x) \leq n\}}{n} > 0.9,$$

and if $n_i(x) \geq n$ then

$$\bar{f}(x, n_i(x)) = n_i(x)\int f\, d\mu^{\bullet}(1 \pm 0.1).$$

Now pick $x_0 \in B$ and define inductively a sequence $x_i \in B$ all in the positive orbit of x by $x_{i+1} = T^{a_{i+1}}(x_i)$, where

$$a_{i+1} = n_{j(i+1)}(x_i) = n_{k(i+1)}(x_0) - \sum_{j=1}^{i} a_j$$

is the smallest such value N so that $T^{a_{i+1}}(x_i) \in B$. Now

$$a_{i+1} = n_{j(i+1)}(x_i) \geq N > 10\alpha,$$

and

$$\bar{f}(x_i, a_{i+1}) = a_{i+1}\int f\, d\mu^{\bullet}(1 \pm 0.1).$$

Let $b_i = \sum_{j=1}^{i} a_i$ and now

$$\bar{f}(x_0, b_i) = \sum_{j=0}^{i-1} \bar{f}(x_i, a_i)$$

and the intervals

$$J_i = (\bar{f}(x_0, b_i) - \alpha, \bar{f}(x_0, b_i) + \alpha)$$

are all disjoint. Further, as all $b_i = n_{k(i+1)}(x_0)$, $\bigcup_i J_i \subset I$.

We now show $\bigcup_i J_i$ has positive density by showing $\sup b_n/n < \infty$, as then the fraction of $[0, \bar{f}(x_0, b_n) + \alpha)$ occupied by $\bigcup_{n=1}^{n} J_i$ is asymptotically at least

$2\alpha n/b_n \int_T \bar{f}(1.1) > 0$, which is in conflict with I having zero density. To see $\sup b_n/n < \infty$, notice that any value j, $b_n + N < j < b_{n+1}$, either is not an $n_k(x_0)$, is not an $n_k(x_n)$, or is not in B. This is a block of $a_j - N$ indices in $[0, b_n]$. In the entire block $[0, b_n]$, the fraction of indices not an $n_k(x_0)$ is $(0.1)b_n$, and in each block b_{j-1} to b_j the fraction of indices not an $n_k(x_{j-1})$ is at most $(0.1)a_j$. We conclude

$$\sum_{k=0}^{b_n} \chi_{A^c}(T^k(x_0)) \geq \left(\sum_{j=1}^{n} (a_j - N) - (0.2)b_n \right),$$

and so for n sufficiently large

$$\frac{(0.1)b_n}{n} \geq \frac{(0.8)b_n}{n} - N$$

as $N/0.7 \geq b_n/n$. Thus we must conclude

$$\int \bar{f}\, d\mu^{\cdot} = \int \tilde{f}\, d\mu^{\cdot} = \int f\, d\mu^{\cdot} - c = 0$$

or

$$c = \int f\, d\mu.$$

The second half of the theorem is very simple. Letting

$$\tilde{f}(x) = f(x) - \int f\, d\mu,$$

$$\hat{\tilde{T}} = \hat{T}\hat{S}_{-\int f d\mu},$$

hence still preserves α. As $\int \tilde{f}\, d\mu = 0$,

$$\hat{\tilde{S}} = \tilde{S}\hat{S}_{-\int f d\mu} = \mathrm{id},$$

hence

$$\tilde{S} = S_{\int f d\mu}. \qquad \square$$

Proof of theorem 12.

