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ALMOST ALL NORMAL SETS ARE STRICTLY NORMAL

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We consider the space S_n of all nonempty bounded closed normal subsets of the cone R^n_+ where R^n_+ is the set of all vectors $x \in R^n$ with nonnegative coordinates. We equip the space S_n with the Hausdorff metric and show that most elements of S_n are, in fact, strictly normal. More precisely, we show that the complement of the collection of all stricly normal elements of S_n is a σ-porous subset of S_n.

INTRODUCTION

In this paper we consider the space S_n of all nonempty bounded closed normal subsets of the cone R^n_+ where R^n_+ is the set of all vectors $x \in R^n$ with nonnegative coordinates. The space S_n is an important class of sets which is used in mathematical economics [**6, 7, 8**], abstract convexity [**9**], approximation theory [**10, 12**] and in monotonic analysis [**10, 12**]. For instance, level sets of increasing functions are normal. We equip the space S_n with the Hausdorff metric and show that a generic bounded closed normal subset of R^n_+ is strictly normal.

When we say that a certain property holds for a generic element of a complete metric space Y we mean that the set of points which have this property contains a G_δ everywhere dense subset of Y. Such an approach, when a certain property is investigated for the whole space Y and not just for a single point in Y, has already been successfully applied in many areas of Analysis [**1, 2, 3, 4, 5, 11, 12, 13**]. The first generic result in monotonic analysis was obtained in [**11**] where we showed that a generic increasing function defined on an ordered Banach space has a point of minimum. In [**12**] we showed that a generic increasing function is strictly increasing. We considered a space of increasing functions equipped with a natural metric and showed that the complement of the subset of all strictly increasing functions is not only of the first category but also a σ-porous set [**12**]. There exists a natural one-to-one correspondence Ψ between the collection of all closed normal subsets of R^+_n and the space of increasing positively homogeneous functions [**12**, Propositions 1.4 and 1.5]. In [**12**, Section 5] we showed that the set of all strictly normal subsets has a σ-porous complement in an important subspace of S_n equipped with a metric induced by the mapping Ψ. In this

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paper we show that the complement of the set of all strictly normal elements of S_n is a σ-porous subset of S_n with respect to the Hausdorff distance.

We now recall the concept of porosity [**4, 5, 12**].

Let (Y, d) be a complete metric space. We denote by $B(y, r)$ the closed ball of centre $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called porous in (Y, d) if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$ there exists $z \in Y$ for which

$$B(z, \alpha r) \subset B(y, r) \setminus E.$$

A subset of the space Y is called σ-porous in (Y, d) if it is a countable union of porous subsets in (Y, d).

Remark 1. It is known that in the above definition of porosity, the point y can be assumed to belong to E.

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite-dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have measure 0 and are of the first category.

To point out the difference between porous and nowhere dense sets note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there is a point $z \in Y$ and a number $s > 0$ such that $B(z, s) \subset B(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which depends only on E.

The paper is organised as follows. In the first section we show that the space S_n equipped with the Hausdorff metric is complete and state our main result. The main result is established in Section 2.

1. Preliminaries and the main result

We consider the Euclidean space R^n with vectors $x = (x_1, \ldots, x_n) \in R^n$ and the norm $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$, $x \in R^n$. Set $\mathbf{1} = (1, \ldots, 1)$.

Denote by R^n_+ the cone of positive elements:

$$R^n_+ = \{x = (x_1, \ldots, x_n) \in R^n : x_i \geqslant 0,\ i = 1, \ldots, n\}.$$

The following definition will be used in the sequel (see [**6, 8, 9, 10**]).

A set $E \subset R^n_+$ is called normal if $x \in E$, $y \in R^n_+$ and $y \leqslant x$ imply that $y \in E$.

A point $x \in E \subset R^n_+$ is called a boundary point of the set E if for each $\varepsilon > 0$ there is $y \in R^n_+ \setminus E$ such that $|x - y| < \varepsilon$.

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The following definition was introduced in [**12**].

A set $E \subset R_+^n$ is called strictly normal if for each boundary point $x \in E$ the inequality $x < y$ implies that $y \notin E$.

