IMPRIMITIVELY GENERATED
LIE-ALGEBRAIC HAMILTONIANS
AND SEPARATION OF VARIABLES

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ABSTRACT. Turbiner's conjecture posits that a Lie-algebraic Hamiltonian operator whose domain is a subset of the Euclidean plane admits a separation of variables. A proof of this conjecture is given in those cases where the generating Lie-algebra acts imprimitively. The general form of the conjecture is false. A counter-example is given based on the trigonometric Olshanetsky-Perelomov potential corresponding to the A_2 root system.

1. **Introduction.**

1.1. *Motivation.* There are profound connections between quantum mechanics and the theory of Lie algebras, and their representations. Typically, one looks for a way to relate a given Hamiltonian operator to a finite-dimensional Lie-algebra, and then uses information about the algebra's representations to solve the corresponding spectral problem. This general philosophy manifests itself in a number of distinct approaches [3] [1] [2]. The context for the present article is the application of Lie theory to the study of quasi-exactly solvable spectral problems. In order to motivate the questions dealt with here it will be useful to briefly review the relevant background.

A spectral problem is called *quasi-exactly solvable* (Q.E.S. for short) if there exists a method for explicitly obtaining eigenvalues and eigenvectors for a finite subset of an otherwise infinite spectrum [25]. Typically, quasi-exact solvability amounts to the existence of an explicitly describable basis of a finite-dimensional invariant subspace. The Q.E.S. problems related to 1-dimensional non-relativistic quantum mechanics are profoundly related to the following realization of $\mathfrak{sl}_2\mathbb{R}$ by first-order differential operators [24] [11]:

$$T_1 = \partial_z, \ T_2 = 2z\partial_z - n, \ T_3 = z^2\partial_z - nz, \quad n \in \mathbb{N}.$$

The crucial fact regarding these operators is that the vector space of polynomials of degree n or less is an invariant subspace; indeed this subspace realizes the $n + 1$ dimensional, irreducible representation of $\mathfrak{sl}_2\mathbb{R}$.

It therefore stands to reason that every differential operator generated by the T_i's will admit the same finite-dimensional invariant subspace. Consider, for instance, the

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operator given by

$$-H_0 = \frac{1}{4}(T_1T_2 + T_2T_1) + 16bT_3 + 8cT_2 + \frac{n}{2}T_1,$$

where b, c are real parameters. A change of coordinates, $z = x^2/4$, and a gauge transformation, $H = \mu \cdot H_0 \cdot \mu^{-1}$, where

$$\mu = \exp\left(\frac{b}{4}x^4 + \frac{c}{2}x^2\right),$$

yields the Hamiltonian for the sextic anharmonic potential [26] [14]:

$$H = -\partial_{xx} + b^2x^6 + 2bcx^4 + \left(c^2 + b(3 + 4n)\right)x^2 + c(1 + 2n).$$

This Hamiltonian operator is quasi-exactly solvable because the method of construction guarantees that the vector space spanned by $\{\mu, \mu x^2, \ldots, \mu x^{2n}\}$ is an invariant subspace of the operator.

The above construction of a Q.E.S. Hamiltonian has the following generalization to higher dimensions [20] [8]. One starts with a realization of a finite-dimensional Lie algebra $\mathfrak{g}$ by first order differential operators

$$T_a = V_a + \lambda_1\eta_{1a} + \cdots + \lambda_k\eta_{ka}, \quad a = 1, \ldots, \dim(\mathfrak{g}),$$

where the V_a's are a realization of $\mathfrak{g}$ by vector fields, where the η_{ia} are functions, and where the λ_i are real parameters such that for certain values there exists a finite-dimensional invariant subspace of functions, W_λ.

The general second-order order, Lie algebraic operator is given by:

$$\tag{1} H_0 = \sum_{ab} C^{ab}T_aT_b + \sum_a B^aT_a,$$

where the C^{ab} and B^a are real numbers with $C^{ab} = C^{ba}$. In the rest of this article operators like H_0, *i.e.* operators that can be generated by a finite-dimensional Lie algebra of first-order differential operators, will be called *Lie-algebraic*. A Lie algebraic Hamiltonian/Schrödinger operator, H, is a formally self-adjoint (*i.e.* Laplacian plus potential) Lie algebraic operator. One should note that the class of Lie algebraic operators is closed under gauge transformations. One is therefore allowed to construct a Lie algebraic Hamiltonian by gauge-transforming an arbitrary Lie algebraic H_0 into self-adjoint form (whenever such a transformation is possible). Of course, if W_λ is an invariant subspace for H_0, then W_λ multiplied by the gauge factor will be an invariant subspace for the Hamiltonian, H.

The above generalized construction involves two complications not encountered in the 1-dimensional case. First, every 1-dimensional, second-order differential operator can be related to a self-adjoint operator by a gauge transformation. Essentially, the reason for this is that all 1-dimensional 1-forms are closed, and of course this is no longer true in higher dimensions.

Every locally defined second order differential operator can be expressed in local coordinates as

$$H_0 = \sum_{ij} g^{ij} \partial_{ij} + \sum_i h^i \partial_i + f.$$

If its symbol, g^{ij} is non-degenerate, then the operator can also be expressed invariantly as

$$H_0 = \Delta + \vec{V} + F,$$

where Δ is the Laplace-Beltrami operator associated with the pseudo-Riemannian metric structure g^{ij}, and where $\vec{V}$ and F are, respectively, a vector field and a function. It is not hard to check that H_0 can be gauge-transformed into a Schrödinger operator if and only if $\vec{V}$ is a gradient vector field (w.r.t. the g^{ij} metric structure). Thus, the condition that H_0, defined as per (1), be gauge-equivalent to a Schrödinger operator imposes a severe restriction on the choice of the C^{ab} and B^a coefficients.

Second, in higher dimensions the metric geometry engendered by the symbol of a second-order differential operator is not in general Euclidean. If one imposes the condition of vanishing curvature, then the choice of admissible C^{ab} coefficients is further restricted.

1.2. *Turbiner's Conjecture.* The above considerations make clear that Euclidean, Lie algebraic Hamiltonians are a rather small and particular class of operators. Surprisingly, in 2 dimensions this class appears to possess an additional property, namely the corresponding Schrödinger equation can be solved by a separation of variables. This observation was first made by A. Turbiner, who conjectured the following [23]:

CONJECTURE 1.1 (TURBINER). *Let H be a Lie-algebraic Schrödinger operator defined on a 2-dimensional manifold. If the symbol of H engenders a Euclidean geometry, i.e. if the corresponding Gaussian curvature is zero, then the spectral equation $H\psi = E\psi$ can be solved by a separation of variables.*

Note that in the above statement and throughout the remainder of the article, separation of variables will be taken to mean separation relative to an orthogonal coordinate system [15].

The following example will serve to illustrate the conjecture and will provide a good reference point for further discussion. Let α_1 denote the Lie algebra of affine transformation of the real line. The usual realization of this Lie algebra is by vector fields ∂_u and $u\partial_u$. The present example is based on the related realization of $\alpha_1 \oplus \alpha_1$ by vector fields on $\mathbb{R}^2$:

$$T_1 = \partial_u, \quad T_2 = u\partial_u, \quad T_3 = \partial_v, \quad T_4 = v\partial_v.$$

Consider the following Lie algebraic operator:

$$-H_0 = T_1^2 + 2\{T_2 + T_4, T_3\} + a(2T_2 + 4T_4) + (4b + 2)T_3,$$

where the curly brackets denote the symmetric anti-commutator, and where a and b are real parameters. Taking (u, v) as the local coordinates, the symbol of this operator (up to sign) is given by

$$(2) \qquad \begin{pmatrix} 1 & 2u \\ 2u & 4v \end{pmatrix}.$$

Interpreting the above matrix as the contravariant form of a pseudo-Riemannian metric tensor, g^{ij}, one can easily check that the curvature is zero. Cartesian coordinates, call them (x, y), are given by

$$(3) \qquad u = x, \quad v = x^2 + y^2$$

A straight-forward calculation shows that $-H_0$ can be rewritten as

$$\Delta + \nabla\big(av + b\log(v - u^2)\big),$$

where Δ and ∇ are given with respect to the metric tensor (2). Switching to Cartesian coordinates, one obtains

$$-H_0 = \partial_{xx} + \partial_{yy} + \nabla\big(a(x^2 + y^2) + 2b\log|y|\big).$$

Clearly, H_0 is gauge-equivalent to a Schrödinger operator. The necessary gauge factor is e^{σ}, where

$$\sigma = \frac{a}{2}(x^2 + y^2) + b\log|y|,$$

and the corresponding Schrödinger operator is

$$H = -\partial_{xx} - \partial_{yy} + 2a(1 + b) + a^2(x^2 + y^2) + b(b - 1)y^{-2}.$$

Notice that both H_0 and the corresponding Schrödinger operator separate in Cartesian coordinates.

