5

LIMITS IN CATEGORIES

Although it is natural to commence the study of modules with finitely generated modules, there are many occasions when it is necessary to consider modules that are not be finitely generated. For example, we shall see that many common and useful examples of flat modules are not finitely generated. The main topic of this chapter, the formation of the direct limit of a set of modules, provides a useful tool for the construction and analysis of non-finitely generated modules in terms of finitely generated modules. In the next chapter, we see how localizations of rings and modules are obtained as direct limits.

The general properties of direct limits are discussed in the first section of this chapter, with an emphasis on limits of modules. We then give two applications of these results. The first is to the theory of flat modules, where we obtain several characterizations of such modules, including Lazard's Theorem. The second is to direct limits of rings, where we give examples of von Neumann regular rings, which can be regarded as 'infinite-dimensional' analogues of Artinian semisimple rings.

In the brief final section, we consider the dual construction, that of an inverse limit. This is used to construct the completions of rings and modules in Chapter 7.

The immediate precursors of the direct and inverse limit constructions are to be found in the attempt to compute the homology theory of topological spaces by an approximation procedure ([Alexandroff 1926], [Alexandroff 1929]). The formal definition of a direct limit was given in [Pontrjagin 1931] for groups. In this paper, Pontrjagin also discusses inverse systems but overlooks the possibility of constructing the inverse limit (see [Dieudonné 1989] p. 73), which was accomplished, in a special case, independently by [Čech 1932]. These strands were tied together in [Steenrod 1936], who considers topological spaces as well as groups. Of course, in that far-off time before the birth of category

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theory, the topological and group-theoretic expositions had to be presented separately.

5.1 DIRECT LIMITS

In this section we give the construction of direct limits, together with their basic functorial properties. For our applications, we need consider only direct limits which are indexed by directed sets and which are formed in one of the categories $\mathcal{S}\varepsilon\tau$, $\mathcal{G}\mathcal{P}$, $\mathcal{R}\mathcal{I}\mathcal{N}\mathcal{G}$, $\mathcal{R}\mathcal{N}\mathcal{G}$ or $\mathcal{M}\mathcal{O}\mathcal{D}_R$, where R is an arbitrary ring. Thus our approach can be fairly explicit, and we usually supply details of arguments only for the category $\mathcal{M}\mathcal{O}\mathcal{D}_R$.

However, as we also need to work with left modules, we devote some attention to setting up the appropriate notation.

We commence with a review of the properties of the sets that we use to index our direct limits.

5.1.1 Directed sets

A *directed set* is a partially ordered set (1.1.2) Λ in which every pair of elements has an upper bound. Thus, there is a relation $\leq$ between the elements of Λ which satisfies the conditions for a partial order, namely

PO1 *Reflexivity.*
$\lambda \leq \lambda$ always;

and

PO2 *Transitivity.*
If $\lambda \leq \mu$ and $\mu \leq \nu$, then $\lambda \leq \nu$;

and which also obeys the following axiom.

DS *Existence of upper bounds.*
If $\lambda \in \Lambda$ and $\mu \in \Lambda$, then there is some $\nu \in \Lambda$ with $\lambda \leq \nu$ and $\mu \leq \nu$.

(Sometimes for variety, we write $\nu \geq \lambda$ to mean that $\lambda \leq \nu$.)

Thus, schematically, in a directed set one has pictures like that in Fig. 5.1. We allow the possibility that a directed set is not proper, that is, it may happen that $\lambda \leq \mu$ and $\mu \leq \lambda$ but $\lambda \neq \mu$.

Notice that an ordered set Ω can be regarded as a proper directed set such that, for any two members ω and θ, either $\omega \leq \theta$ or $\theta \leq \omega$.

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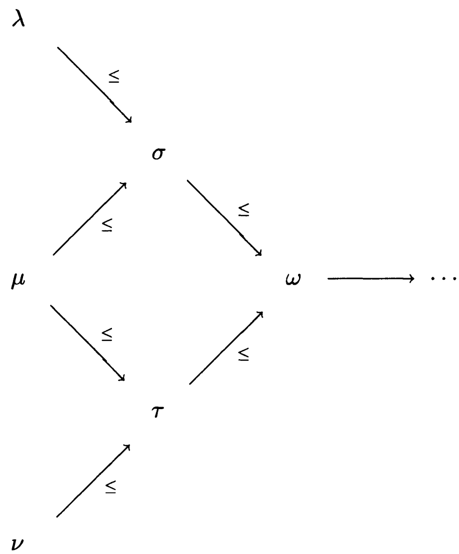

Fig. 5.1.

We view a directed set Λ as a left category as in (1.1.2). Thus the elements $\lambda, \mu, \ldots$ of Λ are its objects, there is a unique morphism $\iota^{\lambda\mu} : \lambda \to \mu$ whenever $\lambda \le \mu$, and transitivity becomes the law of composition

$$\iota^{\lambda\mu}\iota^{\mu\nu} = \iota^{\lambda\nu}.$$

5.1.2 Examples

1. The usual ordering on the set $\mathbb{N}$ of natural numbers gives the directed set $\omega : 1 \le 2 \le 3 \le \cdots$.
2. For any ring R, introduce a partial ordering on the set $R \setminus \{0\}$ of nonzero elements of R by the rule that $r \le s$ if and only if $rx = s$ for some x in R, that is, r is a left divisor of s. We write this ordered set as $\mathrm{Lat}(R)$, the *division lattice* on R. This set is not always a directed set. For example, if $R = Re \times Rf$, an internal direct product of rings, with orthogonal central idempotents e, f, then e and f have no common right multiple in $\mathrm{Lat}(R)$. If R is a commutative domain, then $\mathrm{Lat}(R)$ is a directed set, and we later (6.2.16) meet some noncommutative domains for which $\mathrm{Lat}(R)$ is also directed. However, even when $\mathrm{Lat}(R)$ is a directed set, it is rarely proper.

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Since only multiplication is used in the definition, this ordering can be defined for any monoid. Thus it makes $\mathbb{N}$ into a directed set in a different way from the first example.

3. Next, let X be any set and let $\mathbf{P}(X)$ be the power set on X, that is, the set of all subsets of X. We regard $\mathbf{P}(X)$ as a proper directed set by inclusion: $S \leq T$ if and only if $S \subseteq T$.
4. The subset $\mathbf{P}_f(X)$ of $\mathbf{P}(X)$ comprising the finite subsets of X is also a directed set, under the same ordering.

5.1.3 Direct systems

Let $\mathcal{C}$ be a category selected from among $\mathcal{S}\mathcal{ET}$, $\mathcal{G}\mathcal{P}$, $\mathcal{R}\mathcal{ING}$, $\mathcal{R}\mathcal{NG}$ or $\mathcal{M}\mathcal{OD}_R$, where R is an arbitrary ring. Given a directed set Λ, a Λ-*direct system* (or simply direct system) C_Λ of objects in $\mathcal{C}$ is a set

$$\{C_\lambda, \phi_{\mu\lambda} \mid \lambda, \mu \in \Lambda\}$$

consisting of objects C_λ in $\mathcal{C}$, together with a family of morphisms $\phi_{\mu\lambda} : C_\lambda \to C_\mu$ with the following properties.

DSys 1. $\phi_{\mu\lambda}$ is defined only when $\lambda \leq \mu$.
DSys 2. $\phi_{\lambda\lambda} = id$, the identity morphism on C_λ.
DSys 3. If $\lambda \leq \mu$ and $\mu \leq \nu$, then $\phi_{\nu\lambda} = \phi_{\nu\mu}\phi_{\mu\lambda}$.

Condition DSys 3 is sometimes called the *coherence condition*, which can be illustrated by the commutative diagram

$$C_\lambda \xrightarrow{\phi_{\mu\lambda}} C_\mu \xrightarrow{\phi_{\nu\mu}} C_\nu, \qquad C_\lambda \xrightarrow{\phi_{\nu\lambda}} C_\nu$$

while a typical pattern of objects and morphisms is illustrated in Fig. 5.2.

In category-theoretic terms, these axioms state that a Λ-direct system C_Λ in $\mathcal{C}$ is a covariant functor from Λ to $\mathcal{C}$. (The choice of notation $\phi_{\mu\lambda}$ when $\lambda \leq \mu$ is largely forced on us by the contrachirality of this functor – see (1.1.4).)

Given two direct systems C_Λ and $D_\Lambda = \{D_\lambda, \psi_{\mu\lambda}\}$ (over the same directed set Λ), we define a *morphism* γ_Λ from C_Λ to D_Λ to be a collection of morphisms $\gamma_\lambda : C_\lambda \to D_\lambda$ in $\mathcal{C}$ such that

$$\psi_{\mu\lambda}\gamma_\lambda = \gamma_\mu\phi_{\mu\lambda} : C_\lambda \longrightarrow D_\mu \text{ when } \lambda \leq \mu.$$

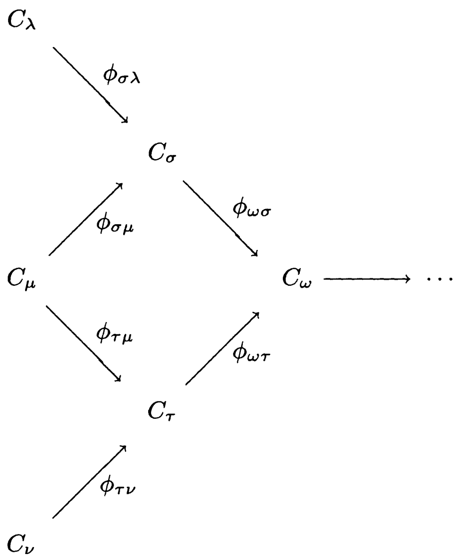

Fig. 5.2.

In other words, γ_Λ is just a natural transformation from C_Λ to D_Λ.

Thus the collection of direct systems over Λ forms the functor category $[\Lambda, \mathcal{C}]$.