Note that a subset $E \subset R_+^n$ is strictly normal if and only if for each $x, y \in E$ satisfying $x < y$ there is $r > 0$ such that

$$\left\{z \in R_+^n : \ |x - z| \leqslant r\right\} \subset E.$$

For each $x \in X$ and each $A \subset X$ set

$$\rho(x, A) = \inf\left\{|x - y| : \ y \in A\right\}.$$

Denote by S the family of all nonempty bounded closed subsets of R_+^n. For each $A, B \in S$ define the Hausdorff distance

$$H(A, B) = \max\left\{\sup\left\{\rho(x, B) : \ x \in A\right\}, \ \sup\left\{\rho(y, A) : \ y \in B\right\}\right\}. \tag{1.1}$$

It is known that the metric space (S, H) is complete. Denote by S_n the family of all normal sets $A \in S$.

Proposition 1.1. *S_n is a closed subset of (S, H).*

Proof: Let $A \in S$, $A_k \in S_n$, $k = 1, 2, \ldots$ and let $H(A_k, A) \to 0$ as $k \to \infty$. We may assume without loss of generality that

$$H(A, A_k) \leqslant 1/k, \ k = 1, 2, \ldots \tag{1.2}$$

Let $0 \leqslant x \leqslant y$ and $y \in A$. We shall show that $x \in A$.

By (1.2) for each natural number k there exists $y^{(k)} = \left(y_1^{(k)}, \ldots, y_n^{(k)}\right) \in A_k$ such that

$$|y - y^{(k)}| \leqslant 2/k. \tag{1.3}$$

Let $k \geqslant 1$ be an integer. Define $x^{(k)} = \left(x_1^{(k)}, \ldots, x_n^{(k)}\right) \in R^n$ by

$$x_i^{(k)} = \max\{x_i - 2/k, 0\}, \ i = 1, \ldots, n. \tag{1.4}$$

It follows from (1.3) and (1.4) that for $i = 1, \ldots, n$

$$|y_i^{(k)} - y_i| \leqslant 2/k, \ y_i^{(k)} \geqslant y_i - 2/k \geqslant x_i - 2/k$$

and

$$y_i^{(k)} \geqslant \max\{x_i - 2/k, 0\} = x_i^{(k)}.$$

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Since A_k is normal we obtain that $x^{(k)} \in A_k$. On the other hand in view of (1.4)

$$|x_i^{(k)} - x_i| \leqslant 2/k, \; i = 1, \ldots, n,$$
$$|x^{(k)} - x| \leqslant 2n/k.$$

This implies that $x \in A$. The proposition is proved. ∎

For each $x \in R^n$ and each $r > 0$ set

$$B_{\|\cdot\|}(x, r) = \{y \in R^n : \; |y - x| \leqslant r\}.$$

A set $A \in S_n$ is called strictly normal in the strong sense if for each natural number k there exists $\gamma_k > 0$ such that for each $x, y \in A$ satisfying

$$|x - y| \geqslant 1/k, \; y > x$$

the following relation holds:

$$B_{\|\cdot\|}(x, \gamma_k) \cap R^n_+ \subset A.$$

Clearly, each strictly normal in the strong sense set $A \in S_n$ is strictly normal.

We shall establish the following result.

Theorem 1.1. *There exists a set $\mathcal{F} \subset S_n$ such that the complement $S_n \setminus \mathcal{F}$ is σ-porous in (S_n, H) and each $A \in \mathcal{F}$ is strictly normal in the strong sense.*

2. Proof of Theorem 1.1

For each natural number k denote by $\mathcal{F}_k$ the set of all $A \in S_n$ which have the following property:

(P1) There is $\gamma_k > 0$ such that for each $x, y \in A$ satisfying

$$y > x, \; |y - x| \geqslant 1/k,$$

the relation $B_{\|\cdot\|}(x, \gamma_k) \cap R^n_+ \subset A$ holds.

Define

$$\mathcal{F} = \bigcap_{k=1}^{\infty} \mathcal{F}_k.$$

Clearly any element of $\mathcal{F}$ is strictly normal in the strong sense. Therefore in order to prove the theorem it is sufficient to show that for each natural number k the set $S_n \setminus \mathcal{F}_k$ is σ-porous in (S_n, H).

Fix a natural number k. For each natural number m set

(2.1) $$E_m = \left\{A \in S_n : \sup\{\|x\| : x \in A\} \leqslant m\right\}.$$

Since

$$S_n \setminus \mathcal{F}_k = \bigcup_{m=1}^{\infty} (E_m \setminus \mathcal{F}_k)$$

in order to prove the theorem it is sufficient to show that for each natural number m the set $E_m \setminus \mathcal{F}_k$ is porous in (S_n, H).