Switching to polar coordinates, r and θ, one has

$$H_0 = -\partial_{rr} - r^{-1}\partial_r - r^{-2}\partial_{\theta\theta} - \nabla\big(ar^2 + 2b\log(r) + 2b\log|\sin(\theta)|\big),$$
$$H = -\partial_{rr} - r^{-1}\partial_r + 2a(1 + b) + a^2r^2 + r^{-2}\{-\partial_{\theta\theta} + b(b - 1)\sin^{-2}(\theta)\}.$$

It is evident from the above expressions that H_0 and H also separate in polar coordinates.

1.3. *Summary of results.* The purpose of the present article is to show that Turbiner's conjecture is in general false, but that one can salvage the conjecture by adding two extra assumptions. The first of these assumptions is a somewhat technical compactness condition that serves to guarantee the completeness of the metric structure induced by the operator's symbol. The second assumption is that the generating Lie algebra acts imprimitively, *i.e.* that there exists an invariant foliation. Imprimitivity is an indispensable assumption; a counter-example to the original conjecture based on primitive actions will be given in Section 5.2. Indeed, the strength of the imprimitivity assumption is such that in addition to implying separation of variables, it also implies that the coordinates of separation will be either Cartesian or polar.

Imprimitivity plays such a decisive role for Turbiner's conjecture because of the following fact (Corollary 2.16): the perpendicular distribution of the invariant foliation is totally geodesic. In other words, a geodesic that is "launched" in a direction perpendicular to the invariant foliation will remain perpendicular throughout its evolution. Now on a 2-dimensional manifold a non-trivial foliation must be 1-dimensional. Hence the perpendicular distribution must be 1-dimensional as well, and is therefore a foliation in its own right. Hence, if the metric tensor is Euclidean, and comes from the symbol of a Lie algebraic operator generated by imprimitive actions, then the leaves of the perpendicular foliation will be straight lines.

Now there are infinitely many non-isomorphic ways to foliate a small neighborhood of the Euclidean plane by straight lines. A foliation of the full plane is a different matter; one can prove that the leaves of a global foliation by lines must be mutually parallel. If one adopts a slightly more general definition of foliation, then one can prove that a global foliation must be a pencil of either parallel or coincident lines (Theorem 4.1). Thus, if one could somehow globalize the setting of the Lie-algebraic operator to the full Euclidean plane, then one could prove that the invariant foliation consists of either parallel lines, or of concentric circles. The existence of a separating coordinate system—Cartesian in the first case, and polar in the second—readily follows from this fact.

It turns out that such a globalization is always possible. The symbol of a Lie algebraic operator is a tensor that may possess degenerate points. Indeed, the signature of the tensor can change as one "crosses" the locus of degeneracy. One must therefore throw away the locus of degeneracy—it is at most a codimension 1 subvariety—and take the domain of the Lie-algebraic Hamiltonian to be one of those remaining open components where the signature is positive-definite. Such a domain is not, in general, isometric to the Euclidean plane, but with the help of the completeness assumption alluded to earlier, one can prove that the domain plus a portion of its boundary is isometric to the Euclidean plane modulo a discrete group of isometries.

It may help to think of this result as a generalization of the classical Killing-Hopf theorem [9] [21], which states that a complete Riemannian manifold of constant curvature is isometric to the quotient of one the standard space-forms: a sphere, Euclidean space, or hyperbolic space. However, the present context demands a generalization of the notion of pseudo-Riemannian manifold—one that permits the metric tensor to have certain well-behaved singularities. For lack of better terminology, the name almost-Riemannian manifold will be used here to refer to manifolds equipped with such a metric tensor. It turns out that the symbol of a Euclidean Lie algebraic operator is an almost Riemannian metric tensor, and one can therefore use the generalized Killing-Hopf theorem to obtain the globalization mentioned above.

The proof of the generalized Killing-Hopf Theorem is too long, and too distinct in scope to be reasonably included in the present article. The proof is available in the author's PhD dissertation [16], and will appear in a subsequent publication. The present article will therefore limit itself to the relevant definitions and to some hopefully illuminating examples.

The example of Section 1.2 conveniently illustrates the preceding summary. Regarding the geometric manifestation of imprimitivity, note that the action of $\alpha_1 \oplus \alpha_1$ is doubly imprimitive. The two invariant foliations are $\{u = \mathrm{const.}\}$ and $\{v = \mathrm{const.}\}$. The perpendicular foliations are, respectively, $\{v - u^2 = \mathrm{const.}\}$ and $\{v/u^2 = \mathrm{const.}\}$, or in Cartesian coordinates, $\{y = \mathrm{const.}\}$ and $\{y/x = \mathrm{const.}\}$ respectively. Evidently, both are foliations by straight lines.

Regarding the notion of an almost-Riemannian manifold, consider the metric tensor given in (2). This contravariant metric tensor has a locus of degeneracy, namely $\{v = u^2\}$. The signature is positive definite for $v > u^2$ and mixed for $v < u^2$. The (u, v) plane equipped with this metric tensor turns out to be an instance of a complete almost-Riemannian manifold. The domain $\{v \geq u^2\}$ is isometric to the Euclidean plane modulo the reflection $y \mapsto -y$, with (3) giving the corresponding projection. This is the sort of phenomenon described by the generalized Killing-Hopf theorem.

How does separation of variables follow from all this? Imprimitivity and the generalized Killing-Hopf theorem allow one to conclude that the leaves of the invariant foliation are either parallel lines, or concentric circles. Indeed, the example of Section 1.2 was chosen to conveniently illustrate both of these possibilities at the same time. Now the infinitesimal criterion for the invariance of a foliation $\{\lambda = \mathrm{const.}\}$ is that the action of a vector field on λ must give back a function of λ. It follows immediately that an operator generated by imprimitively acting vector fields will enjoy the same property. Therefore, for the example under discussion one is guaranteed that $H_0(x)$ is a function of x and that $H_0(r)$, where $r^2 = x^2 + y^2$, is a function of r. Since H_0 is gauge-equivalent to a Schrödinger operator, it must be of the form $-\Delta + \nabla \sigma$. Now certainly $\Delta(x)$ is a function of x and $\Delta(r)$ is a function of r, and hence the same is true if one operates on these two functions with $\nabla \sigma$. But, $\nabla \sigma(x) = \partial \sigma / \partial x$ and $\nabla \sigma(r) = \partial \sigma / \partial r$, where partial derivatives are taken with respect to Cartesian coordinates in the first instance, and with respect to polar coordinates in the second. Hence $\partial \sigma / \partial x$ must be a function of x and $\partial \sigma / \partial r$ must be a function of r. Therefore, one can break up σ into a sum of a function of x and a function of y, or into a sum of a function of r and a function of θ. This makes it quite clear why H_0 and the corresponding Schrödinger operator separate in Cartesian coordinates, and in polar coordinates. The crucial idea of the above argument bears repeating: if a 2-dimensional, Euclidean, Lie-algebraic operator is imprimitively generated, then the leaves of the invariant foliation must either be parallel lines, or concentric circles.

The organization of the remainder of the article is as follows. Section 2.1 introduces the notation and concepts necessary for the discussion of Lie-algebraic operators. Section 2.2 is concerned with the relevant properties of the corresponding pseudo-Riemannian geometry. Section 3.1 gives the definition of an almost Riemannian manifold and Section 3.2 illustrates the definition with examples. Section 4 is devoted to the proof of the result about generalized foliations of the Euclidean plane by straight lines. Section 5.1 proves the conjecture with the extra imprimitivity and completeness assumptions, while Section 5.2 gives a counter-example to the general form of Turbiner's conjecture.

2. **The geometry of Lie-algebraic metrics.**

2.1. *Preliminaries.* Recall from the introduction that a differential operator is called *Lie-algebraic* if it can be generated by a finite-dimensional Lie algebra of first-order differential operators. The purpose of the present section is to describe some properties of the metric geometry induced by the symbol of a second-order Lie algebraic operator.

The general form of such an operator, H_0, is given in (1). The symbol, $\sigma(H_0)$, is completely specified by the second-order coefficients, C^{ab}. Indeed,

$$(4) \qquad\qquad \sigma(H_0) = \sum_{ab} C^{ab} V_a \otimes V_b.$$

One endows the domain of H_0 with the structure of a pseudo-Riemannian manifold by interpreting $\sigma(H_0)$ as the contravariant form of a metric tensor. A metric tensor obtained in this manner will be henceforth referred to as Lie algebraic.