5.1.4 Construction of the direct limit

Let $C_\Lambda = \{C_\lambda, \phi_{\mu\lambda} \mid \lambda, \mu \in \Lambda\}$ be a direct system. We define its *direct limit* as follows.

First, we form the disjoint union $\bigsqcup_\Lambda (C_\lambda, \lambda)$ as in Exercise 1.4.9. Thus (C_λ, λ) is the set of pairs (c_λ, λ) where $c_\lambda \in C_\lambda$.

We next introduce a relation $\sim$ on the disjoint union by the rule that, for $c_\lambda \in C_\lambda$ and $c_\mu \in C_\mu$, we have $(c_\lambda, \lambda) \sim (c_\mu, \mu)$ if $\phi_{\nu\lambda} c_\lambda = \phi_{\nu\mu} c_\mu$ for some $\nu \geq \lambda, \mu$. It is easy to verify that $\sim$ is an equivalence relation.

The direct limit is defined as

$$\operatorname{dir\,lim}_\Lambda C_\lambda = \left(\bigsqcup\nolimits_\Lambda (C_\lambda, \lambda)\right) / \sim,$$

the set of equivalence classes under this equivalence relation. The class of (c_λ, λ) is denoted $[c_\lambda]$, which we informally call the class of c_λ. Alternative

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terms are *inductive limit* and, in the context of category theory, *colimit*; notation can also be ind $\lim_\Lambda$, $\operatorname{colim}_\Lambda$ or $\varinjlim$.

A different method for constructing the direct limit is indicated in Exercise 5.1.1 below.

We now show that $\operatorname{dir}\lim_\Lambda C_\lambda$ is also an object of the category $\mathcal{C}$ in the cases that $\mathcal{C}$ is one of $\mathcal{S}\mathit{et}$, $\mathcal{G}\mathit{p}$, $\mathcal{R}\mathit{ing}$, $\mathcal{R}\mathit{ng}$ or $\mathcal{M}\mathit{od}_R$. For $\mathcal{S}\mathit{et}$, there is nothing to do. The details for the remaining categories are similar, so we outline the argument for $\mathcal{M}\mathit{od}_R$ only.

Let C_Λ be a direct system of right R-modules. To define addition, take c_λ in C_λ and c_μ in C_μ, choose some ν with $\nu \geq \lambda, \mu$, and then put

$$[c_\lambda] + [c_\mu] = [\phi_{\nu\lambda}c_\lambda + \phi_{\nu\mu}c_\mu].$$

Scalar multiplication is given by

$$[c_\lambda] \cdot r = [(c_\lambda \cdot r)_\lambda] \text{ for } r \in R.$$

Some routine checking verifies that we have a well-defined right R-module structure on $\operatorname{dir}\lim_\Lambda C_\lambda$.

It is easy to verify that a morphism γ_Λ from C_Λ to D_Λ in $[\Lambda, \mathcal{C}]$ induces a morphism $\operatorname{dir}\lim_\Lambda \gamma_\lambda$ from $\operatorname{dir}\lim_\Lambda C_\lambda$ to $\operatorname{dir}\lim_\Lambda D_\lambda$, given by the rule

$$(\operatorname{dir}\lim_\Lambda \gamma_\lambda)[c_\lambda] = [\gamma_\lambda c_\lambda].$$

We summarize the discussion as follows.

5.1.5 Theorem

Let Λ be a directed set and let $\mathcal{C}$ be one of the categories $\mathcal{S}\mathit{et}$, $\mathcal{G}\mathit{p}$, $\mathcal{R}\mathit{ing}$, $\mathcal{R}\mathit{ng}$ or $\mathcal{M}\mathit{od}_R$, where R is a ring. Then the direct limit defines a functor

$$\operatorname{dir}\lim_\Lambda : [\Lambda, \mathcal{C}] \longrightarrow \mathcal{C}. \qquad \square$$

5.1.6 Some examples

1. In general, we cannot expect a module category to contain the direct limit of all the direct systems of modules in the given category. This is typified by (5.1.12), which tells us that any R-module is a direct limit of members of the category $\mathcal{M}_R$ of all finitely generated R-modules. In particular, let $\mathcal{K}$ be a field, and let

$$\mathcal{K} \longrightarrow \mathcal{K}^2 \longrightarrow \mathcal{K}^3 \longrightarrow \cdots$$

   be the direct system of standard inclusions of each finite-dimensional vector space in its successor. This is clearly a direct system in $\mathcal{M}_\mathcal{K}$ which fails to have a direct limit in $\mathcal{M}_\mathcal{K}$.

2. An arbitrary direct sum may be regarded as a direct limit of finite direct sums. For example, let C_λ be a collection of right R-modules, indexed by an ordered set Λ. Then the finite direct sums of modules C_λ form a direct system over the directed set $\mathbf{P}_f(\Lambda)$ as in (5.1.2). The embedding of each finite direct sum in a larger one is achieved by setting the extra components to zero. We define a map

$$\Phi : \operatorname{dir\,lim}_{\mathbf{P}_f(\Lambda)} \left(\bigoplus\nolimits_{\Lambda'} C_\lambda\right) \longrightarrow \bigoplus\nolimits_{\Lambda} C_\lambda$$

where in the above expression Λ' ranges over the finite subsets of Λ (with ordering inherited from Λ). The map Φ serves to locate each finite subset of Λ in the infinite direct sum; it is given by

$$[(c_{\lambda_1}, \dots, c_{\lambda_n})] \longmapsto (x_\lambda)$$

where x_λ takes the value c_{λ_i} when $\lambda = \lambda_i$ lies in the finite subset $(\lambda_1, \dots, \lambda_n)$ of Λ, and 0 otherwise. It is easy to see that Φ is an isomorphism. The same argument works for categories other than $\mathcal{M}od_R$.

3. For a given prime number p, let $\mathbb{F}_{p^m}$ denote the finite field with p^m elements. Standard facts of Galois theory tell us that $\mathbb{F}_{p^m}$ is uniquely determined to within isomorphism of fields by its order p^m, and that such a field exists for each p^m. Further, $\mathbb{F}_{p^m}$ is a subfield of $\mathbb{F}_{p^n}$ precisely when m divides n ([Cohn 1979] §5.7). There is consequently a direct system of field inclusions

$$\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2} \hookrightarrow \mathbb{F}_{p^6} \hookrightarrow \cdots \hookrightarrow \mathbb{F}_{p^{m!}} \hookrightarrow \mathbb{F}_{p^{(m+1)!}} \hookrightarrow \cdots$$

whose direct limit is again a field, denoted $\overline{\mathbb{F}}_p$.

We note that $\overline{\mathbb{F}}_p$ is an *algebraically closed field*, that is, any nonconstant polynomial $f(X)$ with coefficients in $\overline{\mathbb{F}}_p$ splits into linear factors over $\overline{\mathbb{F}}_p$. To see this, first observe that the coefficients of $f(X)$ must already be contained in some $\mathbb{F}_{p^k}$. A splitting field of $f(X)$ (see [BK: IRM] Exercise 5.2.3) must be a finite extension of $\mathbb{F}_{p^k}$, and therefore isomorphic to some $\mathbb{F}_{p^h}$, which is in turn contained in some $\mathbb{F}_{p^{h!}}$ and hence in $\overline{\mathbb{F}}_p$.

The field $\overline{\mathbb{F}}_p$ is also known as the *algebraic closure* of $\mathbb{F}_p$, since it is (evidently) the smallest algebraically closed field containing $\mathbb{F}_p$.

4. Let R be a ring and let $M_n(R)$ be the $n \times n$ matrix ring over R, as usual. We use the directed set ω and define, for each $n \geq 1$ and $k \geq 0$, a nonunital ring homomorphism $\rho_{n+k,n} : M_n(R) \to M_{n+k}(R)$ by sending an $n \times n$ matrix A to the $(n+k) \times (n+k)$ matrix $A \oplus O_k$ where O_k is the $k \times k$ zero matrix. Thus, although each object is a ring, we obtain a direct system only in $\mathcal{R}_{NG}$. (Evidently, when $k \geq 1$ this map fails to send the identity

matrix I_n to I_{n+k}.) Then $\operatorname{dir\,lim}_\omega M_n(R)$ is the nonunital ring mR of all 'finite' matrices.

5. Modifying the previous example, we may pass from an $n \times n$ matrix A to the $(n+k)\times(n+k)$ matrix $A \oplus I_k$. This procedure preserves multiplication and does send I_n to I_{n+k}, but behaves poorly with respect to addition. When we restrict attention to (twosided) invertible matrices A, we obtain a homomorphism $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+k}(R)$ of the general linear groups (1.3.2)(vii). We obtain a direct system in $\mathcal{G}_\mathcal{P}$, with $\operatorname{dir\,lim}_\omega \mathrm{GL}_n(R) = \mathrm{GL}(R)$. The group $\mathrm{GL}(R)$ consists of all twosided invertible matrices in CR that differ from the infinite identity matrix in only a finite number of entries. This example is of central importance in algebraic K-theory.

5.1.7 Basic properties of direct limits

We now establish some basic properties of the functor $\operatorname{dir\,lim}_\Lambda$. We confine the discussion to right modules; the reader should have little difficulty in supplying the analogous results for groups and rings.

5.1.8 Lemma

Let $L_\Lambda = \{L_\lambda, \phi_{\mu\lambda}\}$ and $M_\Lambda = \{M_\lambda, \psi_{\mu\lambda}\}$ be direct systems of right R-modules and let γ_Λ be a morphism from L_Λ to M_Λ.

Then the following assertions hold.

(i) $[\ell_\lambda] = 0$ *in* $\operatorname{dir\,lim}_\Lambda L_\lambda$ *if and only if* $\phi_{\mu\lambda}\ell_\lambda = 0$ *for some* μ *with* $\lambda \le \mu$.

(ii) *If each* γ_λ *is injective, so is* $\operatorname{dir\,lim}_\Lambda \gamma_\lambda$.