Let m be a natural number. Choose a positive number

(2.2) $$\alpha < \left(16^3 n^3 km\right)^{-1}.$$

Assume that

(2.3) $$A \in E_m \setminus \mathcal{F}_k \text{ and } r \in (0, 1/k].$$

Denote by $\widetilde{A}$ the set of all $z \in R^n_+$ for which there exists $y \in A$ such that

(2.4) $$z \leqslant y + (4n)^{-1} r \left[1 - (4mn)^{-1} \sum_{i=1}^{n} y_i\right] \mathbf{1}.$$

It is not difficult to see that $\widetilde{A}$ is bounded closed normal set and satisfies

(2.5) $$H\left(A, \widetilde{A}\right) \leqslant r/4.$$

Assume that $C \in S_n$ and

(2.6) $$H\left(\widetilde{A}, C\right) \leqslant \alpha r.$$

We shall show that $C \in \mathcal{F}_k$ with $\gamma_k = \alpha r$.

Assume that $x, y \in C$,

(2.7) $$y > x, \ |x - y| \geqslant 1/k$$

and that

(2.8) $$z \in R^n_+, \ |x - z| \leqslant \alpha r.$$

We shall show that $z \in C$. By (2.6) there are

(2.9) $$\widetilde{x}, \widetilde{y} \in \widetilde{A}$$

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such that

$$|\widetilde{y} - y|, |\widetilde{x} - x| \leqslant \alpha r. \tag{2.10}$$

It follows from (2.7), (2.10), (2.3) and (2.2) that

$$|\widetilde{x} - \widetilde{y}| \geqslant |x - y| - |\widetilde{x} - x| - |\widetilde{y} - y| \geqslant 1/k - 2\alpha r \geqslant (2k)^{-1},$$

$$|\widetilde{x} - \widetilde{y}| \geqslant (2k)^{-1}, \tag{2.11}$$

$$\widetilde{y} \geqslant y - \alpha r\mathbf{1} > x - \alpha r\mathbf{1} \geqslant \widetilde{x} - 2\alpha r\mathbf{1},$$

$$\widetilde{y} \geqslant \widetilde{x} - 2\alpha r\mathbf{1}. \tag{2.12}$$

By (2.9) and the definition of $\widetilde{A}$ (see (2.4)) there exists $u \in A$ such that

$$\widetilde{y} \leqslant u + (4n)^{-1} r \left[1 - (4mn)^{-1} \sum_{i=1}^{n} u_i\right] \mathbf{1}. \tag{2.13}$$

In view of (2.11) there exists an integer $j_0 \in \{1, \ldots, n\}$ such that

$$|\widetilde{x}_{j_0} - \widetilde{y}_{j_0}| \geqslant (2kn)^{-1}. \tag{2.14}$$

It follows from (2.12), (2.2) and (2.3) that

$$\widetilde{x}_{j_0} - \widetilde{y}_{j_0} \leqslant 2\alpha r < (2kn)^{-1}.$$

Combined with (2.14) this inequality implies that

$$\widetilde{y}_{j_0} - \widetilde{x}_{j_0} \geqslant (2kn)^{-1}. \tag{2.15}$$

By (2.13), (2.3) and (2.15),

$$u_{j_0} \geqslant \widetilde{y}_{j_0} - (4nk)^{-1} \geqslant \widetilde{x}_{j_0} + (2nk)^{-1} - (4nk)^{-1}$$

and

$$u_{j_0} \geqslant \widetilde{x}_{j_0} + (4nk)^{-1}. \tag{2.16}$$

Define $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n) \in R^n_+$ by

$$\widetilde{u}_i = u_i,\ i \in \{1, \ldots, n\} \setminus j_0,\ \widetilde{u}_{j_0} = u_{j_0} - (16nk)^{-1}. \tag{2.17}$$

Clearly $\widetilde{u} \in R^n_+$,

$$\widetilde{u} \leqslant u \text{ and } \widetilde{u} \in A. \tag{2.18}$$

Define $\widehat{u} = (\widehat{u}_1, \dots, \widehat{u}_n) \in R^n$ by

$$\widehat{u} = \widetilde{u} + (4n)^{-1} r \left[1 - (4mn)^{-1} \sum_{i=1}^{n} \widetilde{u}_i \right] \mathbf{1}. \tag{2.19}$$

By the definition,

$$\widehat{u} \in \widetilde{A}. \tag{2.20}$$

The equations (2.19) and (2.17) imply that

$$\widehat{u} = \widetilde{u} + (4n)^{-1} r \left[1 - (4mn)^{-1} \Big(\sum_{i=1}^{n} u_i - (16nk)^{-1} \Big) \right] \mathbf{1}. \tag{2.21}$$