In order for a Lie algebraic metric tensor to be non-degenerate it is necessary that the vector fields V_a span the tangent space of the underlying manifold. This is nothing but the infinitesimal criterion for transitive action, and therefore the natural setting for a Lie-algebraic metric is a homogeneous space. To that end, let $\mathbf{G}$ be a real Lie group and $\mathbf{H}$ a closed subgroup. Let $\mathbf{M} = \mathbf{G}/\mathbf{H}$ and $\pi\colon \mathbf{G} \to \mathbf{M}$ denote, respectively, the homogeneous space of right cosets, and the canonical projection. For $a \in \mathfrak{g}$ let a^{L} and a^{R} denote, respectively, the corresponding left- and right- invariant vector fields on $\mathbf{G}$, and $\mathfrak{g}^{\mathrm{L}}$ and $\mathfrak{g}^{\mathrm{R}}$ the collections of all such. To avoid any possible confusion, it should be noted that $\mathfrak{g}^{\mathrm{L}}$ corresponds to *right* group actions, and $\mathfrak{g}^{\mathrm{R}}$ to *left* ones. Let $a^{\pi} = \pi_*(a^{\mathrm{L}})$, $a \in \mathfrak{g}$ denote the realization of $\mathfrak{g}$ by projected vector fields (*i.e.* by infinitesimal automorphisms). It will also be assumed that $\mathfrak{h}$ does not contain any ideals of $\mathfrak{g}$. This will ensure that $a \mapsto a^{\pi}$ is a faithful realization.

The coefficients C^{ab} that specify a Lie-algebraic metric tensor can be described in a basis-independent manner as an element $C \in \mathrm{Sym}^2 \mathfrak{g}$. The corresponding Lie algebraic metric tensor is nothing but $\pi_*(C^{\mathrm{L}})$; henceforth it will be denoted simply as C^{π}.

One must still contend with the fact that C^{π} may be a degenerate tensor. The projection $\pi\colon \mathbf{G} \to \mathbf{M}$, naturally induces a vertical distribution, $\mathfrak{h}^{\mathrm{R}}$, on $\mathbf{G}$. Dually, there is the cotangent sub-bundle of horizontal 1-forms, $(\mathfrak{h}^{\perp})^{\mathrm{R}}$. This sub-bundle is spanned by right-invariant, differential 1-forms α^{R}, such that $\alpha \in \mathfrak{g}^*$ annihilates $\mathfrak{h}$. The extra information given by C induces a decomposition of the tangent bundle of $\mathbf{G}$, and allows one to speak of horizontal vectors and vertical 1-forms. The decomposition is given by

$$(5) \qquad\qquad \mathbf{T}\mathbf{G} = \mathfrak{h}^{\mathrm{R}} \oplus C^{\mathrm{L}}(\mathfrak{h}^{\perp})^{\mathrm{R}}.$$

PROPOSITION 2.1. *The above decomposition fails precisely in those fibers of $\pi\colon \mathbf{G} \to \mathbf{M}$, where the metric tensor, C^{π}, is degenerate. In other words, the projection of the horizontal distribution, $C^{\mathrm{L}}(\mathfrak{h}^{\perp})^{\mathrm{R}}$, spans the tangent space of $\mathbf{M}$ precisely at those points where the metric tensor is non-degenerate.*

For the remainder of this section fix a $C \in \mathrm{Sym}^2 \mathfrak{g}$ such that the decomposition (5) does not fail identically. In particular, the rank of C must be greater or equal to $\dim(\mathfrak{g}) - \dim(\mathfrak{h})$. Let $\mathbf{M}_0 \subset \mathbf{M}$ denote the submanifold where C^π is non-degenerate, and regard $\mathbf{M}_0$ as a pseudo-Riemannian manifold with metric tensor C^π.

2.2. *The adapted frame.* The goal of the present section is to show that the geodesics of a Lie-algebraic metric can be described as projections of certain vector fields on the the group $\mathbf{G}$. The key tool in this section is a frame of $\mathbf{TG}$ adapted to the horizontal-vertical decomposition given in (5).

DEFINITION 2.2. A vector field on $\mathbf{G}$ of the form $C^{\mathrm{L}} \alpha^{\mathrm{R}}$, where $\alpha \in \mathfrak{h}^\perp$, will be called *horizontal*. A vector field of the form a^{R}, where $a \in \mathfrak{h}$, will be called *vertical*. Consider an adapted basis, $a_1, \ldots, a_r$, of $\mathfrak{g}$, where the last $r - n$ entries form a basis of $\mathfrak{h}$. Let $\alpha^1, \ldots, \alpha^r$ be the adapted dual basis, where the first n entries span the space of annihilators of $\mathfrak{h}$. With respect to such a basis, denote the horizontal vector fields by $H^i = C^{\mathrm{L}}(\alpha^i)^{\mathrm{R}}$, where $i = 1, \ldots, n$, and the vertical vector fields by $V_i = a_i^{\mathrm{R}}$, where $i = n + 1, \ldots, r$. As per Proposition 2.1, away from the degenerate fibers the following vector fields form a basis of $\mathbf{TG}$:

$$H^1, \ldots, H^n, V_{n+1}, \ldots, V_r.$$

This basis will be called *an adapted frame of $\mathbf{G}$ relative to C.*

The structure equations of the adapted frame naturally break up into three types: vertical-vertical, vertical-horizontal, and horizontal-horizontal. The first two types are essentially uninteresting. They are related to the structure constants, call them c_{ij}^k, of $\mathfrak{g}$:

$$[a_i, a_j] = \sum_k c_{ij}^k a_k, \quad \text{where } i, j, k = 1, \ldots, r.$$

PROPOSITION 2.3. *The vertical-vertical, and the vertical-horizontal structure constants of the adapted frame are given by*

$$[V_i, V_j] = -\sum_k c_{ij}^k V_k, \quad \text{and} \quad [V_i, H^k] = \sum_j c_{ij}^k H^j.$$

By contrast, the horizontal-horizontal type of structure coefficients are not, in general, constants. They will be denoted by A and B according to

$$[H^i, H^j] = \sum_k 2 A_k^{ij} H^k + \sum_l B^{ijl} V_l.$$

The factor of 2 in the above equation is there to simplify some later formulas. These structure coefficients turn out to play a fundamental role in the description of the metric geometry of C^π.

The horizontal vector fields are not, in general, projectable. Therefore, an expression of the form $\pi_*(H^i)$ is, at best, a section of the pullback bundle, $\pi^*(\mathbf{TM})$. To describe the

metric geometry, it is necessary to pull back to $\mathbf{G}$ the covariant derivative operator, ∇, of the Levi-Civita connection on $\mathbf{M}_0$, so that it can operate on such sections. Speaking geometrically, this is equivalent to pulling back to $\mathbf{G}$ the parallel transport operators along paths on $\mathbf{M}_0$.

DEFINITION 2.4. Let γ be a path on $\mathbf{G}$, and let X be a section of $\mathbf{TM}_0$ along $\pi \circ \gamma$. The pullback of the covariant derivative, it will be denoted by $\tilde{\nabla}$, is defined by

$$\tilde{\nabla}_{\dot{\gamma}}(X \circ \pi) = (\nabla_{\pi_*(\dot{\gamma})}X) \circ \pi.$$

Let X be a vector field on $\mathbf{G}$. In the sequel it will be convenient to abbreviate $\pi_*(X)$ as X^π, and $\tilde{\nabla}X^\pi$ simply as $\tilde{\nabla}X$. The distinction is important. The reader should keep in mind that $\tilde{\nabla}$ operates on sections of $\pi^*(\mathbf{TM}_0)$, and *not* on sections of $\mathbf{TG}$. The following abbreviation will also be useful:

$$X \cdot Y = X^\pi \cdot Y^\pi,$$

where X, Y are vector fields on $\mathbf{G}$, and the dot on the right refers to the inner product on $\mathbf{M}_0$.

PROPOSITION 2.5. *If X is a projectable vector field on $\mathbf{G}$, then*

$$\tilde{\nabla}X = (\nabla X^\pi) \circ \pi.$$

PROPOSITION 2.6. *Let X, Y be vector fields, and f a function on $\mathbf{G}$. The pullback operator satisfies the following analogues of the standard identities for the covariant derivative:*

$$\tilde{\nabla}_{fX}Y = f\tilde{\nabla}_X Y,$$
$$\tilde{\nabla}_X(fY) = f\tilde{\nabla}_X Y + X(f)Y^\pi.$$

PROPOSITION 2.7. *The inner product on $\mathbf{M}_0$ is compatible with $\tilde{\nabla}$. Furthermore, $\tilde{\nabla}$ acts in a torsionless manner. More formally, let X, Y_1, Y_2 be vector fields on $\mathbf{G}$. Then,*

$$X(Y_1 \cdot Y_2) - (\tilde{\nabla}_X Y_1) \cdot Y_2 - Y_1 \cdot (\tilde{\nabla}_X Y_2) = 0,$$
$$\tilde{\nabla}_{Y_1}Y_2 - \tilde{\nabla}_{Y_2}Y_1 - [Y_1, Y_2]^\pi = 0.$$

PROOF.   This is a direct consequence of the preceding two propositions.   ∎

With these preliminaries out of the way, it is possible to derive the connection coefficients in terms of the adapted frame.

PROPOSITION 2.8. *The parallel translation of a horizontal vector in a vertical direction is given by the flow of the corresponding vertical vector field. More formally,*

$$\tilde{\nabla}_{V_i}H^j = [V_i, H^j]^\pi = \sum_k c_{ik}^j (H^k)^\pi.$$

PROOF. This is a consequence of the second identity in Proposition 2.7 ∎

To derive the formula for the horizontal-horizontal connection coefficients will require a slight technical diversion. For i, j, k between 1 and n, define

$$T^{ijk} = d(\alpha^k)^R(H^i, H^j),$$

where d is the usual exterior derivative. Note that the resulting expression is skew-symmetric in the first two indices.