(iii) *If each* γ_λ *is surjective, so is* $\operatorname{dir\,lim}_\Lambda \gamma_\lambda$.

(iv) *If* δ_Λ *is a morphism from* M_Λ *to* $N_\Lambda = \{N_\lambda, \tau_{\mu\lambda}\}$ *such that each*

$$L_\lambda \xrightarrow{\gamma_\lambda} M_\lambda \xrightarrow{\delta_\lambda} N_\lambda$$

is exact, then

$$\operatorname{dir\,lim}_\Lambda L_\lambda \xrightarrow{\operatorname{dir\,lim}_\Lambda \gamma_\lambda} \operatorname{dir\,lim}_\Lambda M_\lambda \xrightarrow{\operatorname{dir\,lim}_\Lambda \delta_\lambda} \operatorname{dir\,lim}_\Lambda N_\lambda$$

is also exact.

Proof

For (i), note that the zero in $\operatorname{dir\,lim}_\Lambda L_\lambda$ is $[0]$, where 0 can be taken to be the zero of any of the modules L_ν, $\nu \in \Lambda$. For (ii), if $(\operatorname{dir\,lim}_\Lambda \gamma_\lambda)[\ell_\lambda] = 0$, then $0 = \psi_{\mu\lambda}\gamma_\lambda\ell_\lambda = \gamma_\mu\phi_{\mu\lambda}\ell_\lambda$, whence $[\ell_\lambda] = 0$. Part (iii) is obvious.

Finally, if $[m_\lambda]$ lies in the kernel of dir $\lim_\Lambda \delta_\lambda$, then, for some μ with $\lambda \leq \mu$, we have $\delta_\mu\psi_{\mu\lambda}m_\lambda = 0$. Therefore there exists ℓ_μ in L_μ with $\psi_{\mu\lambda}m_\lambda = \gamma_\mu\ell_\mu$, so that $[m_\lambda] = (\text{dir}\lim_\Lambda \gamma_\lambda)[\ell_\mu]$. □

Next, we note a universal property of the direct limit. Given a direct system $C_\Lambda = \{C_\lambda, \phi_{\mu\lambda}\}$, we define a collection of morphisms $\phi_{\bullet\lambda} : C_\lambda \to \text{dir}\lim_\Lambda$ by $\phi_{\bullet\lambda}(c_\lambda) = [c_\lambda]$. A straightforward verification gives the following result.

5.1.9 Theorem

(i) *If $\lambda \leq \mu$, then $\phi_{\bullet\mu}\phi_{\mu\lambda} = \phi_{\bullet\lambda}$.*

(ii) *If D is a fixed object of $\mathcal{C}$ and $\{\chi_\lambda \mid \lambda \in \Lambda\}$ is any collection of morphisms $\chi_\lambda : C_\lambda \to D$ such that $\chi_\mu\phi_{\mu\lambda} = \chi_\lambda$ whenever $\lambda \leq \mu$, then there is a unique morphism $\chi : \text{dir}\lim_\Lambda C_\lambda \to D$ with $\chi\phi_{\bullet\lambda} = \chi_\lambda$ for all λ.* □

In many applications, as in the examples in (5.1.6), the maps $\phi_{\mu\lambda}$ are all injective. We then have

5.1.10 Proposition

Suppose that each map $\phi_{\mu\lambda}, \lambda \leq \mu$, is injective. Then

$$\text{dir}\lim_\Lambda C_\lambda = \bigcup\nolimits_\Lambda \phi_{\bullet\lambda}C_\lambda,$$

the ordinary union. □

An easy consequence of the preceding proposition is the following partial converse to (5.1.8)(ii) above which is immediate from the commutativity relation $\psi_{\bullet\lambda}\gamma_\lambda = (\text{dir}\lim_\Lambda \gamma_\lambda)\phi_{\bullet\lambda}$.

5.1.11 Corollary

Let $L_\Lambda = \{L_\lambda, \phi_{\mu\lambda}\}$ and $M_\Lambda = \{M_\lambda, \psi_{\mu\lambda}\}$ be direct systems of right R-modules and let γ_Λ be a morphism from L_Λ to M_Λ. Suppose that each $\phi_{\mu\lambda}$ is injective.

Then, if dir $\lim_\Lambda \gamma_\lambda$ *is injective, so also is each γ_λ.* □

Here is an important application of the preceding proposition.

5.1.12 Proposition

Let M be a right R-module. Then M is the direct limit of its finitely generated R-submodules.

Proof

Take Λ to be the set of finite subsets of M, ordered by inclusion. Clearly, Λ is a directed set. For $\lambda \in \Lambda$, let M_λ be the submodule generated by that

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subset. The inclusion maps make $\{M_\lambda\}$ into a direct system, with direct limit $\bigcup_\Lambda M_\lambda = M$. □

It is worth recording that a similar argument works for groups.

5.1.13 Proposition

Every group is the direct limit of its finitely generated subgroups. □

5.1.14 Quasicyclic groups

As an illustration, we give two constructions for the quasicylic p-group, where p is a fixed prime number.

The elementary construction is as follows. For each integer $i > 0$, let C_{p^i} be the (additive) subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of all those elements which have order dividing p^i. For each i, C_{p^i} is cyclic of order p^i, and $C_{p^i} \subset C_{p^{i+1}}$. The *quasicyclic p-group* (or *Prüfer group of type* p^∞) is then $G = \bigcup C_{p^i}$.

Alternatively, G can be represented as the direct limit of the direct system of $\mathbb{Z}$-modules $\{\mathbb{Z}/p^i\mathbb{Z}, \phi_{ji}\}$ where, for $i \leq j$,

$$\phi_{ji} : \mathbb{Z}/p^i\mathbb{Z} \longrightarrow \mathbb{Z}/p^j\mathbb{Z}$$

is the injective $\mathbb{Z}$-homomorphism sending the generator of $\mathbb{Z}/p^i\mathbb{Z}$ to p^{j-i} times the generator of $\mathbb{Z}/p^j\mathbb{Z}$. Diagrammatically, the system is

$$\mathbb{Z}/p\mathbb{Z} \xrightarrow{\phi_{21}} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\phi_{32}} \mathbb{Z}/p^3\mathbb{Z} \longrightarrow \cdots$$

We obtain

$$\text{dir lim}_{\mathbb{N}} (\mathbb{Z}/p^i\mathbb{Z}) \cong \text{dir lim}_{\mathbb{N}} C_{p^i} = \bigcup_i C_{p^i}.$$

5.1.15 Cofinality

In applications of the theory such as (5.1.12) above, it is often convenient to replace the given direct system (for example, all the finitely generated submodules of a given module) by a more sparse system (such as a certain selection of the finitely generated submodules) in such a way that the same limit is obtained. This is achieved as follows.

We say that a subset Ω of a directed set Λ is *cofinal* in Λ if for any λ in Λ there exists some ω in Ω with $\lambda \leq \omega$. This terminology is extended in the obvious way to a direct system indexed by Λ and its subsystem indexed by Ω. It is easily seen that in this event dir $\lim_\Omega$ and dir $\lim_\Lambda$ coincide.

As an example, consider a free module $M = \text{Fr}_R(X)$. Let N be any finitely generated submodule of M and choose some finite generating set of N. Then

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each generator of N can be expressed as a linear combination of a finite set of elements of X, and so all the given generators of N can be expressed in terms of a finite subset Y of X. Thus N is a submodule of $\mathrm{Fr}_R(Y)$. Since N is arbitrary, we see that the set of all finitely generated free submodules of M is cofinal in the set of all finitely generated submodules, and that M is also the direct limit of its finitely generated free submodules.

5.1.16 Generalizations

1. There are many cases when a direct system has objects in a given category, but the colimit of the system lies outside the category. As we have noted already, by (5.1.12) the category $\mathcal{M}_R$ of finitely generated right R-modules provides such examples.
2. Our construction of the direct limit relies on the fact that in any of the categories under consideration, an 'object' is a set with some extra structure. It is also possible to define direct limits for an abstract category, where the term 'colimit' becomes more conventional: a *colimit* is an object which satisfies the universal property given in (5.1.9), if such an object exists.
3. We can also generalize from the standpoint that a Λ-direct system is a covariant functor from Λ. It thereby becomes possible to form direct limits over an arbitrary category $\mathcal{D}$ rather than a directed set. Again, the term 'colimit' is now preferred for the universal object defined as in (5.1.9). (A formal statement is provided in Exercise 5.1.7.)

   For example, take $\mathcal{D}$ to be a set Λ regarded as a discrete category (Exercise 1.1.1). Then a colimit over $\mathcal{D}$ is just a coproduct.

   A colimit over the category

$$\bullet \rightrightarrows \bullet$$

   that has two objects and two non-identity morphisms is called a *coequalizer*. For instance, the coequalizer of two homomorphisms

$$\alpha, \beta : M \longrightarrow N$$

   in $\mathcal{M}\mathrm{OD}_R$ is the cokernel of the difference $\alpha - \beta$.

   It can be shown that a category $\mathcal{C}$ possesses colimits over arbitrary indexing categories precisely when it posseses all coproducts and coequalizers ([Eisenbud 1995], Theorem A6.1); see also Exercise 5.1.7.
4. In the generality allowed in (3) above, one cannot expect that results such as Lemma 5.1.8 will continue to hold: a direct limit of monomorphisms need no longer be a monomorphism. However, such results can be preserved if the indexing category Λ is a *filtered* category. (The term *cofiltering* is also

met.) Such a (left) category may have several morphisms $\lambda \to \mu$, but the following conditions must hold.

Filt 1. Given $\lambda, \mu \in \Lambda$, there exists an object ω in Λ together with morphisms $\lambda \to \omega$ and $\mu \to \omega$.

Filt 2. If $\alpha, \beta : \lambda \to \mu$ are morphisms in Λ, then there is a morphism $\gamma : \mu \to \nu$ for some $\nu \in \Lambda$ with $\alpha\gamma = \beta\gamma$.