LEMMA 13. *If f_1 and f_2 are two $\mathbb{R}$ cocycles and there is a bound b and set A, $\mu(A) > 0$, so that whenever x, $T^n(x) \in A$, then*

$$|f_1(x, n) - f_2(x, n)| < b,$$

then there is a measurable ϕ with

$$f_1(x) - f_2(x) = \phi(T(x)) - \phi(x).$$

Proof. Let

$$\phi(x) = \sup_{n, T^n(x) \in A} (f_1(x, n) - f_2(x, n)).$$

As T is ergodic, for a.e. x, there is an n_0 with $x_0 = T^{n_0}(x) \in A$ and now

$$\begin{aligned} &\sup_{n, T^n(x) \in A} (f_1(x, n) - f_2(x, n)) \\ &\quad \leq \sup_{n, T^n(x) \in A} ((f_1(x, n_0) - f_2(x, n_0)) + (f_1(x_0, n - n_0) - f_2(x_0, n - n_0))) \\ &\quad \leq f_1(x, n_0) - f_2(x, n_0) + b, \end{aligned}$$

hence ϕ is finite a.e.

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For $\varepsilon > 0$, select an n so that $T^n(x) \in A$ and

$$0 \leq \phi(x) - (f_1(x, n) - f_2(x, n)) < \varepsilon.$$

Now

$$0 \leq \phi(T(x)) - (f_1(T(x), n-1) - f_2(T(x), n-1))$$

so

$$\phi(T(x)) - \phi(x) - (f_1(x) - f_2(x)) > -\varepsilon$$

for all ε. Hence

$$\phi(T(x)) - \phi(x) - (f_1(x) - f_2(x)) \geq 0.$$

On the other hand, selecting n, $T^n(x) \in A$,

$$0 \leq \phi(T(x)) - (f_1(T(x), n-1) - f_2(T(x), n-1)) < \varepsilon$$

and as $0 \leq \phi(x) - (f_1(x, n) - f_2(x, n))$,

$$\phi(T(x)) - \phi(x) - (f_1(x) - f_2(x)) < \varepsilon$$

for all ε hence ≤ 0 and the lemma follows. □

It is easy to show that the condition of this lemma is also necessary if f_1 and f_2 are to differ by a coboundary.

It is well known that if $g \in L^1(\mu)$, then we can modify g by a coboundary to $\bar{g}$ so that for a.e. x, $|\bar{g}(x)| \leq 2\int g\, d\mu = 2\int \bar{g}\, d\mu$. Hence we can assume g is bounded.

Furthermore, as $\mathscr{A}$ is an algebra, if we solve the problem for cg, we solve it for g. Hence we assume $|g(x)| \leq 1$.

Now to describe the format of our argument. We select parameters ε_i with $\sum_i \sqrt{\varepsilon_i} < 1/24$, any such will do. We will select inductively a sequence $N_i \nearrow \infty$, $1/N_i < \varepsilon_i$, and pairs of sets (b_i, t_i) standing for bottom and top,

$$b_{i+1} \subset b_i, \quad t_{i+1} \subset t_i, \quad b_i \cap t_i = \varnothing,$$

and for each $x \in b_i$ there is an $N_i \geq n_i(x) \geq N_i/2$ with $T^{n_i(x)}(x) \in t_i$ and if $0 < n < n_i(x)$, $T^n(x) \notin t_i \cup b_i$.

Furthermore $B_i = \bigcup_{x \in b_i} (\bigcup_{n=0}^{n_i(x)-1} T^n(x))$ has $\mu(B_i) > 1 - \varepsilon_i$. Notice that t_i and B_i are disjoint, t_i occupying one level beyond B_i.

This is simply a special sort of block structure which, as we shall see, is not difficult to construct. If $x \in b_i$, then the set of points $x, T(x), \ldots, T^{n_i(x)-1}$ we will call an i-block.