LEMMA 2.9. *The symbol T^{ijk} satisfies the following identities:*

(6)
$$H^i(H^j \cdot H^k) = T^{ijk} + T^{ikj},$$

(7)
$$[H^i, H^j] \cdot H^k = T^{ijk} - T^{jki} - T^{kij}.$$

PROOF. Let $a \in \mathfrak{g}$, and $\alpha \in \mathfrak{h}^\perp$. From $L_{a^L}(\alpha^R) = 0$, and from the homotopy formula for the Lie derivative it follows that for every vector field, X, on **G**

(8)
$$X(\alpha^R a^L) = d\alpha^R(X, a^L).$$

From

$$H^i \cdot H^j = \sum_{k,l=1}^r C^{kl} \langle (\alpha^i)^R ; (a^k)^L \rangle \langle (\alpha^j)^R ; (a^l)^L \rangle.$$

and from (8) one derives (6). Using the fact that

$$[H^i, H^j] \cdot H^k = \alpha^{kR}([H^i, H^j]),$$

and the standard formula for the exterior derivative, one obtains

$$[H^i, H^j] \cdot H^k = H^i(H^j \cdot H^k) - H^j(H^i \cdot H^k) - d(\alpha^k)^R(H^i, H^j).$$

From this and from (6), Equation (7) follows immediately. ∎

PROPOSITION 2.10. *The covariant derivative of a horizontal vector-field in a horizontal direction is given by*

$$\tilde{\nabla}_{H^i} H^j = \frac{1}{2}[H^i, H^j]^\pi = \sum_k A_k^{ij}(H^k)^\pi.$$

PROOF. As a consequence of Proposition 2.7, the analogue of the standard formula for the covariant derivative of the Levi-Civita connection remains valid for non-projectable vector fields. The formula in question is

$$2\tilde{\nabla}_{H^i} H^j \cdot H^k = H^i(H^j \cdot H^k) + H^j(H^i \cdot H^k) - H^k(H^i \cdot H^j)$$
$$- H^i \cdot [H^j, H^k] - H^j \cdot [H^i, H^k] + H^k \cdot [H^i, H^j]$$

Combining the above with the identities in Lemma 2.9 gives

$$2(\alpha^k)^{\mathrm{R}}(\nabla_{H^i}H^j) = T^{ijk} - T^{jki} - T^{kij}$$
$$= (\alpha^k)^{\mathrm{R}}([H^i, H^j])$$

By fixing i, j, and varying k, one sees that $\nabla_{H^i}H^j$ must match $\frac{1}{2}[H^i, H^j]^\pi$. ∎

The effort that went into the development of the adapted frames machinery is justified by the next theorem.

THEOREM 2.11. *The integral curves of horizontal vector fields project down to geodesics on* $\mathbf{M}_0$.

PROOF. This theorem is a direct consequence of Proposition 2.10, which implies that $\tilde{\nabla}_{H^i}H^i = 0$. ∎

A word of caution is required at this point. A horizontal vector field is not, in general, projectable, and thus does not give a foliation of $\mathbf{M}_0$ by geodesic trajectories. The theorem merely states that if γ is a path in $\mathbf{G}$ such that $\dot{\gamma} = (C^{\mathrm{L}}\alpha^{\mathrm{R}}) \circ \gamma$ for some fixed $\alpha \in \mathfrak{h}^\perp$, then the projection $\pi \circ \gamma$ is a geodesic down on $\mathbf{M}_0$.

Theorem 2.11 highlights the group-theoretic origins of the geometry induced by a Lie-algebraic metric tensor. Indeed, in order to obtain the geodesics on $\mathbf{M}_0$ there is no need to compute the Christoffel symbols and then to solve the second order geodesic equation. The theorem shows that one need only integrate a certain vector field on the group, and then project the resulting trajectories down to $\mathbf{M}_0$. Such a computation will be illustrated in Section 2.4.

2.3. *Geometric consequences of imprimitivity.* In the preceding section it was indicated that the horizontal vector fields, H^i, are not, in general, projectable. This is a pity, because, otherwise there would be a foliation of $\mathbf{M}_0$ by geodesic trajectories. The purpose of the present section is to discuss a condition that allows for something almost as good—the projectability of a portion of the horizontal distribution. The condition in question is the imprimitivity of the group action.

Recall that $\mathbf{G}$ is said to act imprimitively if there exists a $\mathbf{G}$-invariant foliation on $\mathbf{M}$. A more algebraic criterion is given by the following [17] [7].

PROPOSITION 2.12. *Suppose that the isotropy subgroup,* $\mathbf{H}$*, is connected. The* $\mathbf{G}$-*action on* $\mathbf{M} = \mathbf{G}/\mathbf{H}$ *is imprimitive if and only if* $\mathfrak{h}$ *is not a maximal subalgebra of* $\mathfrak{g}$*, i.e. if and only if there exists a Lie algebra* $\mathfrak{f}$ *that is properly intermediate between* $\mathfrak{h}$ *and* $\mathfrak{g}$. *If an intermediate subalgebra,* $\mathfrak{f}$*, does exist, then the invariant distribution is given by* $\pi_*(\mathfrak{f}^{\mathrm{R}})$.

For the rest of the section suppose that $\mathbf{G}$ acts imprimitively on $\mathbf{M}$. Fix an intermediate subalgebra, $\mathfrak{f}$. Let $\Lambda = \pi_*(\mathfrak{f}^{\mathrm{R}})$ be the $\mathbf{G}$-invariant integrable distribution on $\mathbf{M}$, and let $\Lambda^\perp$ denote the distribution of tangent vectors that are perpendicular to Λ.

PROPOSITION 2.13. *The projection of a horizontal vector field* $C^{\mathrm{L}}\alpha^{\mathrm{R}}$*, where* $\alpha \in \mathfrak{f}^\perp$ *belongs to* $\Lambda^\perp$. *Indeed,* $\Lambda^\perp$ *is spanned by these projections.*

PROOF. Fix an $\alpha \in \check{\mathfrak{f}}^{\perp}$, a $p \in \mathbf{M}_0$, and consider $(C^{\mathrm{L}}\alpha^{\mathrm{R}})_q$ at various points, $q \in \mathbf{G}$, in the fiber above p. From Proposition 2.12 one has $\pi_*(\check{\mathfrak{f}}_q^{\mathrm{R}}) = \Lambda_p$ at all q above p. Since $(C^{\mathrm{L}}\alpha^{\mathrm{R}})_q \cdot u$, where $u \in \mathrm{T}_q\mathbf{G}$, is just $\alpha^{\mathrm{R}}{}_q(u)$, one can infer that the projection of $(C^{\mathrm{L}}\alpha^{\mathrm{R}})_q$ is perpendicular to Λ_p for all q above p. The non-degeneracy assumption on C_p^{π} implies that $\dim(\Lambda^{\perp})$ is equal to the codimension of $\mathfrak{f}$ in $\mathfrak{g}$, and hence is equal to $\dim(\check{\mathfrak{f}}^{\perp})$. Therefore, the projection of $C^{\mathrm{L}}(\check{\mathfrak{f}}^{\perp})^{\mathrm{R}}$ spans $\Lambda^{\perp}$. ∎

The present context demands the following generalization of the usual notion of a totally geodesic submanifold.

DEFINITION 2.14. A distribution, D, of a pseudo-Riemannian manifold will be called *totally geodesic* whenever the following is true: if a geodesic belongs to D at one point then that geodesic is an integral manifold of D.

It is now possible to state the key result of the present section.

THEOREM 2.15. *If Λ is an invariant foliation, then $\Lambda^{\perp}$ is totally geodesic.*

PROOF. Let a geodesic, γ be given. By Theorem 2.11, γ is the projection of an integral curve of a horizontal vector field, $C^{\mathrm{L}}\alpha^{\mathrm{R}}$, $\alpha \in \mathfrak{h}^{\perp}$. If a geodesic belongs to $\Lambda^{\perp}$ at one point, then α must be in $\check{\mathfrak{f}}^{\perp}$. Consequently, by Proposition 2.13, γ belongs to $\Lambda^{\perp}$ everywhere. ∎

The following result is needed in the proof of Turbiner's conjecture. Recall that if $\Lambda^{\perp}$ is rank 1, *i.e.* if the codimension of $\mathfrak{f}$ in $\mathfrak{g}$ is equal to 1, then $\Lambda^{\perp}$ is integrable. In particular this occurs if the codimension of $\mathfrak{h}$ in $\mathfrak{g}$ is 2, *i.e.* when $\mathbf{M}$ is two-dimensional.