5.1.17 Notation for left modules

So far, we have dealt only with categories whose morphisms appear on the left of the objects on which they operate, and our notation has been designed to read well in this situation. However, there are important applications in which we need to discuss direct limits of systems of left modules, and it is useful to have an alternative notation which is adapted to left modules and which also enables us to recognise when we are working with them. We do this by 'raising the indices'. Recall that homomorphisms of left modules are written on the right and compose accordingly.

Given a directed set Λ, a *direct system* M_Λ of left R-modules is a set $\{M_\lambda, \phi^{\lambda\mu} \mid \lambda, \mu \in \Lambda\}$, consisting of left R-modules M_λ and homomorphisms $\phi^{\lambda\mu} : M_\lambda \to M_\mu$, with the properties

DSys$^\odot$ 1. $\phi^{\lambda\mu}$ is defined only when $\lambda \leq \mu$.
DSys$^\odot$ 2. $\phi^{\lambda\lambda} = id$, the identity on M_λ.
DSys$^\odot$ 3. If $\lambda \leq \mu$ and $\mu \leq \nu$, then $\phi^{\lambda\nu} = \phi^{\lambda\mu}\phi^{\mu\nu}$.

The symbol $\odot$ refers to the mirror category which we introduced in (1.1.5); all the results on direct systems of right modules can be transcribed to left modules by using the formal identity ${}_R\mathcal{M}od = (\mathcal{M}od_{(R^\circ)})^\odot$, where R° is the opposite ring of R – see (1.1.5). (The attractive appearance of DSys$^\odot$ 3 is due to the direct system functor from Λ to ${}_R\mathcal{M}od$ being both covariant and cochiral.) For convenience, we list the main effects of the alteration of the notation.

Given two direct systems M_Λ and $N_\Lambda = \{N_\lambda, \psi^{\lambda\mu}\}$ of left R-modules, a *morphism* γ_Λ from M_Λ to N_Λ is now a collection of homomorphisms

$$\gamma^\lambda : M_\lambda \longrightarrow N_\lambda$$

such that

$$\gamma^\lambda\psi^{\lambda\mu} = \phi^{\lambda\mu}\gamma^\mu : M_\lambda \longrightarrow N_\mu \text{ whenever } \lambda \leq \mu.$$

The collection of direct systems of left R-modules over Λ is again a category, $[\Lambda, {}_R\mathcal{M}od]$.

The direct limit $\operatorname{dir\,lim}_\Lambda M_\lambda$ of a system of left R-modules has addition and scalar multiplication given by

$$[m_\lambda] + [m_\mu] = [m_\lambda \phi^{\lambda\nu} + m_\mu \phi^{\mu\nu}] \text{ where } \lambda, \mu \leq \nu,$$

and

$$r \cdot [m_\lambda] = [(rm_\lambda)_\lambda].$$

The universal property of the direct limit now reads as follows. Given a direct system $M_\Lambda = \{M_\lambda, \phi^{\lambda\mu}\}$, there is a collection of homomorphisms

$$\phi^{\lambda\bullet} : M_\lambda \longrightarrow \operatorname{dir\,lim}_\Lambda M_\lambda \text{ with } (m_\lambda)\phi^{\lambda\bullet} = [m_\lambda].$$

5.1.18 Theorem

(i) *If $\lambda \leq \mu$, then $\phi^{\lambda\mu}\phi^{\mu\bullet} = \phi^{\lambda\bullet}$.*

(ii) *If N is a fixed left R-module and $\{\chi^\lambda \mid \lambda \in \Lambda\}$ is any collection of homomorphisms $\chi^\lambda : M_\lambda \to N$ such that $\chi^\lambda = \phi^{\lambda\mu}\chi^\mu$ when $\lambda \leq \mu$, then there is a unique homomorphism $\chi : \operatorname{dir\,lim}_\Lambda M_\lambda \to N$ with $\phi^{\lambda\bullet}\chi = \chi^\lambda$ for all λ.* □

We now consider the relation between direct limits and tensor products of modules. The fundamental result is as follows.

5.1.19 Theorem

(i) *Let $N_\Lambda = \{N_\lambda, \phi^{\lambda\mu}\}$ be a direct system of left R-modules and let M be an L-R-bimodule.*

Then $M \otimes_R N_\Lambda = \{M \otimes_R N_\lambda, id \otimes \phi^{\lambda\mu}\}$ is a direct system of left L-modules, and there is an isomorphism

$$\operatorname{dir\,lim}_\Lambda(M \otimes_R N_\lambda) \cong M \otimes_R (\operatorname{dir\,lim}_\Lambda N_\lambda),$$

which is natural both for morphisms of direct systems and for homomorphisms of L-R-bimodules.

(ii) *Let $M_\Lambda = \{M_\lambda, \phi_{\lambda\mu}\}$ be a direct system of right R-modules and let N be an R-S-bimodule.*

Then $M_\Lambda \otimes_R N = \{M_\lambda \otimes_R N, \phi_{\mu\lambda} \otimes id\}$ is a direct system of right S-modules, and there is an isomorphism

$$\operatorname{dir\,lim}_\Lambda(M_\lambda \otimes_R N) \cong (\operatorname{dir\,lim}_\Lambda M_\lambda) \otimes_R N,$$

which is natural both for morphisms of direct systems and for homomorphisms of R-S-bimodules.

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Proof

We prove (i) only, since this is the form which appears most frequently in applications. We take as granted the functorial properties of the tensor product which are summarized in (3.1.9).

It is trivial to verify that we have a direct system as claimed. By (5.1.18), there are natural maps from $M \otimes_R N_\lambda$ to $M \otimes_R (\operatorname{dir\,lim}_\Lambda N_\lambda)$ for each λ in Λ, and hence there is an induced homomorphism θ from $\operatorname{dir\,lim}_\Lambda (M \otimes_R N_\lambda)$ to $M \otimes_R (\operatorname{dir\,lim}_\Lambda N_\lambda)$, given explicitly by

$$([m \otimes n_\lambda])\theta = m \otimes [n_\lambda].$$

For the inverse, we define a map from the cartesian product $M \times (\operatorname{dir\,lim}_\Lambda N_\lambda)$ to $\operatorname{dir\,lim}_\Lambda (M \otimes_R N_\lambda)$ by sending $(m, [n_\lambda])$ to $[m \otimes n_\lambda]$. It is straightforward to check that this map extends to a homomorphism on $M \# (\operatorname{dir\,lim} N_\Lambda)$ that vanishes on its relation subgroup $B(M, \operatorname{dir\,lim} N_\Lambda)$, and so induces an inverse to θ (see (3.1.2) and (3.1.3)).

It is clear that the constructions for both θ and its inverse are natural. □

5.1.20 Matrices again

Recall from (3) of (5.1.6) that there is a direct system of field inclusions

$$\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2} \hookrightarrow \cdots \hookrightarrow \mathbb{F}_{p^{m!}} \hookrightarrow \cdots$$

whose direct limit is the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. Now, as an abelian group, the set $M_n(F)$ of $n \times n$ matrices over a field F can be identified with the vector space F^{n^2}, which in turn is isomorphic to $F \otimes_{\mathbb{Z}} \mathbb{Z}^{n^2}$ as an abelian group. We therefore deduce from the above theorem that the direct system of inclusions

$$M_n(\mathbb{F}_p) \hookrightarrow M_n(\mathbb{F}_{p^2}) \hookrightarrow \cdots \hookrightarrow M_n(\mathbb{F}_{p^{m!}}) \hookrightarrow \cdots$$

has $M_n(\overline{\mathbb{F}}_p)$ as its direct limit in the category of abelian groups. Since any pair of matrices in $M_n(\overline{\mathbb{F}}_p)$ has only finitely many entries altogether, the matrix product effectively takes place within some $M_n(\mathbb{F}_{p^{m!}})$. It follows that $M_n(\overline{\mathbb{F}}_p)$ also satisfies the universal property for the direct limit in the category $\mathcal{R}\textsc{ing}$, and is the direct limit there too.

Exercises

5.1.1 This exercise is in some sense a converse to (5.1.6)(2) in that it constructs the direct limit from the direct sum.

Let $C_\Lambda = \{C_\lambda, \phi_{\mu\lambda} \mid \lambda, \mu \in \Lambda\}$ be a direct system of right R-modules. For an alternative construction of $\operatorname{dir\,lim}_\Lambda C_\lambda$, first take the

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direct sum $D = \bigoplus_\Lambda C_\lambda$, with canonical inclusion maps $\iota_\lambda : C_\lambda \hookrightarrow D$. Now let E be the submodule of D generated by all elements of the form

$$\iota_\lambda c_\lambda - \iota_\mu \phi_{\mu\lambda} c_\lambda \text{ where } \lambda \leq \mu \text{ and } c_\lambda \in C_\lambda.$$

Show that D/E, together with the homomorphisms

$$\phi_{\bullet\lambda} : C_\lambda \xhookrightarrow{\iota_\lambda} D \twoheadrightarrow D/E$$

induced by ι_λ, satisfies the universal property of (5.1.9), and hence deduce that D/E is canonically isomorphic to $\operatorname{dir\,lim}_\Lambda C_\lambda$ as constructed in the text.

5.1.2 There is an obvious notion of a map $F : \Omega \to \Lambda$ of directed sets. In category-theoretic terms F is a functor between directed sets viewed as categories.

Show that, given directed sets Λ and Ω, the direct product

$$\Delta = \Lambda \times \Omega$$

can be made into a directed set so that the projections $P_\Lambda : \Delta \to \Lambda$ and $P_\Omega : \Delta \to \Omega$ are surjective maps of directed sets.

Show further that for any fixed element ω of Ω, there is an injective map $I_{\Lambda,\omega} : \Lambda \to \Delta$ and likewise for the other term.