Along with this block structure we will have a sequence of functions $f_i \in \mathscr{A}$ and sets A_i, $b_i \cup T^{-1}(t_i) \subset \bigcup_{j=1}^{i} A_i$, so that:

(i) $|f_i(x)| \leq 12\sqrt{\varepsilon_{i-1}}$ uniformly on X for $i > 1$; and for $i \geq 1$

(ii) $\mu(A_i) \geq 1 - 10\sqrt{\varepsilon_{i-1}}$; and

(iii) if $x \in b_i$, $T^n(x) \in \bigcap_{j=1}^{i} A_i$, $n \leq n_i(x)$, then

$$\left| f_i(x, n) - (g(x, n) - \sum_{j=1}^{i-1} f_j(x, n)) \right| \leq \sum_{j=1}^{i} \varepsilon_i.$$

LEMMA 14. *If we can construct the functions $f_i \in \mathscr{A}$ and sets A_i as above, theorem* 12 *follows.*

Proof. Applying lemma 13, let $A=\bigcap_{i=1}^{\infty} A_i$, $\mu(A)\geq \frac{1}{2}$, and let $f=\sum_{j=1}^{\infty} f_j\in\mathscr{A}$, as $\mathscr{A}$ is closed under uniform convergence.

Suppose x, $T^n(x)\in A=\bigcap_{i=1}^{\infty} A_i$. The set of points

$$\bar{B}_i=\bigcup_{x\in X}\left(\bigcup_{j=0}^{n_i(x)-n} T^j(x)\right)$$

has $\mu(\bar{B}_i)\geq 1-n\varepsilon_i$, so for a.e. x, x lies in infinitely many $\bar{B}_i$. Hence for i_0, $n\sum_{j=i_0+1}^{\infty}\sqrt{\varepsilon_i}<\frac{1}{4}$, and $\bar{x}\in b_{i_0}$, $x=T^{n_1}(\bar{x})$ and n_1, $n_1+n<n_{i_0}(\bar{x})$. As both x, $T^n(x)\in A_{i_0}$,

$$\begin{aligned}
&\left|g(x,n)-\sum_{j=1}^{\infty} f_j(x,n)\right| \\
&\quad=\left|g(x,n)-\left(\sum_{j=1}^{i_0} f_j(x,n)+\sum_{j=i_0+1}^{\infty} f_j(x,n)\right)\right| \\
&\quad=\Bigg|g(\bar{x},n_1+n)-\sum_{j=1}^{i_0} f_j(\bar{x},n_1+n)-\left(g(\bar{x},n_1)-\sum_{j=1}^{i_0} f_j(\bar{x},n_1)\right) \\
&\qquad-\sum_{j=i_0+1}^{\infty}\left(\sum_{k=0}^{n-1} f_j(T^k(x))\right)\Bigg| \\
&\quad\leq 2\sum_{j=1}^{i_0}\varepsilon_i+n\sum_{j=i_0+1}^{\infty}\sqrt{\varepsilon_i}\leq\tfrac{1}{2}. \qquad\square
\end{aligned}$$

It remains to construct inductively our towers given by (b_i, t_i), functions $f_i\in\mathscr{A}$ and sets A_i.

To get the construction started, we can set $A_0=X$, and now construct a Rochlin tower $\bar{B}_1, T(\bar{B}_1)\ldots T^{N_1-1}(\bar{B}_1)$, where

$$\mu\left(\bigcup_{j=0}^{N_1-2} T^i(\bar{B}_1)\right)>1-\sqrt{\varepsilon_0},$$

$$N_1>1/\varepsilon_2.$$

Select $f_1\in\mathscr{A}$, $\|f_1-g\|_1<\varepsilon_1/N_1$, $|f_1|\leq 1$, uniformly on X. Thus for a subset $b_1\subset\bar{B}_1$, $\mu(b_1)>(1-\varepsilon_0)\mu(B_1)$, if $x\in b_1$, $0\leq n\leq N_1$,

$$|f_1(x,n)-g(x,n)|\leq\varepsilon_1.$$

Let $t_1=T^{N_1-1}(b_1)$ and

$$A_1=\textstyle\bigcup_{i=0}^{N_1-1} T^i(b_1).$$

Thus $(t_1, b_1), f_1$ and A_1 satisfy conditions (ii) and (iii). Now suppose we have constructed $(t_j, b_j), f_j, A_j$ for $j=1,\ldots,(i-1)$ satisfying (i), (ii) and (iii). To construct the next level, select N_i so large:

$$N_i>\frac{N_i-1}{6\varepsilon_i},$$

that we can build a Rochlin tower of height N_i, $\bar{B}_i$, $T(\bar{B}_i),\ldots, T^{N_i-1}(\bar{B}_i)$, with