COROLLARY 2.16. *If $\mathrm{rank}(\Lambda^{\perp}) = 1$, then the integral curves of $\Lambda^{\perp}$ are geodesic trajectories. Indeed, in this case the geodesics are given by the projection of integral curves of $C^{\mathrm{L}}\alpha^{\mathrm{R}}$ where $\alpha \in \mathfrak{g}^*$ is any non-zero annihilator of $\mathfrak{f}$.*

2.4. *An example.* At this point it will be helpful to illustrate the concepts and formulas of the preceding sections with a concrete example. This example will be based on the two-dimensional linear representation of $\mathrm{GL}_2\,\mathbb{R}$. This group is sufficiently "small" so as to permit concrete, manageable formulas.

The computations will be based on the group coordinates,

$$(9) \qquad \begin{pmatrix} x & y \\ z & w, \end{pmatrix}$$

and on the following basis of the lie algebra, $\mathfrak{gl}_2\mathbb{R}$:

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad a_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad a_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The homogeneous space, $\mathbf{M}$, is $\mathbb{R}^2$ minus the origin, and the projection from the group to $\mathbf{M}$ will be the operation of taking the first row of the coordinate matrix (9). As such, the

group coordinates x, y also serve as coordinates on $\mathbf{M}$. This setup induces the following vector field realization of $\mathfrak{gl}_2\mathbb{R}$:

$$a_1^\pi = x\partial_x, \quad a_2^\pi = x\partial_y, \quad a_3^\pi = y\partial_x, \quad a_4^\pi = y\partial_y.$$

The natural basepoint of $\mathbf{M}$ is $x = 1$, $y = 0$. The isotropy algebra at this point is spanned by a_3 and a_4.

Define $C \in \mathrm{Sym}^2\,\mathfrak{g}$ by

$$C = a_1^2 + a_4^2 - a_1 \odot a_4 + a_1 \odot a_3 + a_2 \odot a_4,$$

and consider the corresponding Lie algebraic metric tensor

$$C^\pi = \begin{pmatrix} x^2 + 2xy & -xy \\ -xy & 2xy + y^2 \end{pmatrix}.$$

This is a Euclidean metric with Cartesian coordinates (ξ, η) given by

(10) $$x = e^\xi \sin^2(\eta), \quad y = e^\xi \cos^2(\eta).$$

Since $\mathrm{GL}_2\,\mathbb{R}$ is an open subset of the affine space of two-by-two matrices, one can represent the tangent vectors of the group by matrices, and conveniently describe vector fields as matrices with entries that are functions of x, y, z, w. Thus, to get a left- (respectively right-) invariant vector field one simply left (respectively right) multiplies a constant matrix by the generic group element (9). For instance, the right-invariant vector fields, $a_1^\mathrm{R}, \ldots, a_4^\mathrm{R}$, are represented by

$$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} z & w \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ z & w \end{pmatrix}.$$

To describe the horizontal vector fields it is necessary to have an expression for the contraction of right-invariant vector fields and left-invariant 1-forms. To this end one uses the formula $(\alpha^i)^\mathrm{R}(a_j)^\mathrm{L} = \mathrm{Ad}_j^i$. The adjoint representation matrix is

$$\frac{1}{xw - yz} \begin{pmatrix} xw & -xz & yw & -yz \\ -xy & x^2 & -y^2 & xy \\ wz & -z^2 & w^2 & -wz \\ -yz & xz & -yw & xw \end{pmatrix}.$$

From this one computes the horizontal vector fields to be

$$H^1 = \begin{pmatrix} x & y \\ z & w \end{pmatrix} + \frac{w + z}{xw - yz} \begin{pmatrix} 2xy & -2xy \\ xw + yz & -xw - yz \end{pmatrix}$$

$$H^2 = -\frac{x + y}{xw - yz} \begin{pmatrix} 2xy & -2xy \\ xw + yz & -xw - yz \end{pmatrix}.$$

By Lemma 2.9, $H^i(H^i \cdot H^i) = 0$, and hence

$$\kappa_1 = H^1 \cdot H^1 + 1 = \frac{2xy(w + z)^2}{(xw - yz)^2}, \quad \kappa_2 = H^2 \cdot H^2 = \frac{2xy(x + y)^2}{(xw - yz)^2},$$

are constants of motion of H^1, H^2, respectively.

The next step will be to illustrate Theorem 2.11 by integrating the horizontal vector fields and showing that their integral curves project to straight lines on **M**. The projections of H^1, H^2, are represented by the first rows of the respective matrix representations. Hence, the projection of H^1 is given by

$$\frac{dx}{dt} = x + \sqrt{2xy\kappa_1}, \quad \frac{dy}{dt} = y - \sqrt{2xy\kappa_1},$$

These equations can be solved by rewriting them as

$$\frac{d}{dt}(x+y) = x+y, \quad \frac{d}{dt}\left(\sqrt{\frac{x}{y}}\right) = \sqrt{\frac{\kappa_1}{2}}\left(\frac{x}{y}+1\right).$$

The solutions in Cartesian coordinates are

$$\eta = \sqrt{\frac{\kappa_1}{2}}\,\xi + \text{const.}$$

The projection of H^2 is given by

$$\frac{dx}{dt} = -\sqrt{2xy\kappa_2}, \quad \frac{dy}{dt} = \sqrt{2xy\kappa_2}.$$

The solutions are simply

$$\xi = \text{const.}$$

Thus one sees that the integral curves of H^1 and H^2 project down to straight lines.

The linear $\mathrm{GL}_2\,\mathbb{R}$ actions considered here are imprimitive. The invariant foliation is given by the radial lines, $y/x = \text{const}$. In Cartesian coordinates it is given by $\eta = \text{const}$. According to Proposition 2.12 the invariant foliation corresponds to the subalgebra spanned by

$$a_1^{\mathrm{R}},\ a_3^{\mathrm{R}},\ a_4^{\mathrm{R}}.$$

The annihilators of this subalgebra are spanned by $(\alpha^2)^{\mathrm{R}}$. Thus, according to Corollary 2.16, H^2 must project to a foliation by straight lines that are perpendicular to the invariant foliation. This is in accordance with the above calculations, which show that projections of the integral curves of H^2, namely $\xi = \text{const}$, are perpendicular to the invariant foliation, namely $\eta = \text{const}$.

3. **Almost-Riemannian manifolds.**

3.1. *Definitions and motivation.* As was mentioned in the introduction, one cannot directly interpret a type $(2,0)$ Lie algebraic tensor as a conventional metric tensor. The difficulty is caused by the presence of points where the contravariant tensor is degenerate. The inverse tensor is singular at such points, and consequently the tangent space lacks a meaningful inner product. The present context therefore requires a suitable generalization of pseudo-Riemannian structure, one that will embrace the presence of

degeneracies in the contravariant metric tensor, but do so in a way that results in objects that are reasonably well behaved.

To this end let $\mathbf{M}$ be a real, analytic manifold and g^{ij} a type $(2, 0)$ tensor field. Let $\mathbf{S}_g \subset \mathbf{M}$ denote the corresponding locus of degeneracy; in a chart of local coordinates this is just the set of points where $\det g^{ij} = 0$. The analyticity requirement means that $\mathbf{S}_g$ is either the empty set, a codimension 1 subvariety, or all of $\mathbf{M}$. Set $\mathbf{M}_0 = \mathbf{M} \setminus \mathbf{S}_g$ and suppose that g^{ij} is not identically degenerate. Consequently, $\mathbf{M}_0$ is an open, dense subset of $\mathbf{M}$, the elements of $\mathbf{S}_g$ are boundary points of $\mathbf{M}_0$, and the connected components of $\mathbf{M}_0$ are pseudo-Riemannian manifolds.

Let u, v be analytic vector fields such that the corresponding plane section, $u \wedge v$, is non-degenerate on $\mathbf{M}_0$, *i.e.* such that $|u|^2|v|^2 - (u \cdot v)^2 \neq 0$. Let $K(u \wedge v)$ denote the corresponding sectional curvature function.

DEFINITION 3.1. The pair $(\mathbf{M}, g^{ij})$ will be called an *almost-Riemannian manifold* whenever for all u, v as above, $K(u \wedge v)$ has removable singularities at points of $\mathbf{S}_g$.

The following two facts follow immediately from the definition. First, if $\mathbf{M}$ is 2-dimensional, then it is enough to suppose that the Gaussian curvature has removable singularities at $\mathbf{S}_g$. Second, if sectional curvature is constant on the connected components of $\mathbf{M}_0$, then $\mathbf{M}$ is almost-Riemannian.

The degenerate points of an almost-Riemannian manifold naturally break up into two classes.

DEFINITION 3.2. A boundary point, $p \in \mathbf{S}_g$, will be called *unreachable* if all smooth curves with p as an endpoint have infinite length. Conversely, a boundary point will be called *reachable* if it can be attained by a finite length curve.

The existence of reachable boundary points necessitates a suitable generalization of the notion of completeness. For a geodesic segment $\gamma\colon (0, 1) \longrightarrow \mathbf{M}_0$, let $T > 1$ be the largest number, possibly ∞, such that γ can be extended to a geodesic with domain $(0, T)$.