Is Δ a product or coproduct in the category of directed sets?

5.1.3 Let Λ be a directed set. Show that the following statements are equivalent for elements $\lambda, \omega \in \Lambda$.

(i) Viewing Λ as a category, $\lambda \cong \omega$.
(ii) We have both $\lambda \leq \omega$ and $\omega \leq \lambda$.

(Note that the relation $\lambda \leq \lambda$ must be interpreted as the identity map on λ.)

Let Δ be the set of isomorphism classes of elements of Λ. Verify that Δ can be regarded as a directed set such that the canonical map from Λ to Δ is a surjective map of directed sets.

Prove also that Λ, Δ and the skeleton (1.3.15) $\operatorname{Sk}(\Lambda)$ of Λ are equivalent as categories, the proper directed sets Δ and $\operatorname{Sk}(\Lambda)$ being isomorphic.

5.1.4 Let $F : \Omega \to \Lambda$ be a map of directed sets. Show that F induces, by composition, a morphism $[F, -] : [\Lambda, \mathcal{C}] \to [\Omega, \mathcal{C}]$ of the categories of direct systems in $\mathcal{C}$ over Λ and Ω, which gives a morphism F^* in $\mathcal{C}$ from $\operatorname{colim}_\Lambda C_\lambda$ to $\operatorname{colim}_\Omega C_\omega$ for every direct system $\{C_\lambda, \phi_{\lambda\mu}\}$ in $\mathcal{C}$.

Prove further that if $F : \Omega \to \Lambda$ and $G : \Omega \to \Lambda$ are naturally isomorphic (as functors), then $F^* = G^*$ on any direct system in $\mathcal{C}$.

5.1.5 Suppose that Ω is cofinal in Λ. Show that $\operatorname{colim}_\Lambda C_\lambda \cong \operatorname{colim}_\Omega C_\lambda$ for any direct system C_Λ over Λ, and that the inclusion map induces an equivalence between $[\Lambda, \mathcal{C}]$ and $[\Omega, \mathcal{C}]$.

5.1.6 Let Latid(R) denote the set of nonzero ideals of a ring R, regarded as an ordered set by reverse inclusion: $I \leq J$ if and only if $I \supseteq J$. Clearly, Latid(R) is proper. Show that there is a map of directed sets from Lat(R) to Latid(R). When is this map surjective?

5.1.7 Define the 'constant functor' $\mathrm{Cnst}_\mathcal{C} : \mathcal{C} \to [\Lambda, \mathcal{C}]$ to be that sending an object C to the constant direct system $\{C, id_C\}$ (that is, each $C_\lambda = C$), with similar behaviour on morphisms. Let $\{C_\lambda\}$ be a direct system in $\mathcal{C}$, regarded as an object in $[\Lambda, \mathcal{C}]$. Show that its colimit, together with the canonical morphisms from each C_λ to $\operatorname{colim}_\Lambda C_\lambda$, corresponds to an initial object in the right-fibre category $\{C_\lambda\} \backslash \mathrm{Cnst}_\mathcal{C}$. Apply (1.4.7) to conclude that the category $\mathcal{C}$ has Λ-colimits (that is, every direct system over Λ in $\mathcal{C}$ has a colimit in $\mathcal{C}$) precisely when $\mathrm{Cnst}_\mathcal{C}$ has a left adjoint $\mathrm{Colim} : [\Lambda, \mathcal{C}] \to \mathcal{C}$.

Remark. The results in this and the next exercise are readily seen to remain true if we consider colimits over an arbitrary category $\mathcal{D}$ as in (3) of (5.1.16).

5.1.8 Combine the conclusion of the preceding exercise with Exercise 1.3.9 to deduce that any functor which has a right adjoint (1.3.7) commutes up to natural isomorphism with the functor Colim. Here are two applications.

(a) Using (3.1.19), give an alternative proof for (5.1.19): if $N_\Lambda = \{N_\lambda, \phi^{\lambda\mu}\}$ is a direct system of left R-modules and M is an L-R-bimodule, there is an isomorphism

$$\operatorname{dir\,lim}_\Lambda (M \otimes_R N_\lambda) \cong M \otimes_R (\operatorname{dir\,lim}_\Lambda N_\lambda).$$

(b) Let Λ and Ω be directed sets and let $\Delta = \Lambda \times \Omega$ be viewed as a directed set (see Exercise 5.1.2 above).
Show that if $\{C_{\lambda,\omega}\}$ is a direct system over Δ in some category $\mathcal{C}$, then, for each choice of λ, $\{C_{\lambda,\omega}\}$ forms an Ω-direct system in $\mathcal{C}$, while for each fixed ω, $\{C_{\lambda,\omega}\}$ forms a Λ-direct system in $\mathcal{C}$.
Thus we can attempt to form 'the' colimit of $\{C_{\lambda,\omega}\}$ by three routes. Verify that if $\mathcal{C}$ is a category in which all colimits exist, then all the possibilities agree, and we may write

$$\operatorname{colim}_\Lambda \operatorname{colim}_\Omega C_{\lambda,\omega} = \operatorname{colim}_\Omega \operatorname{colim}_\Lambda C_{\lambda,\omega} = \operatorname{colim}_\Delta C_{\lambda,\omega}.$$

5.1.9 We work in $\mathcal{M}_{\mathcal{OD}R}$ and let $C_\Lambda = \{C_\lambda, \phi_{\mu\lambda}\}$ and $\chi_\lambda : C_\lambda \to D$ be as in (5.1.9). By applying (5.1.8) with D replaced by the direct system $\{D_\lambda = D, id\}$, show that if each χ_λ is injective (or surjective), then so is $\chi : \operatorname{dir}\lim_\Lambda C_\lambda \to D$.

5.2 DIRECT LIMITS, FLAT MODULES AND RINGS

We now consider two related applications of the direct limit construction. The first concerns flatness. Many important constructions involve the formation of a direct limit of free or projective modules, and so give rise to flat modules ([Serre 1956]). This observation is now exploited in many areas of mathematics. In the remaining chapters, we see how it is used in the study of localizations and completions. Here, we prove that direct limits of flat modules are again flat, and we prove an extension of Lazard's Theorem, which shows that flat modules can be characterized as the limits of finitely generated free modules. This section also contains a discussion of direct limits of rings, leading to the class of Von Neumann regular rings. Such a ring has the property that any (left) module is flat, and moreover, any direct limit of such rings again belongs to the class.

5.2.1 Construction of flat modules

We start by using Theorem 5.1.19 to produce some flat modules.

5.2.2 Theorem

(i) *Suppose that $N_\Lambda = \{N_\lambda, \phi^{\lambda\mu}\}$ is a direct system of left R-modules and that each N_λ is flat. Then* $\operatorname{dir}\lim_\Lambda N_\lambda$ *is a flat left R-module.*

(ii) *Suppose that $M_\Lambda = \{M_\lambda, \phi_{\mu\lambda}\}$ is a direct system of right R-modules and that each M_λ is flat. Then* $\operatorname{dir}\lim_\Lambda M_\lambda$ *is a flat right R-module.*

Proof

By (3.2.4), it is enough to show that if $\alpha : M' \to M$ is injective, then so also is

$$\alpha \otimes id_\Lambda : M' \otimes_R \operatorname{dir}\lim{}_\Lambda N_\lambda \longrightarrow M \otimes_R \operatorname{dir}\lim{}_\Lambda N_\lambda,$$

where we write id_Λ for the identity map on $\operatorname{dir}\lim_\Lambda N_\lambda$; we also write id_λ for the identity map on N_λ. By naturality,

$$\alpha \otimes id_\Lambda = \operatorname{dir}\lim{}_\Lambda(\alpha \otimes id_\lambda).$$

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But for each λ, $\alpha \otimes id_\lambda : M' \otimes_R N_\lambda \to M \otimes_R N_\lambda$ is injective, so the result follows from (5.1.8). □

The above result will be very useful in our treatment of localization. Here are some preliminary applications, the first of which is immediate from (5.1.12).

5.2.3 Corollary

Suppose that M is a (right) R-module and that every finitely generated submodule of M is flat. Then M is flat. □

5.2.4 Corollary

Let $\mathcal{K}$ be the field of fractions of a commutative domain $\mathcal{O}$. Then $\mathcal{K}$ is flat as an $\mathcal{O}$-module.

Proof

Let Λ be $\mathrm{Lat}(\mathcal{O})$, the set of nonzero elements of $\mathcal{O}$ ordered by division (5.1.2). For d in Λ, let M_d be the left $\mathcal{O}$-module

$$M_d = \mathcal{O}d^{-1} = \{xd^{-1} \mid x \in \mathcal{O}\},$$

and if d divides e, let ϕ^{de} be the obvious inclusion. Then $\mathcal{K} = \mathrm{dir\,lim}\, M_d$. But each M_d is isomorphic to $\mathcal{O}$; hence it is projective and thus flat by (3.2.5). □

5.2.5 Corollary

Let $\mathcal{O}$ be a Dedekind domain and let M be a torsion-free $\mathcal{O}$-module. Then M is flat.

Proof

Every finitely generated submodule of M is projective (2.3.20 - B), so the result follows from (3.2.5) again. □

An even easier argument leads to a useful observation.

5.2.6 Corollary

Let $\mathcal{K}$ be a field. Then any vector space over $\mathcal{K}$ is flat as a $\mathcal{K}$-module. □

As a further development of the above theorem, we can now characterize flat modules as direct limits of certain further classes of modules. In the result below, the equivalence of (i) and (ii) is known as Lazard's Theorem ([Lazard 1964]). The equivalence of (i) and (iii) is basic to Quillen's work on the K-theory of nonunital rings ([Quillen]).

5.2.7 Theorem

Let M be a right R-module. Then the following are equivalent.

(i) *M is flat;*

(ii) *M is a direct limit of projective modules;*

(iii) *M is a direct limit of finitely generated free modules.*

Proof

That each of (ii) and (iii) implies (i) follows by combining the previous theorem with (3.2.5). We proceed from (i) to (iii), passing through (ii) en route.