(a) $\bigcup_{j=0}^{N_i-1} T^j(\bar{B}_i))>1-\varepsilon_i/3$;
(b) for all $x\in\bar{B}_i$, $\sum_{j=0}^{N_i-1}\chi_{B_{i-1}}(T^j(x))>(1-2\varepsilon_{i-1})N_i$; and
(c) for each $x\in\bar{B}_i$ there is $0\leq j(x)\leq\varepsilon_i N_i/6$, $N_i(1-(\varepsilon_i/6))\leq k(x)\leq N_i$, so that $T^{j(x)}(x)\in b_{i-1}$ and $T^{k(x)}(x)\in t_{i-1}$.

Conditions (b) and (c) are fairly simple applications of the pointwise ergodic theorem. Let

$$\bar{b}_i = \bigcup_{x \in \bar{B}_i} T^{j(x)}(x),$$

and

$$\bar{t}_i = \bigcup_{x \in \bar{B}_i} T^{k(x)}(x),$$

and if $x = T^{j(\bar{x})}(\bar{x}) \in \bar{b}_i$ then

$$n_i(x) = k(x) - j(x).$$

Letting

$$\tilde{B}_i = \left(\bigcup_{x \in \bar{b}_i} \bigcup_{j=0}^{n_i(x)-1} T^j(x) \right),$$

$$\mu(\tilde{B}_i) \geq 1 - 2\varepsilon_i/3.$$

For each $x \in \bar{b}_i$ we still have

(b′) $\sum_{j=0}^{n_i(x)-1} \chi_{B_{i-1}}(T^j(x)) > (1 - 2(\varepsilon_{i-1} + \varepsilon_i/3))n_i(x)$.

For each $x \in b_i$, the integers $0, 1, \ldots, n_i(x) - 1$ can be broken into consecutive intervals $I_0, J_1, I_1, J_2, I_2, \ldots, J_s, I_s$ where $I_0, I_1, \ldots, I_s$ correspond to $(i-1)$-blocks in the orbit of x, and $J_1, \ldots J_s$ are the intervals in between, hence each begins in t_{i-1} and ends just before the next b_{i-1}. Expand each J_i to a minimal $\bar{J}_i$ with

$$\sum_{j \in \bar{J}_i} \chi_{B_{i-1}}(T^j(x)) > (1 - 3\sqrt{\varepsilon_{i-1}}) \operatorname{card}(\bar{J}_i)$$

The $\bar{J}_i \subset \{0, \ldots, n_i(x) - 1\}$ certainly exist but need not be disjoint. Write

$$\bigcup_{i=1}^{s} \bar{J}_i = \bigcup_{j=1}^{t} \tilde{J}_j,$$

a disjoint union where each $\bar{J}_i$ is contained in some $\tilde{J}_j$, and the $\tilde{J}_j$ are a maximal such collection of disjoint intervals. Each $\tilde{J}_j$ can be written as an ordered union of $\bar{J}_i$'s where any point in $\tilde{J}_j$ is in at most two consecutive elements of the union. Consecutive $\bar{J}_i$ must overlap in at least one point by maximality. Thus

$$\sum_{j \in \tilde{J}_i} \chi_{B_{i-1}}(T^j(x)) > (1 - 6\sqrt{\varepsilon_{i-1}}) \operatorname{card}(\tilde{J}_i).$$