DEFINITION 3.3. Let $\mathbf{R}$ be an open connected component of $\mathbf{M}_0$. One will say that $\mathbf{M}$ is complete within $\mathbf{R}$ whenever for all geodesic segments lying within $\mathbf{R}$, either $T = \infty$, or $\lim_{t \to T} \gamma(t)$ (relative to the manifold topology) is a reachable boundary point of $\mathbf{R}$.

The following result is very useful in establishing completeness of an almost-Riemannian manifold.

PROPOSITION 3.4. *Let $\mathbf{R}$ be as above. Suppose that the signature of g^{ij} is positive definite within $\mathbf{R}$, and that $\mathbf{R}$ is contained in a compact (relative to the manifold topology) subset of $\mathbf{M}$. Then, $\mathbf{M}$ is complete within $\mathbf{R}$.*

The proof of the imprimitive case of Turbiner's conjecture relies on the following generalization of the Killing-Hopf theorem [9] [21] to the almost-Riemannian context [16]. Let $\mathbf{R}$ be an open connected component where the signature of the metric is positive definite, and let $\bar{\mathbf{R}}$ denote the union of $\mathbf{R}$ and the reachable points of its boundary.

THEOREM 3.5. *Assume the following to be true:* $\dim \mathbf{M} = 2$*; the Gaussian curvature is constant;* $\mathbf{M}$ *is complete within* $\mathbf{R}$*. Let* $\mathbf{F}$ *denote one of* $\mathbb{R}^2$*,* S^2*, or* $\mathbb{H}^2$ *according to the sign of the curvature. Then, there exists an analytic map* $\Pi: \mathbf{F} \to \mathbf{M}$*, such that* $\Pi(\mathbf{F}) = \bar{\mathbf{R}}$*, and such that* g^{ij} *is the push-forward of the metric tensor on* $\mathbf{F}$*. Furthermore,* $\bar{\mathbf{R}}$ *is isometric to the quotient* $\mathbf{F}/\Gamma$*, where* Γ *is the group of isometries* ϕ *such that* $\Pi = \Pi \circ \phi$*.*

The proof of the above theorem is rather involved, and will be given in a subsequent publication. The present article will limit itself to a number of examples illustrating the salient features of the almost-Riemannian formalism. One also expects that the above theorem continues to hold in dimensions greater than 2, as well as for mixed signatures. However, at the present time this must be left as a conjecture.

3.2. *Examples.* One naturally encounters the notion of an almost-Riemannian manifold when considering the standard models of constant curvature spaces. For instance, in the Poincare model of the hyperbolic plane one has $ds^2 = v^{-2}(du^2 + dv^2)$. Note that the corresponding contravariant inverse, $v^2(\partial_u{}^2 + \partial_v{}^2)$, is analytic. Although the convention is to restrict one's attention to the domain $\{v > 0\}$, one can just as well regard the whole (u, v) plane with the given metric tensor as an instance of an almost-Riemannian manifold. The u-axis need not be discarded. It is the locus of degeneracy and consists entirely of unreachable points.

Next, consider $\mathbb{H}^2$ modeled as a hyperboloid in Lorenzian 3-space. Let u, v, w denote the coordinates on the latter, and consider pushing forward the hyperboloid's metric structure onto the (u, v) plane via the obvious projection. Equivalently, one can pull back the covariant metric tensor via the map

$$(u, v) \longmapsto (u, v, \sqrt{1 + u^2 + v^2})$$

and invert. The end result is the following contravariant tensor:

$$\begin{pmatrix} 1 + u^2 & uv \\ uv & 1 + v^2 \end{pmatrix}.$$

The determinant of the above matrix is $1 + u^2 + v^2$, and hence the locus of degeneracy is empty; it consists of "imaginary points". Consequently one can regard the (u, v) plane with the above contravariant metric tensor as an instance of an ordinary Riemannian manifold.

The situation becomes more interesting when one considers the analogous construction for positive curvature. Consider the projection of the unit sphere in Euclidean (u, v, w) space to the (u, v) plane. Pulling back the Euclidean metric along the map

$$(u, v) \longmapsto (u, v, \sqrt{1 - u^2 - v^2})$$

and inverting, one obtains the following contravariant tensor:

$$\begin{pmatrix} 1 - u^2 & -uv \\ -uv & 1 - v^2 \end{pmatrix}.$$

Now there is a non-empty locus of degeneracy, namely the circle $u^2 + v^2 = 1$, and one has no choice but to regard the (u, v) plane as an instance of an almost-Riemannian manifold. As an illustration of the generalized Killing-Hopf theorem note that the closed disk $\{u^2 + v^2 \leq 1\}$ is isometric to the quotient of the 2-sphere by the reflection along the w axis. This example also illustrates the behaviour of an almost-Riemannian manifold at the reachable locus of degeneracy. Reachable boundary points are precisely the places where the rank of the projection from the Euclidean plane to the closed disk is less than maximal, *i.e.* the places where the 2-sphere is "folded" by the projection.

Turning next to the case of zero curvature, let W be a finite reflection group acting on $\mathbb{R}^n$. The current example gives a procedure for realizing $\mathbb{R}^n/W$ as an almost-Riemannian manifold. It is well known [10] that the invariants of W are a polynomial algebra in certain basic invariants. To obtain an almost-Riemmanian manifold one simply treats the basic invariants as if they were coordinates. Consider, for example, the k-th dihedral group acting on the (x, y) plane. The algebra of invariants is generated by $u = x^2 + y^2$ and by $v = \Re\big((x + iy)^k\big)$. Pushing forward the Euclidean metric tensor via the map $(x, y) \longmapsto (u, v)$ one obtains a tensor whose entries are the 3 possible products of ∇u and ∇v expressed as functions of u and v. An easy calculation shows that this contravariant tensor is

$$(11) \qquad \begin{pmatrix} 4u & 2kv \\ 2kv & k^2 u^{k-1} \end{pmatrix}.$$

The locus of degeneracy is the cusp curve $v^2 = u^k$. The region $\{u^k \geq v^2\}$ is isometric to one of the $2k$ closed wedges carved out by the mirror lines of the reflections in the dihedral group. Once again one sees that reachable boundary points "downstairs" correspond to mirror lines "upstairs".

It is also possible to use an almost-Riemannian manifold to realize the quotient of Euclidean space by an infinite reflection group. The approach is the same as in the preceding example; one finds a set of basic invariants and uses these as coordinates. As an example let W be the group of plane isometries generated by reflections through the sides of an equilateral triangle. Equivalently, W is the affine Weyl group corresponding to the root system of the $\tilde{s}l_3$ Lie algebra [10]. Let $\mathfrak{h}$ denote the diagonal Cartan subalgebra equipped with the usual Killing inner product, and let L_1, L_2, L_3 denote the weights corresponding to, respectively, the first, second, and third diagonal entry of a trace-free diagonal matrix. Throughout one should keep in mind that $L_3 = -L_1 - L_2$. Taking L_1 and L_2 as non-orthogonal coordinates of $\mathfrak{h}$, the contravariant form of the metric tensor reads

$$\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}.$$

Set $z_k = \exp(2\pi i L_k)$, and note that symmetric polynomials of the z_k's give W-invariant functions of $\mathfrak{h}$. The z_k's generate the coordinate ring of the corresponding torus of diagonal unimodular matrices, and it is known that the algebra of invariant elements of the complexified coordinate ring is generated by the characters of the two fundamental representations of $\tilde{s}l_2\mathbb{C}$ [6]:

$$\chi_1 = z_1 + z_2 + z_3, \quad \chi_{1,1} = z_1 z_2 + z_2 z_3 + z_1 z_3.$$

Calculating formally one obtains

$$\nabla \chi_1 \cdot \nabla \chi_1 = 4\pi^2 \left(-\frac{2}{3} \chi_1^2 + 2\chi_{1,1} \right),$$

$$\nabla \chi_1 \cdot \nabla \chi_{1,1} = 4\pi^2 \left(-\frac{1}{3} \chi_1 \chi_{1,1} + 3 \right),$$

$$\nabla \chi_{1,1} \cdot \nabla \chi_{1,1} = 4\pi^2 \left(-\frac{2}{3} \chi_{1,1}^2 + 2\chi_1 \right).$$

On the real torus the two characters are complex conjugates, and so fundamental invariants are given by the real and imaginary parts of χ_1, call them respectively u and v. The corresponding contravariant metric tensor in (u, v) coordinates is given by:

$$(12) \qquad \frac{2\pi^2}{3} \begin{pmatrix} -3u^2 + v^2 + 6u + 9 & -4uv - 6v \\ -4uv - 6v & -3v^2 + u^2 - 6u + 9 \end{pmatrix}.$$

The locus of degeneracy of the above matrix is given by

$$(u^2 + v^2)^2 - 8(u^3 - 3uv^2) + 18(u^2 + v^2) - 27 = 0.$$

The above is the Cartesian equation of the Euler deltoid [13], the curve obtained by rolling a unit circle inside a circle of radius 3. For this reason the tensor in (12) will henceforth be referred to as the deltoid metric. In Section 5.2 it will serve as the basis for a counter-example to Turbiner's conjecture.