We take a surjective homomorphism $\epsilon : F \to M$ where F is a free module satisfying the conditions of (3.2.13); for example, $F = \mathrm{Fr}_R(M \times \mathbb{N})$ with $\epsilon : M \times \mathbb{N} \to M$ the projection $(m, n) \mapsto m$. Let Λ be some directed set which labels the direct system (under inclusion) of finitely generated submodules of $\operatorname{Ker} \epsilon$. Then, by (3.2.13), each finitely generated submodule L_λ of $\operatorname{Ker} \epsilon$ is contained in a finitely generated direct summand L'_λ of F with $L'_\lambda \subseteq \operatorname{Ker} \epsilon$, and clearly each quotient module F/L'_λ is projective. Now

$$L'_\lambda + L'_\mu \subseteq (L'_\lambda + L'_\mu)' \text{ for any } \lambda, \mu,$$

so the set of all such L'_λ is a direct subsystem of the system of all finitely generated submodules of $\operatorname{Ker} \epsilon$. It is evidently cofinal, and hence $\operatorname{Ker} \epsilon = \operatorname{dir\,lim}_\Lambda L'_\lambda$. Then the commuting diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L'_\lambda & \longrightarrow & F & \longrightarrow & F/L'_\lambda & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow id & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} \epsilon & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

combine to yield a commuting diagram of direct systems, whose direct limits form the commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{dir\,lim}_\Lambda L'_\lambda & \longrightarrow & F & \longrightarrow & \operatorname{dir\,lim}_\Lambda F/L'_\lambda & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow id & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker} \epsilon & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

with exact rows, by (5.1.8). Hence the right-hand vertical arrow is also an isomorphism, which gives (ii).

To prove (iii), we first show that M is a direct limit of a direct system all of whose terms can be taken to be the (non-finitely generated) free module

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F. For each λ in Λ, set $F_\lambda = F$ and let $\iota_\lambda : F/L'_\lambda \to F$ and $\pi_\lambda : F \to F/L'_\lambda$ with $\pi_\lambda \iota_\lambda = id$ be the homomorphisms which give the splitting of the upper exact sequence in the first diagram above. For $\lambda \le \mu$ in Λ, let

$$\psi_{\mu\lambda} : F/L'_\lambda \longrightarrow F/L'_\mu$$

be the surjection induced from the inclusion of L'_λ in L'_μ. We therefore obtain a morphism

$$\pi_\Lambda : \{F_\lambda, \iota_\mu \psi_{\mu\lambda} \pi_\lambda\} \longrightarrow \{F/L'_\lambda, \psi_{\mu\lambda}\}$$

of direct systems, which by (5.1.8) induces a surjection of direct limits

$$\operatorname{dir\,lim}_\Lambda \pi_\lambda : \operatorname{dir\,lim}_\Lambda F_\lambda \longrightarrow M.$$

To see that $\operatorname{dir\,lim}_\Lambda \pi_\lambda$ is in fact an isomorphism, consider an element $[x_\lambda]$ in its kernel. Then by (5.1.8) again, for each λ there is a μ with $\lambda \le \mu$ and $\psi_{\mu\lambda}(\pi_\lambda x_\lambda) = 0$. But then $(\iota_\mu \psi_{\mu\lambda} \pi_\lambda)(x_\lambda) = 0$, which forces $[x_\lambda]$ to be zero.

Finally, we show that M can be obtained as a direct limit of a system of finitely generated free submodules of F. By (5.1.15), F is the direct limit of the direct system comprising its finitely generated free submodules F'_ω, indexed by some directed set Ω (and ordered by inclusion). We can form a directed set Φ whose underlying set is $\Lambda \times \Omega$ with $(\lambda, \chi) \le (\mu, \omega)$ precisely when both $\lambda \le \mu$ and $\iota_\mu \psi_{\mu\lambda} \pi_\lambda (F'_\chi) \subseteq F'_\omega$. (To see that this is a directed set, recall that the image of a finitely generated module is again finitely generated and apply the argument of (5.1.15) again.) For each λ, take $F_{\lambda,\chi} = F'_\chi$, with homomorphisms given by restriction of $\iota_\mu \psi_{\mu\lambda} \pi_\lambda$. A simple chase of elements in the system of diagrams

$$\begin{array}{ccc} F_{\lambda,\chi} & \longrightarrow & \operatorname{dir\,lim}_\Phi F_{\lambda,\chi} \\ \downarrow & & \downarrow \\ \operatorname{dir\,lim}_\Omega F_{\lambda,\chi} = F_\lambda & \longrightarrow & \operatorname{dir\,lim}_\Lambda F_{\lambda,\chi} = M \end{array}$$

(as in the previous paragraph) shows that the right-hand vertical arrow is an isomorphism. (A more general version of this kind of argument occurs in Exercise 5.1.8 above.) □

5.2.8 *Direct limits of rings*

Suppose that a ring R is the direct limit $\operatorname{dir\,lim}_\Lambda R_\lambda$ of a direct system $\{R_\lambda, \rho_{\mu\lambda}\}$ of rings. Then there is a method of constructing R-modules in terms of R_λ-modules, as follows.

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A *coherent system* of right modules is a direct system $M_\Lambda = \{M_\lambda, \psi_{\mu\lambda}\}$, where each M_λ is now a right R_λ-module, and for $\lambda \leq \mu$, $\psi_{\mu\lambda} : M_\lambda \to M_\mu$ is an R_λ-module homomorphism, where M_μ is regarded as an R_λ-module by restriction of scalars through $\rho_{\mu\lambda} : R_\lambda \to R_\mu$. In formulas, this means that

$$\psi_{\mu\lambda}(m_\lambda r_\lambda) = \psi_{\mu\lambda}(m_\lambda)\rho_{\mu\lambda}(r_\lambda)$$

whenever $\lambda \leq \mu$ and $m_\lambda \in M_\lambda$ and $r_\lambda \in R_\lambda$.

It follows that the direct limit of $\mathbb{Z}$-modules $\operatorname{dir\,lim}_\Lambda M_\lambda$ is in fact an R-module by means of the scalar multiplication

$$[m_\lambda] \cdot [r_\mu] = [\psi_{\nu\lambda}(m_\lambda) \cdot \rho_{\nu\mu}(r_\mu)]$$

where ν is chosen such that $\lambda \leq \nu$ and $\mu \leq \nu$. (The point of the coherence conditions is to make this expression independent of the choice of such an element ν.)

Note that any R-module M arises in this way on taking each M_λ to be M, viewed as an R_λ-module by restriction, and taking all maps $\psi_{\mu\lambda}$ to be the identity homomorphism.

We leave to the reader the tasks of defining morphisms between coherent families, and verifying that they induce homomorphisms of R-modules.

5.2.9 Examples: yet more matrices

We have already formed two direct systems of matrix rings $M_n(R)$, by varying R and fixing n, and vice versa. We now exploit the fact that the matrix ring $M_n(R)$ operates by right multiplication on the free module of row vectors ${}^nR$, to obtain a coherent system of modules in each case.

1. Fix n and a prime number p. We continue the example of (5.1.20) in which the direct system of matrix ring inclusions

   $$M_n(\mathbb{F}_p) \hookrightarrow M_n(\mathbb{F}_{p^2}) \hookrightarrow \cdots \hookrightarrow M_n(\mathbb{F}_{p^{m!}}) \hookrightarrow \cdots$$

   was seen to have direct limit $M_n(\overline{\mathbb{F}}_p)$. Then the corresponding inclusions of (row) vector spaces

   $${}^n(\mathbb{F}_p) \hookrightarrow {}^n(\mathbb{F}_{p^2}) \hookrightarrow \cdots \hookrightarrow {}^n(\mathbb{F}_{p^{m!}}) \hookrightarrow \cdots$$

   (with each inclusion being given by extension of scalars) form a coherent system of right modules. From (5.1.19) it follows that the direct limit is just the right $M_n(\overline{\mathbb{F}}_p)$-module ${}^n(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$, in other words, ${}^n(\overline{\mathbb{F}}_p)$.

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2. We now develop Example 5.1.6(4), where we formed the nonunital ring mR of all 'finite' matrices as the direct limit dir $\lim_\omega M_n(R)$ in $\mathcal{R}_{\mathcal{NG}}$ of the nonunital ring homomorphisms

$$\rho_{n+k,n} : M_n(R) \longrightarrow M_{n+k}(R), \quad A \longmapsto A \oplus O_k.$$

If we embed ${}^nR$ in ${}^{n+k}R$ by adjoining zero entries, then a coherent system of right modules is obtained.

The limit of this system is naturally isomorphic to the direct sum ${}^\omega R$ (recall that elements of ${}^\omega R$ are row vectors indexed by the natural numbers and having only finitely many nonzero entries). Thus ${}^\omega R$ is a right mR-module in a natural way.

5.2.10 Von Neumann regular rings

The von Neumann regular rings, a generalization of Artinian semisimple rings, play an important role in functional analysis. We show how some von Neumann regular rings can be obtained as direct limits. A comprehensive account of the theory of von Neumann regular rings is given by [Goodearl 1979].

First, the definition. A ring R is *von Neumann regular* if for any $a \in R$, there is an $x \in R$ with $a = axa$. We record some rephrasings of this definition.

5.2.11 Proposition

The following statements are equivalent.