But as decreasing $\bar{J}_i$ by one term reverses the inequality,

$$\sum_{j \in \bar{J}_i} \chi_{B_{i-1}}(T^j(x)) \leq (1 - 3\sqrt{\varepsilon_{i-1}}) \operatorname{card}(\bar{J}_i) + 3\sqrt{\varepsilon_{i-1}})$$

and of course as card $J_i \geq 1$, card $(\bar{J}_i) \geq 1/3\sqrt{\varepsilon_{i-1}}$ so

$$\begin{aligned}\sum_{j \in \bar{J}_i} \chi_{B_{i-1}}(T^j(x)) &\leq (1 - 3\sqrt{\varepsilon_{i-1}} + 3\varepsilon_{i-1}) \operatorname{card}(\bar{J}_i) \\ &\leq (1 - 3\sqrt{\varepsilon_{i-1}}(1 - \sqrt{\varepsilon_{i-1}})) \operatorname{card}(\bar{J}_i) \\ &\leq (1 - \tfrac{3}{2}\sqrt{\varepsilon_{i-1}}) \operatorname{card}(\bar{J}_i),\end{aligned}$$

and again as $\tilde{J}_i$ can be written as an ordered union of $\bar{J}_i$ with each point in at most two $\bar{J}_i$'s,

$$\sum_{j \in \tilde{J}_i} \chi_{B_{i-1}}(T^j(x)) \leq (1 - \tfrac{3}{4}\sqrt{\varepsilon_{i-1}}) \operatorname{card}(\tilde{J}_i)$$

so as the $\tilde{J}_i$ are disjoint and cover the J_i,

$$3\varepsilon_{i-1}n_i(x) > \sum_{j=0}^{n_i(x)-1} \chi_{B_{i-1}^c}(T^j(x))$$
$$= \sum_i (\text{card } J_i) = \sum_i \sum_{j\in\tilde{J}_i} \chi_{B_{i-1}^c}(T^j(x))$$
$$> \tfrac{3}{4}\sqrt{\varepsilon_{i-1}} \sum_i \text{card } (\tilde{J}_i),$$

and so

$$\text{card } (\bigcup_i \tilde{J}_i) < 4\sqrt{\varepsilon_{i-1}}\, n_i(x),$$

i.e. all the $\tilde{J}_i$ occupy a small fraction of $0,\ldots,n_i(x)-1$. Certainly the $\tilde{J}_i$ can be selected measurably in $x\in\bar{B}_i$. Write $\bigcup_i \tilde{J}_i = \tilde{J}(x)$ and define

$$\bar{A}_i = \bigcup_{x\in\bar{B}_i} \bigcup_{j\in\tilde{J}(x)} T^j(x).$$

This is our first approximation to A_i. Certainly

$$\mu(\bar{A}_i) \geq (1-4\sqrt{\varepsilon_{i-1}}) - (2\varepsilon_{i-1}/3) \geq (1-5\sqrt{\varepsilon_{i-1}}).$$

We now define a first approximation to f_i. Again fix an x. We think of the integers $0, 1, \ldots, n_i(x)-1$ as also representing the orbit points $x, T(x), \ldots, T^{n_i(x)-1}$. We break this piece of orbit into disjoint blocks $I_1, I_2, \ldots, I_s$, where each I_k is constructed by taking a $\tilde{J}_k$, deleting the partial $(i-1)$-block at its right end and completing the $(i-1)$-block at its left end. Thus I_k can be written as

$$I_{1,k}, J_{1,k}, I_{2,k}, J_{2,k}, \ldots, I_{t(k),k}, J_{t(k),k},$$

where each $I_{l,k}$ is an $(i-1)$-block and $J_{l,k}$ are the intervening integers. The corresponding $\tilde{J}_k$ can be written

$$B_k, J_{1,k}, I_{2,k}, \ldots, I_{t(k),k}, J_{t(k),k}, \bar{B}_k$$

where B_k and $\bar{B}_k$ are partial $(i-1)$-blocks. Let

$$\bar{g}(x,n) = g(x,n) - \sum_{j=1}^{i-1} f_j(x,n),$$

and we know

$$\bar{g}(x) \leq 1 - \sum_{j=1}^{i-1} 10\sqrt{\varepsilon_{j-1}} \leq 2$$

so as

$$\sum_{j=1}^{t(k)} \text{card } (J_{j,k}) \leq 6\sqrt{\varepsilon_{i-1}}\, \text{card } (\tilde{J}_i),$$
$$\sum_{j=1}^{t(k)} \sum_{x\in J_{j,k}} \bar{g}(x) \leq 12\sqrt{\varepsilon_{i-1}}\, \text{card } (\tilde{J}_i).$$