Finally, it will be instructive to consider how Proposition 3.4 can be used to show the completeness of an almost-Riemannian manifold. Consider again the metric tensor (2) introduced in Section 1.2. One proceeds by compactifying $\mathbb{R}^2$ to $\mathbb{RP}^1 \times \mathbb{RP}^1$. The latter can be covered by the following four coordinate systems: (u, v), (u, V), (U, v), (U, V), where $U = u^{-1}$ and $V = v^{-1}$. It is straightforward to check that the tensor (2) can be continued in a non-singular fashion to each of these charts. In the (U, V) chart, for example, the tensor in question is given by

$$\begin{pmatrix} U^4 & 2UV^2 \\ 2UV^2 & 4V^3 \end{pmatrix}.$$

The locus of degeneracy of the extended metric tensor is the closed curve

$$\{v = u^2\} \cup \{U = 0\} \cup \{V = 0\}.$$

It's not hard to check that the extra points added by the compactification are all unreachable boundary points, and thus do not meaningfully alter the underlying geometry. Since $\mathbb{RP}^1 \times \mathbb{RP}^1$ is compact, one can apply Proposition 3.4 to conclude that the almost-Riemannian manifold in question is complete in the component $\{v > u^2\}$.

4. **Global foliation of the plane by straight lines.** The present section is a discussion of a theorem to the effect that a foliation (in a suitably general sense) of the Euclidean plane by straight lines must be either a pencil of parallel lines or a pencil

of coincident lines. The functions, distributions, and other mathematical objects in the present section are assumed to be real-analytic.

Let D be a distribution on $\mathbb{R}^2$ whose rank at any given point is either 1 or 2. This means that locally D is given by the kernel of a non-vanishing analytic 1-form. Analyticity implies that the points of rank 1 form a dense, open subset of $\mathbb{R}^2$. One should also recall that a rank 1 distribution is automatically integrable.

THEOREM 4.1. *Suppose the integral manifold of D at every rank 1 point is a straight line. Then, there exists a system of Cartesian coordinates, (x, y) such that D contains the kernel of either dx or of $-y\,dx + x\,dy$. In other words, the collection of integral manifolds of D will contain either a pencil of parallel lines or a pencil of coincident lines.*

PROOF. Choose Cartesian coordinates (x, y) such that D has rank one at the origin, and such that the integral line at the origin is vertical. Thus, near the origin D is the kernel of a locally defined 1-form, $f\,dx + g\,dy$, such that $g(0, y)$ is identically zero. In the eventuality that $g(x, y)$ is identically zero, all vertical lines are going to be integral manifolds, and one can conclude that the kernel of dx is contained in D.

Suppose then that $g(x, y)$ is not identically zero, and set

$$h(x, y) = \frac{f(x, y)x + g(x, y)y}{g(x, y)}.$$

On the open set where $g \neq 0$, only one straight line can be an integral manifold of D. The slope of this line is $-f/g$, and h is its y-intercept. Hence for every (x, y) such that $g(x, y) \neq 0$, there will be two integral manifolds of D passing through the point $\big(0, h(x, y)\big)$: a vertical line, and the line with slope $-f(x, y)/g(x, y)$. But, D must have rank 2 at a point where two different integral lines intersect, *i.e.* $f = g = 0$ at such a point. On the other hand, since D was assumed to be rank 1 at the origin, $f(0, y)$ is not identically zero, and hence the zeroes of $f(0, y)$ are isolated points. Consequently, $h(x, y)$ is a constant, call it k, and hence $f(x, y)x + g(x, y)(y - k)$ is identically zero. Therefore, D contains the kernel of $-y'\,dx + x\,dy'$, where $y' = y - k$. ∎

How is the above theorem related to Turbiner's Conjecture? Recall that Corollary 2.16 implies that, if a Lie-algebraic metric tensor is Euclidean, then the invariant foliation is perpendicular to straight lines. The generalized Killing-Hopf theorem turns this into a global statement. Theorem 4.1 is needed in the proof of the conjecture, because it allows one to conclude that the globalization of the invariant foliation is either a pencil of parallel lines, or of concentric circles. The needed argument is assembled in the following corollary.

COROLLARY 4.2. *Let Λ be a rank 1 distribution on a two-dimensional manifold, $\mathbf{M}$. Let $\Pi: \mathbb{R}^2 \to \mathbf{M}$ be a map with the following properties: there exist points where the Jacobian has rank 2, and near such points $\Pi^*(\Lambda)$ is perpendicular to a local foliation by straight lines. Then, there exist Cartesian coordinates (x, y) of $\mathbb{R}^2$, such that $\Pi^*(\Lambda)$ contains either the kernel of dy or the kernel of dr where $r^2 = x^2 + y^2$.*

PROOF. Let $f\,dx + g\,dy$ be a locally defined analytic 1-form whose kernel is $\Pi^*(\Lambda)$. Let D be the distribution that is locally specified by the kernel of $-g\,dx + f\,dy$. Consequently at points where the Jacobian of Π is non-degenerate D is rank 1 and its integral manifolds are straight lines. The set of such points is open and dense, and hence D satisfies the hypotheses of Theorem 4.1. The desired conclusion follows immediately. ∎

5. The conjecture: proof and counter-example.

5.1. *The imprimitive case.* With the tools developed in the preceding sections it is possible to prove a form of Turbiner's Conjecture that incorporates two extra assumptions. The first assumption is that the generating Lie-algebra acts imprimitively. The second assumption is that the domain of the operator is a homogeneous space, $\mathbf{M} = \mathbf{G}/\mathbf{H}$, that is either compact, or failing that, can be compactified in a $\mathbf{G}$-compatible manner. The imprimitivity assumption is indispensable. Indeed, the next section presents a counter-example to the conjecture based on primitive actions. The compactness assumption implies completeness, and is needed in order to apply the generalized Killing-Hopf theorem (Theorem 3.5).

In light of the fact that the generating Lie algebra consists, in general, of inhomogeneous first-order operators one needs to say a bit more about the imprimitivity assumption. The geometric meaning of imprimitivity is that there exists a foliation such that the group actions move one leaf to another. This geometric description cannot be applied to a Lie algebra of inhomogeneous first-order operators. One therefore requires the following generalized notion of imprimitivity.

DEFINITION 5.1. A collection of operators $\{T_\alpha\}$ will be said to act imprimitively if there exists a foliation Λ such that for all locally defined functions, λ whose level sets are the leaves of the foliation, and for all α, it is the case that $T_\alpha(\lambda)$ and λ are functionally dependent.

Let $\mathbf{U}$ be an open subset of a a two-dimensional homogeneous space $\mathbf{M} = \mathbf{G}/\mathbf{H}$, where $\mathfrak{g}$, as usual, denotes the Lie algebra corresponding to $\mathbf{G}$. Analyticity is an indispensable assumption in the present context, and so it is important to recall that a homogeneous space is automatically endowed with a real-analytic structure [5]. Let $\eta\colon \mathfrak{g} \to C^\infty(\mathbf{U})$ be a linear map such that the operators

$$T_a = a^\pi + \eta(a), \quad a \in \mathfrak{g},$$

give a realization of $\mathfrak{g}$ by first order differential operators on $\mathbf{U}$.

PROPOSITION 5.2. *If the operators $\{T_a : a \in \mathfrak{g}\}$ act imprimitively, then so do the operators $\{a^\pi : a \in \mathfrak{g}\}$, i.e. there exists a $\mathbf{G}$-invariant foliation on $\mathbf{M}$.*

PROOF. Let Λ be the invariant foliation demanded by the hypothesis, and λ a locally defined, non-degenerate function such that the level sets of λ are the leaves of Λ. Imprimitivity means that $T_a(\lambda)$ is a function of λ for every $a \in \mathfrak{g}$. Furthermore the same is true for $T_a(\lambda^2)$. Hence

$$a^\pi(\lambda) = T_a(\lambda^2)/\lambda - T_a(\lambda)$$

is also a function of λ.   ∎

Let H_0 be a second-order Lie-algebraic operator generated by the T_a's as per (1). Let $C \in \mathrm{Sym}^2 \mathfrak{g}$ denote the corresponding second order coefficients.

THEOREM 5.3. *Suppose the following statements are true:*
  (i) H_0 is gauge equivalent to a Schrödinger operator;
 (ii) $\big(\mathbf{U}, \sigma(H_0)\big)$ is isometric to a subset of the Euclidean plane;
(iii) the operators $\{T_a : a \in \mathfrak{g}\}$ act imprimitively;
(iv) $\mathbf{M}$ is either compact, or can be compactified in such a way that the $\mathbf{G}$-action on $\mathbf{M}$ extends to a real-analytic action on the compactification.
Then, both the eigenvalue equation $H_0 \psi = E\psi$, and the corresponding Schrödinger equation separate in either a Cartesian, or a polar coordinate system.