(i) *The ring R is von Neumann regular.*
(ii) *For any $a \in R$, we have $Ra = Re$ for some idempotent $e \in R$.*
(iii) *For any $a \in R$, the left ideal Ra is a direct summand of R.*
(iv) *For any $a \in R$, we have $aR = eR$ for some idempotent $e \in R$.*
(v) *For any $a \in R$, the right ideal aR is a direct summand of R.*

Proof

By the symmetry of (i), it is enough to deal with the first three statements.
(i) ⇒ (ii): We have $a = axa$ for some x, whence $e = xa$ is idempotent. Also, $Ra \subseteq Re \subseteq Ra$.
(ii) ⇒ (iii): We have $R = Re \oplus R(1-e)$ as a left module.
(iii) ⇒ (i): We can write $R = Ra \oplus \mathfrak{b}$ for some left ideal $\mathfrak{b}$ of R. Then $1 = xa + b$ with x in R and b in $\mathfrak{b}$, and thus $a = axa + ab$. But ab is in $Ra \cap \mathfrak{b} = 0$. □

5.2.12 Corollary

Let R be an Artinian semisimple ring. Then R is von Neumann regular.

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Proof

By the Complementation Lemma ([BK: IRM] (4.1.14)), any left ideal of R is a direct summand. □

Given that von Neumann regular rings are defined by means of a property of their elements, the following characterization is truly striking.

5.2.13 Theorem

A ring R is von Neumann regular if and only if every left R-module is flat.

Proof

First suppose that every left R-module is flat. Then in particular, for any $a \in R$ the module R/Ra is flat. This allows us to apply (3.2.12) to the surjection $R \to R/Ra$, to see that Ra is a direct summand of R. Thus R is von Neumann regular by (iii) of (5.2.11).

Conversely, suppose that R is von Neumann regular. We first argue by induction on the number of generators to show that every finitely generated R-module is flat. By (iii) of (5.2.11) again, every cyclic module R/Ra is projective and thus, by (3.2.5), flat.

For the inductive step, given a module M with a prescribed set of n generators, let M' be the submodule generated by the first $n-1$ of them. Then by the induction hypothesis M' is flat, as is the cyclic module M/M'. So from (3.2.10), M is also flat.

The final passage, from finitely generated modules to arbitrary modules, is by courtesy of (the left module counterpart of) (5.2.3). □

Next, we see what happens when we take direct limits.

5.2.14 Proposition

Suppose that a ring R is the direct limit $\operatorname{dir\,lim}_\Lambda R_\lambda$ *of a direct system $\{R_\lambda, \rho_{\mu\lambda}\}$ of von Neumann regular rings.*

Then R is von Neumann regular.

Proof

Take $a \in R$. Then there is an index λ with $a = [a_\lambda]$ for some $a_\lambda \in R_\lambda$. But $a_\lambda = a_\lambda x_\lambda a_\lambda$ for some $x_\lambda \in R_\lambda$; take $x = [x_\lambda]$. □

5.2.15 An example: idempotents all decompose

Let $\mathcal{K}$ be a field and let $n > 1$ be an integer. For any integer $i \geq 0$, let the ring R_i be the direct product of n^i copies of $\mathcal{K}$, (as a ring). Thus $R_0 = \mathcal{K}$, $R_1 = \mathcal{K}^n, \ldots$

Define $\rho_0 : R_0 \to R_1$ by

$$\rho_0(x) = (x, \ldots, x).$$

It is easy to see that ρ_0 is a ring homomorphism.

Next, define $\rho_i : R_i \to R_{i+1}$ by

$$\rho_i(x_1, \ldots, x_{(n^i)}) = (\rho_0 x_1, \ldots, \rho_0 x_{(n^i)});$$

each ρ_i is again a ring homomorphism, and so $\{R_i\}$ is a direct system of rings, indexed by $\{0\} \cup \mathbb{N}$.

Let $R = \operatorname{dir\,lim} R_i$. It is clear that R is von Neumann regular and commutative. On the other hand, R is neither Artinian nor semisimple, since any idempotent in R can be written as a sum of n orthogonal idempotents.

Exercises

5.2.1 An element ω of an ordered set Λ is *maximal* if whenever $\omega \leq \lambda$, then $\lambda \leq \omega$. (Note that Λ need not be proper.) Suppose that each element λ of Λ is less than some maximal element. Show that for an appropriate choice Ω of a set of maximal members of Λ, $\operatorname{dir\,lim} M_\Lambda = \bigoplus_\Omega M_\omega$ for any direct system of R-modules over Λ.

Hence find a (trivial) example where $\operatorname{dir\,lim} M_\lambda$ is flat but some M_λ is not.

5.2.2 Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module, although it is flat by (5.2.4).

5.2.3 Let R be a von Neumann regular ring. Show that R is Artinian if and only if R is semisimple.

5.2.4 Let $\mathcal{D}$ be a division ring and let $n > 1$ be an integer. For $i \geq 0$, let R_i be the ring of $n^i \times n^i$ matrices over $\mathcal{D}$.

Define a ring homomorphism $\rho_i : R_i \to R_{i+1}$ for each i by $\rho_i(A) = A \otimes I_n$, the *Kronecker product* or *tensor product* of matrices , where I_n is the $n \times n$ identity matrix. Thus, if $(A)_{jk} = a_{jk}$, then $(A \otimes I_n)_{jk} = a_{\lceil \frac{j}{n} \rceil, \lceil \frac{k}{n} \rceil}$ where $\lceil x \rceil$ denotes the least integer not less than x. (This operation can be interpreted informally as 'replacing the entry a_{jk} of the $n^i \times n^i$ matrix A by the $n \times n$ block $a_{jk} I_n$', which gives an $n^{i+1} \times n^{i+1}$ matrix.) Let R be the corresponding direct limit.

Show that R is von Neumann regular and that R is a simple ring, but that R is not semisimple as an R-module.

5.2.5 Show that R is von Neumann regular if and only if every finitely generated left ideal of R is generated by an idempotent and is thus a direct summand of R.

Hint. Arguing by induction, with part (ii) of (5.2.11) as the starting point, we need only consider ideals of the form $Re_1 + Re_2$ with e_1, e_2 idempotent. In fact, we can assume that $e_1e_2 = 0$; for otherwise replace e_1 by an idempotent generator of $R(e_1 - e_1e_2)$.

Apply the Flat Test (3.2.9) to deduce that every R-module is flat.

5.2.6 Let $R = \text{dir}\lim_\Lambda R_\lambda$ be a direct limit of rings. Show:

(i) each R_λ has invariant basis number (1.3.5) if and only if R has invariant basis number;

(ii) if each R_λ is a principal right ideal ring, then each finitely generated right ideal of R is also principal.

Let $\mathcal{K}$ be a field, and now take

$$R = \mathcal{K}[x_i \mid x_i = x_{i+1}^2, i = 1, 2, \ldots],$$

the 'polynomial' ring over $\mathcal{K}$ in a countably infinite set of variables subject to the relations $x_i = x_{i+1}^2$ for each i.

Show that R is isomorphic to the direct limit (over the natural numbers) of the system of polynomial rings $\{\mathcal{K}[T_i], \rho_{ji}\}$, where

$$\rho_{i+1,i} : \mathcal{K}[T_i] \longrightarrow \mathcal{K}[T_{i+1}]$$

is given by $\rho_{i+1,i}(T_i) = (T_{i+1})^2$.

Verify that the ideal $(x_i \mid i \in \mathbb{N})$ is not a principal ideal of R and therefore cannot be finitely generated.

5.3 INVERSE LIMITS

The inverse limit is the construction dual to the direct limit. It is an important tool which we use to obtain completions of rings and modules in section 7.1. However, we do not need to know its properties in the same detail that is required for the direct limit, and so we give only a brief resumé, supplemented by some exercises on the exactness of the inverse limit. The language of direct and inverse limits also allows us to make a further analysis of the relationship between direct sums and direct products for rings.

We work in the category of right R-modules for some ring R. The reader will have no difficulty supplying the corresponding details for the categories $\mathcal{R}\textsc{ing}$, $\mathcal{R}\textsc{ng}$, $\mathcal{G}\textsc{p}$, ...

5.3.1 The definition

As before, we begin with a directed set Λ which we may view as a left category. Recall that a direct system in a category $\mathcal{C}$ corresponds to a covariant functor

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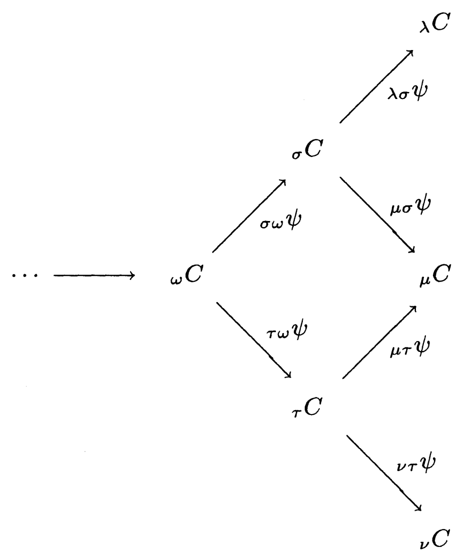
from Λ to $\mathcal{C}$. An *inverse system* in $\mathcal{C}$ corresponds instead to a contravariant functor. Thus $\{{}_\lambda C, {}_{\lambda\mu}\psi\}$ forms an inverse system of right R-modules when it satisfies the axioms:

ISys 1. ${}_{\lambda\mu}\psi : {}_\mu C \to {}_\lambda C$ is defined only when $\lambda \leq \mu$;
ISys 2. ${}_{\lambda\lambda}\psi = id$, the identity on ${}_\lambda C$;
ISys 3. If $\lambda \leq \mu$ and $\mu \leq \nu$, then ${}_{\lambda\mu}\psi_{\mu\nu}\psi = {}_{\lambda\nu}\psi$.

(Note that in this case the inverse system functor from Λ to $\mathcal{M}od_R$ is both contravariant and contrachiral, such that the intuitively appealing formulation of the coherence condition ISys 3 results.)

A typical pattern arising from three objects ${}_\lambda C$, ${}_\mu C$, ${}_\nu C$ is shown in the diagram below.