For each j,

$$\sum_{x\in I_{j,k}} \bar{g}(x) \leq \sum_{j=1}^{i-1} \varepsilon_j \leq 1,$$

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as both b_{i-1} and $T^{-1}(t_{i-1})$ are in $\bigcap_{j=1}^{i-1} A_{i-1}$. Thus

$$\sum_{x\in I_k} \bar{g}(x) \le 6\sqrt{\varepsilon_{i-1}}\,\mathrm{card}\,(\tilde{J}_k) + t(k)$$
$$\le 6\sqrt{\varepsilon_{i-1}}\,\mathrm{card}\,(\tilde{J}_k) + \sum_{j=1}^{t(k)} \mathrm{card}\,(J_{j,k})$$
$$\le 12\sqrt{\varepsilon_{i-1}}\,\mathrm{card}\,(\tilde{J}_k).$$

Define $\bar{f}_i(\bar{x}) = 0$ if $\bar{x} \notin B_i$ and if $x \in A_i$. Otherwise for some $x \in \bar{b}_i$, $\bar{x} \in \tilde{J}_k$ for some k. In this case set

$$\bar{f}_i(\bar{x}) = \sum_{x\in I_k} \bar{g}(x)/\mathrm{card}\,\tilde{J}_k.$$

Thus

$$\bar{f}_i(\bar{x}) \le 12\sqrt{\varepsilon_{i-1}},$$

uniformly on X. Furthermore

$$\sum_{x\in I_k} \bar{g}(x) = \sum_{x\in J_k} \bar{f}_i(x),$$

so if $x \in b_i$,

$$T^n(x) \in \bigcap_{j=1}^{i=1} A_j \cap \bar{A}_i, \qquad n \le n_i(x),$$

then if $x_0 = T^{\bar{n}}(x)$ is the initial point of the $(i-1)$-block containing $\bar{x}$,

$$|\bar{f}_i(x, n) - g(x, n)| = |g(x_0, n - \bar{n})| \le \sum_{j=1}^{i-1} \varepsilon_j.$$

To finish the argument, select $f_i \in \mathscr{A}$ approximating $\bar{f}_i$ so well in $L^1(\mu)$, bounded uniformly by $12\sqrt{\varepsilon_{i-1}}$ that for a subset $b_i \subset \bar{b}_i$, $\mu(b_i)/\mu(\bar{b}_i) > 1 - (\varepsilon_i/2N_i)$, if $x \in b_i$, $n < N_i$,

$$|f_i(x, n) - \bar{f}_i(x, n)| \le \varepsilon_i.$$

We thus have

(i) $|f_i(x)| \le 12\sqrt{\varepsilon_{i-1}}$ uniformly on X.

Now

$$A_i = \bar{A}_i \cap \left(\bigcup_{\substack{x\in \bar{b}_i \\ j<n_i(x)}} T^j(x) \right)$$

so

(ii) $\mu(A_i) \ge 1 - 5\sqrt{\varepsilon_{i-1}} - (\varepsilon_i/2) > 1 - 10\sqrt{\varepsilon_{i-1}}$.

If $x \in b_i$ and $T^n(x) \in \bigcap_{j=1}^{i} A_i$, $n \le n_i(x)$ we conclude

(iii) $|f_i(x, n) - \bar{g}(x, n)| \le |f_i(x, n) - \bar{f}_i(x, n)| + |\bar{f}_i(x, n) - \bar{g}(x, n)| \le \sum_{j=1}^{i} \varepsilon_i$. □

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