PROOF.   By hypothesis (ii), $(\mathbf{M}, C^\pi)$ is a zero-curvature, almost-Riemannian manifold. By hypothesis (iv) and by Proposition 3.4, $\mathbf{M}$ is complete within the open connected component of $\mathbf{M}_0$ containing $\mathbf{U}$. Thus, one can apply Theorem 3.5 to conclude that there exists a real analytic map $\Pi \colon \mathbb{R}^2 \longrightarrow \mathbf{M}$ such that $\mathbf{U}$ is contained in the image, and such that C^π is equal to the push-forward of the Euclidean metric tensor.

Let Λ be the T_a-invariant foliation demanded by hypothesis (iii). By Proposition 5.2, Λ is $\mathbf{G}$-invariant as well. Hence, by Corollary 2.16 $\Pi^*(\Lambda)$, is locally orthogonal to a foliation by straight lines. Next, one can apply Corollary 4.2 to conclude that there exist Cartesian coordinates (x, y) such that $\Pi^*(\Lambda) = \{\lambda = \mathrm{const.}\}$ where λ is either y or $x^2 + y^2$.

Let x and y also denote the corresponding local coordinates down on $\mathbf{U}$; more precisely, one is speaking of the restriction of $x \circ \Pi^{-1}$ and $y \circ \Pi^{-1}$ to $\mathbf{U}$. By hypothesis (i), there exists a function, σ, such that

$$H_0 = -\Delta + \nabla\sigma + U_0.$$

The invariance of Λ means that $H_0(\lambda)$ is a function of λ, and of course

$$\Delta\big(f(y)\big) = f''(y),$$
$$\Delta\big(f(x^2 + y^2)\big) = 4f'(x^2 + y^2) + 4(x^2 + y^2)f''(x^2 + y^2).$$

Consequently Λ is invariant with respect to $\nabla\sigma + U_0$. Next, note that

$$(\nabla\sigma + U_0)(\lambda^2) - \lambda(\nabla\sigma + U_0)(\lambda) = \lambda\nabla\sigma(\lambda),$$

and hence $\nabla\sigma(\lambda)$ and U_0 must both be functions of λ. Hence, as per the discussion in Section 1.3, in the case that $\lambda = y$, one must have $\sigma = \sigma_1(x) + \sigma_2(y)$. Similarly, in the case where $\lambda = x^2 + y^2$, it must be true that U_0 is a function of r and that $\sigma = \sigma_1(r) + \sigma_2(\theta)$, where (r, θ) is the corresponding system of polar coordinates. In the first of the above instances, H_0 and the corresponding Schrödinger operator separate in Cartesian coordinates. In the second case, the two operators separate in polar coordinates   ∎

5.2. *Counter-example.* This final section describes a counter-example to Turbiner's conjecture based on the deltoid metric of Section 3.2. It should be mentioned that the operator constructed here belongs the well-known class of exactly-solvable Hamiltonians described by Olshanetsky and Perelomov in [18]. These operators arise as a natural generalization of the Calogero-Sutherland Hamiltonian [22] and are indexed by the possible finite root systems. The counter-example operator is the Olshanetsky-Perelomov Hamiltonian of trigonometric type corresponding to the A_2 root system.

It will not be necessary to recall the details of the Olshanetsky-Perelomov construction. What is relevant here is the Lie-algebraic nature of these operators [19] [4]. Consider again the deltoid metric tensor (12), but with the $2\pi^2/3$ factor omitted for convenience. Since the entries of the matrix in question are second degree polynomials, this tensor can be generated by the Lie algebra of infinitesimal affine transformations of $\mathbb{R}^2$, or $\mathfrak{a}_2$ for short. The generators of $\mathfrak{a}_2$ are:

$$T_1 = \partial_u, \quad T_2 = \partial_v, \quad T_3 = u\partial_u, \quad T_4 = u\partial_v, \quad T_5 = v\partial_u, \quad T_6 = v\partial_v.$$

It is not hard to verify, either directly or by using Proposition 2.12 that these operators do not admit an invariant foliation, *i.e.* the above realization of $\mathfrak{a}_2$ is primitive.

The Euclidean Laplacian in (u, v) coordinates is given by

$$\begin{aligned}
\Delta = {} & 9T_1^2 + 9T_2^2 - 3T_3^2 + T_4^2 + T_5^2 - 3T_6^2 \\
& + 3\{T_1, T_3\} - 6\{T_1, T_6\} - 3\{T_2, T_5\} - 4\{T_3, T_6\} \\
& - 3T_1 - T_3 - T_6.
\end{aligned}$$

Next, consider the operator

$$H_0 = -\Delta + k\nabla \log \sigma,$$

where

$$\sigma = (u^2 + v^2)^2 - 8(u^3 - 3uv^2) + 18(u^2 + v^2) - 27$$

is, up to a constant factor, the determinant of the tensor matrix (12), and k is a real parameter. Note that σ is just the square of $(z_1 - z_2)(z_2 - z_3)(z_1 - z_3)$ whence a straightforward calculation will show that

$$\nabla \log \sigma = -12(T_3 + T_6).$$

Consequently, H_0 is Lie-algebraic as well. Since H_0 is of the form Laplacian plus gradient, it is gauge-equivalent to a Schrödinger operator. Specifically,

$$e^{-\frac{k}{2}\sigma} \circ H_0 \circ e^{\frac{k}{2}\sigma} = -\Delta + U,$$

where the potential in (u, v) coordinates is given by

$$U = -12k^2 - 12k(1 + k)(u - \sqrt{3}v - 3)(u + \sqrt{3}v - 3)(2u + 3)\sigma^{-1},$$

and in the affine (L_1, L_2) coordinates by

$$(13) \quad U = -12k^2 + 3k(1+k)\left[\frac{1}{\sin^2(\pi L_1 - \pi L_2)} + \frac{1}{\sin^2(2\pi L_1 + \pi L_2)} + \frac{1}{\sin^2(\pi L_1 + 2\pi L_2)} \right]$$

In order to show that the corresponding Schrödinger equation cannot be solved by separation of variables, it will be necessary to recall a few facts regarding this matter [15] [12] (see also 2.7 of [25]). There exists precisely four types of orthogonal coordinate systems that can serve to separate a 2-dimensional Schrödinger equation: they are the Cartesian, polar, parabolic, and elliptic coordinate systems. The first two of these do not require further elaboration. Parabolic coordinates (u, v) are related to Cartesian coordinates (x, y) by

$$x = (u^2 - v^2)/2, \quad y = uv.$$

Elliptic coordinates, (ξ, η), are related to Cartesian coordinates by

$$x = \cosh \xi \cos \eta, \quad y = \sinh \xi \sin \eta.$$

DEFINITION 5.4. Say that a function of 2 variables, f, separates in one of the above four coordinate systems whenever f takes one of the following forms:
- Cartesian coordinates: $f_1(x) + f_2(y)$;
- polar coordinates: $r^{-2}[f_1(r) + f_2(\theta)]$;
- parabolic coordinates: $(u^2 + v^2)^{-1}[f_1(u) + f_2(v)]$;
- elliptic coordinates: $(\cosh^2 \xi - \cos^2 \eta)^{-1}[f_1(\xi) + f_2(\eta)]$.

PROPOSITION 5.5. *A 2-dimensional Schrödinger equation*

$$-\Delta\psi + U\psi = E\psi,$$

can be solved by separation of variables if and only if the potential, U, separates in one of the above mentioned coordinate systems.

The proof that the potential given in (13) does not separate relies on the following two lemmas.

LEMMA 5.6. *Let $f = 1 / \sin^2(ax + by)$ and $D = p(x, y, \partial_x, \partial_y)$, where p is a polynomial. Then $Df = 0$ if and only if $D = D_1 \circ (b\partial_x - a\partial_y)$, for some other linear differential operator, D_1. Furthermore, if D is a non-zero polynomial in the coordinates and coordinate derivations of either the polar or parabolic coordinate systems, or a polynomial of $\cosh \xi$, $\sinh \xi$, $\cos \eta$, $\sin \eta$, ∂_ξ, ∂_η, then $Df \neq 0$.*

Let f_1, f_2, f_3 denote the 3 terms inside the bracketed subexpression of (13), *i.e.* $f_1 = 1 / \sin^2(\pi L_1 - \pi L_2)$, *etc*.

LEMMA 5.7. *Let D be a non-zero polynomial in the coordinates and coordinate derivations of either the Cartesian, polar, parabolic coordinate systems, or a (ξ, η) operator of the type described in the preceding lemma. Then, $D(f_1 + f_2 + f_3) \neq 0$.*

PROPOSITION 5.8. *The Schrödinger equation with the potential given in (13) cannot be solved by separation of variables.*

PROOF. It is necessary to show that the potential, U, in question does not separate in any of the four coordinate systems mentioned above. Since U is a smooth function one can use a second-order derivative of mixed partials to test for separability. In other words U separates in a given Cartesian coordinate system if and only if $\partial_{xy} U = 0$; it separates in a given polar coordinate system if and only if $\partial_{r\theta}(r^2 U) = 0$, *etc*. Thus in each case one has to show that $D(U) = 0$, where D is the appropriate mixed-partials operator. All such operators fit the hypothesis of the preceding lemma, and therefore $D(U)$ can never be zero. ∎

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