It follows from (2.21) and (2.17) that for all $i \in \{1, \dots, n\} \setminus \{j_0\}$

$$\widehat{u}_i = u_i + (4n)^{-1} r \left[1 - (4mn)^{-1} \sum_{j=1}^{n} u_j \right] + (4n)^{-1} r (16nk)^{-1} (4mn)^{-1}, \tag{2.22}$$

(2.23)

$$\widehat{u}_{j_0} = u_{j_0} - (16nk)^{-1} + (4n)^{-1} r \left[1 - (4mn)^{-1} \sum_{j=1}^{n} u_j \right] + (4n)^{-1} r (16nk)^{-1} (4mn)^{-1}.$$

(2.22) and (2.13) imply that for all $i \in \{1, \dots, n\} \setminus \{j_0\}$

$$\widehat{u}_i \geqslant \widetilde{y}_i + r \big(16^2 n^3 mk\big)^{-1}. \tag{2.24}$$

In view of (2.23) and (2.13)

$$\widehat{u}_{j_0} \geqslant \widetilde{y}_{j_0} - (16nk)^{-1} + r \big(16^2 n^3 mk\big)^{-1}. \tag{2.25}$$

(2.8) and (2.10) imply that

$$|z - \widetilde{x}| \leqslant |z - x| + |x - \widetilde{x}| \leqslant \alpha r + \alpha r$$

and

$$\widetilde{x} \geqslant z - 2\alpha r \mathbf{1}. \tag{2.26}$$

By (2.12) and (2.26),

$$\widetilde{y} \geqslant \widetilde{x} - 2\alpha r \mathbf{1} \geqslant z - 4\alpha r \mathbf{1}. \tag{2.27}$$

It follows from (2.20) and (2.6) that there is $v \in R^n_+$ such that

$$v \in C,\ |v - \widehat{u}| \leqslant \alpha r. \tag{2.28}$$

(2.28) implies that

$$v \geqslant \widehat{u} - \alpha r \mathbf{1}. \tag{2.29}$$

(2.29), (2.24), (2.27) and (2.2) imply that for all $i \in \{1, \dots, n\} \setminus \{j_0\}$

$$\begin{aligned} v_i &\geqslant \widehat{u}_i - \alpha r \geqslant \widetilde{y}_i + r\left(16^2 n^3 km\right)^{-1} - \alpha r \\ &\geqslant r\left(16^2 n^3 km\right)^{-1} - \alpha r + z_i - 4\alpha r \\ &= z_i + r\left[\left(16^2 n^3 km\right)^{-1} - 5\alpha\right] > z_i \end{aligned}$$

and

$$v_i > z_i. \tag{2.30}$$

It follows from (2.29), (2.25), (2.15), (2.26) and (2.2) that

$$\begin{aligned} v_{j_0} \geqslant \widehat{u}_{j_0} - \alpha r &\geqslant -\alpha r + \widetilde{y}_{j_0} - (16nk)^{-1} + r\left(16^2 n^3 mk\right)^{-1} \\ &\geqslant \widetilde{x}_{j_0} + (2kn)^{-1} - \alpha r - (16nk)^{-1} + r\left(16^2 n^3 mk\right)^{-1} \\ &\geqslant (2kn)^{-1} - \alpha r - (16nk)^{-1} + r\left(16^2 n^3 mk\right)^{-1} + z_{j_0} - 2\alpha r > z_{j_0} \end{aligned}$$

and

$$v_{j_0} > z_{j_0}. \tag{2.31}$$

By (2.30), (2.31) and (2.28), $z \in C$. Thus we have shown that for each $x, y \in C$ satisfying (2.7) and each $z \in R^n$ satisfying (2.8) the inclusion $z \in C$ holds. Therefore $C \in \mathcal{F}_k$. We have shown that

$$\left\{C \in S_n :\ H(\widetilde{A}, C) \leqslant \alpha r\right\} \subset \mathcal{F}_k.$$

By (2.5) and (2.2),

$$\left\{C \in S_n :\ H(\widetilde{A}, C) \leqslant \alpha r\right\} \subset \left\{C \in S_n :\ H(A, C) \leqslant r\right\}.$$

Therefore the set $E_m \setminus \mathcal{F}_k$ is porous in (S_n, H). This completes the proof of the theorem. ∎

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