Just as the direct limit can be described as a homomorphic image of the direct sum of the modules of the system (Exercise 5.1.1), so dually the *inverse limit* (also known as the *projective limit*, $\varprojlim$ or, in category theory, simply as the *limit*) can be constructed as a submodule of the direct product $\prod_\Lambda {}_\lambda C$. We take

$$\text{inv lim}_\Lambda\, {}_\lambda C = \left\{ (x_\lambda) \in \prod\nolimits_\Lambda {}_\lambda C \mid {}_{\lambda\mu}\psi(x_\mu) = x_\lambda \text{ if } \lambda \leq \mu \right\}.$$

It is routine to verify that $\text{inv lim}_\Lambda\, {}_\lambda C$ is a right R-module, and that a morphism of inverse systems of R-modules leads to a homomorphism of their inverse limits.

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The essential difference between the inverse and the direct limit, which illustrates the duality between them, lies in the universal properties that they satisfy. For the direct limit, this was given in (5.1.9). To state the analogue for the inverse limit, we first define, for an inverse system $\{{}_\lambda C, {}_{\lambda\mu}\psi\}$, a collection of homomorphisms

$${}_{\lambda\bullet}\psi : \operatorname{inv\,lim}_\Lambda {}_\lambda C \longrightarrow {}_\lambda C$$

by

$${}_{\lambda\bullet}\psi((x_\lambda)) = x_\lambda.$$

A straightforward verification then gives the following result.

5.3.2 Theorem

(i) *If $\lambda \leq \mu$, then ${}_{\lambda\mu}\psi_{\mu\bullet}\psi = {}_{\lambda\bullet}\psi$.*

(ii) *If N is a fixed R-module and $\{{}_\lambda\theta \mid \lambda \in \Lambda\}$ is any collection of homomorphisms ${}_\lambda\theta : N \to {}_\lambda C$ such that ${}_\lambda\theta = {}_{\lambda\mu}\psi_\mu\theta$ whenever $\lambda \leq \mu$, then there is a unique morphism $\theta : N \to \operatorname{inv\,lim}_\Lambda {}_\lambda C$ with ${}_{\lambda\bullet}\psi\theta = {}_\lambda\theta$ for all λ.* □

5.3.3 The p-adic integers

This example is dual to (5.1.14). Again fix a prime p. For each $i \leq j$ in $\mathbb{N}$ there is a canonical surjective ring homomorphism ${}_{ij}\psi : \mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$, giving an inverse system

$$\cdots \longrightarrow \mathbb{Z}/p^3\mathbb{Z} \xrightarrow{{}_{23}\psi} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{{}_{12}\psi} \mathbb{Z}/p\mathbb{Z}.$$

Passing to the limit yields as the inverse limit the ring of *p-adic integers* $\widehat{\mathbb{Z}}_p$, which we discuss again in section 7.1.

5.3.4 Sums and coproducts as limits

The names 'direct sum' and 'direct product' are by now due for some clarification, especially when indexed by infinite sets.

For finite collections of objects there is some arbitrariness about the choice of term. For example, as noted in (2.2.11), the direct sum of two right R-modules is both a product and a coproduct in $\mathcal{M}\text{OD}_R$. It could therefore equally be called their direct product. A similar situation holds for nonunital rings. For rings, however, as noted in Exercise 1.4.3, the direct product is a product but not a coproduct, so that the title 'direct sum' is inappropriate (although a few authors use it).

We now analyse the corresponding constructions for a countably infinite set $R_1, R_2, \ldots$ of rings (both in $\mathcal{R}_{\mathcal{ING}}$ and $\mathcal{R}_{\mathcal{NG}}$), using the apparatus of direct and inverse limits. (The uncountable case is similar, albeit messier.)

For each $k \geq 1$, the evident projection

$$R_1 \times \cdots \times R_{k+1} \longrightarrow R_1 \times \cdots \times R_k$$

is a ring homomorphism, giving an inverse system whose inverse limit is the *direct product* $\prod_i R_i$ of the collection $\{R_i\}$; this is the product in the category $\mathcal{R}_{\mathcal{ING}}$ (or $\mathcal{R}_{\mathcal{NG}}$). (The epithet 'direct' here is unfortunate but traditional.)

For nonunital rings, it is possible to form the dual construction, that is, by taking the direct limit of the direct system

$$R_1 \times \cdots \times R_k \longrightarrow R_1 \times \cdots \times R_{k+1}$$

in which the homomorphisms are the inclusion maps

$$(x_1, \ldots, x_k) \longmapsto (x_1, \ldots, x_k, 0).$$

However, even when the rings have unit elements, these maps are merely nonunital ring homomorphisms, since they do not preserve the identity. Thus the direct limit is the nonunital ring direct sum $\bigoplus_i R_i$, which is a coproduct in the category $\mathcal{R}_{\mathcal{NG}}$.

Note that a typical element of $\prod_i R_i$ is an infinite sequence (x_i) with each $x_i \in R_i$, and that a typical element of $\bigoplus_i R_i$ is a sequence (x_i) which has almost all entries $x_i = 0$. Thus we have an evident canonical nonunital ring embedding of $\bigoplus_i R_i$ in $\prod_i R_i$ as an ideal.

In contrast, if in the above discussion we replace the rings R_i by a family of R-modules M_i over a fixed ring R, then $\bigoplus_i M_i$ is the direct sum of the modules, $\prod_i M_i$ is their direct product, and the canonical inclusion is an R-module homomorphism. This also works for families of groups or monoids.

Things go awry for rings since a ring homomorphism is required to preserve two neutral elements, namely the zero and the identity elements. This is illustrated by comparing the last two examples of (5.1.6), which give two embeddings of $M_n(R)$ in $M_{n+1}(R)$, the first of which is taken to preserve the zero matrix and the second to preserve the identity matrix. The first gives rise to a direct system of R-modules and the second to a direct system of monoids.

Exercises: inverse limits and exactness

5.3.1 Let Λ be a directed set, and let ${}_\Lambda M = \{{}_\lambda M, {}_{\lambda\mu}\psi\}$, ${}_\Lambda M'$ and ${}_\Lambda M''$

be inverse systems of right R-modules. Suppose that there is a short exact sequence of inverse systems

$$0 \longrightarrow {}_\Lambda M' \longrightarrow {}_\Lambda M \longrightarrow {}_\Lambda M'' \longrightarrow 0,$$

that is, whenever $\lambda \leq \mu$ in Λ there is a commuting diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}_\mu M' & \longrightarrow & {}_\mu M & \longrightarrow & {}_\mu M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & {}_\lambda M' & \longrightarrow & {}_\lambda M & \longrightarrow & {}_\lambda M'' & \longrightarrow & 0 \end{array}$$

Show that there is an exact sequence of right R-modules

$$0 \longrightarrow \operatorname{inv}\lim_\Lambda {}_\lambda M' \longrightarrow \operatorname{inv}\lim_\Lambda {}_\lambda M \longrightarrow \operatorname{inv}\lim_\Lambda {}_\lambda M''.$$

5.3.2 Now suppose that $\Lambda = \omega$. We then think of the inverse system as a *tower* of modules and we write the inverse limit as $\varprojlim$.

The failure of the right-hand map above to be surjective gives rise to a further functor, the *derived functor* $\varprojlim^1$ of $\varprojlim$, by means of the following construction by [Steenrod 1940].

(i) Define M to be the right R-module $\prod_{i=1}^{\infty} {}_i M$, the direct product of all the ${}_i M$ in ${}_\omega M$, and define $\chi : M \to M$ by

$$\theta({}_i m) = {}_i m - {}_{i,i+1}\psi({}_{i+1}m).$$

Show that $\varprojlim {}_i M \cong \operatorname{Ker}\theta$.

(ii) Now define $\varprojlim^1 {}_i M \cong \operatorname{Cok}\theta$. Similarly, for the short exact sequence of 5.3.1, define M', M'', $\theta' : M' \to M'$ and $\theta'' : M'' \to M''$, $\varprojlim^1 {}_i M'$ and $\varprojlim^1 {}_i M''$. Use the Snake Lemma (Exercise 2.3.13) to construct an exact sequence

$$\begin{array}{l} 0 \longrightarrow \varprojlim {}_i M' \longrightarrow \varprojlim {}_i M \longrightarrow \varprojlim {}_i M'' \longrightarrow \\ \longrightarrow \varprojlim^1 {}_i M' \longrightarrow \varprojlim^1 {}_i M \longrightarrow \varprojlim^1 {}_i M'' \longrightarrow 0. \end{array}$$

As an example, deduce from the short exact sequences

$$0 \longrightarrow p^i\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/p^i\mathbb{Z} \longrightarrow 0$$

that the inverse system $\{p^i\mathbb{Z}\}$ given by inclusions

$$\cdots \longrightarrow p^{i+1}\mathbb{Z} \longrightarrow p^i\mathbb{Z} \longrightarrow \cdots \longrightarrow p^2\mathbb{Z} \longrightarrow p\mathbb{Z}$$

has $\varprojlim^1 \{p^i\mathbb{Z}\} = (\widehat{\mathbb{Z}}_p)/\mathbb{Z}$.

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(iii) It is immediate from this sequence that

$$\varprojlim^1 {}_iM = 0 \quad \text{implies} \quad \varprojlim^1 {}_iM'' = 0,$$

while

$$\varprojlim^1 {}_iM' = 0 \text{ and } \varprojlim^1 {}_iM'' = 0 \text{ imply } \varprojlim^1 {}_iM = 0.$$

For an entry to recent work on this topic, see [McGibbon & Steiner 1995].

5.3.3 After Dieudonné-Grothendieck, the inverse system $\{{}_iM, {}_{i,i+1}\psi\}$ is said to be *Mittag-Leffler* if for each i there is a j with $i \leq j$ such that Im ${}_{ij}\psi =$ Im ${}_{ik}\psi$ whenever $j \leq k$. Show that then $\varprojlim^1 {}_iM = 0$, and that the Mittag-Leffler relationships suggested by (iii) above do in fact hold. For recent developments on this topic, see [Emmanouil 1996].

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