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Uniqueness and representation of a function in terms of its translated averages

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We give the representation formula of a periodic function with period one in terms of its translated means on the unit interval. We also give an application of this formula to a boundary value problem.

Let f be a continuous periodic function on the real line with period one. We consider its α means

$$S_{\alpha,n}(f) = \frac{1}{n} \sum_{k=1}^{n} f\left(\alpha + \frac{k}{n}\right)$$

$n = 1, 2, \ldots$, and

$$S_{\alpha,\infty}(f) = \lim_{n\to\infty} S_{\alpha,n}(f)$$

We denote

$$R_{\alpha,n}(f) = S_{\alpha,n}(f) - S_{\alpha,\infty}(f) .$$

As in [5], a periodic function f with period one is defined to be in B^s if the Fourier coefficients of f satisfy the following condition:

$$a_n(f) = \int_0^1 f(t)e^{-i2\pi nt}dt = O\left(\frac{1}{n^s}\right) .$$

It is easy to prove that $a_n(f)$ and $R_{\alpha,n}(f)$ have the same rate of

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convergence to zero, (*cf.* [4]). Let $\mu(n)$ be the Möbius function of n. We announce the following results.

THEOREM 1. *Let* α *and* β *be any two real numbers in* $[0, 1]$ *such that* $\alpha - \beta$ *is irrational and* f *be a periodic continuous function with period one such that* $S_{\alpha,n}(f) = S_{\beta,n}(f) = 0$ *for all* $n > 0$. *Then* f *is the zero function if* f *satisfies one of the following conditions:*

(a) $f \in B^s$, $s > 1$;

(b) f *has bounded variation in* $[0, 1]$;

(c) $\sum_{|k|=N}^{\infty} |a_n| = O\left(\frac{1}{N}\right)$;

(d) the Fourier series of f *is an absolutely convergent gap series.*

In case f can be regarded as the boundary value of an analytic function, an analogous uniqueness result has been announced in [2] and proved in [3] and [6]. Also if $\alpha - \beta$ is rational, it is easy to find a function g such that $S_{\alpha,n}(g) = S_{\beta,n}(g) = 0$ for all integers n.

THEOREM 2. *Let* α *and* β *be any two real numbers in* $[0, 1]$.

(i) Suppose $\alpha - \beta$ *is irrational and* f *is in* B^s *with* $s > 1$; *then*

$$a_m(f) = \sum_{k=1}^{\infty} \frac{\mu(k)\left[e^{-i2\pi m\beta}R_{\alpha,|mk|}(f) - e^{-i2\pi m\alpha}R_{\beta,|mk|}(f)\right]}{2i\sin 2\pi m(\alpha-\beta)}$$

for all $m = \pm 1, \pm 2, \ldots$.

(ii) Let $\alpha - \beta$ *be an algebraic number of degree* $q \geq 2$ *and* f *be in* B^s *with* $s > \sqrt{2q}$. *Then*

$$(1) \quad f(t) = R_{\alpha,\infty}(f) + \sum_{n=1}^{\infty} \left[R_{\alpha,n}(f)P_n(\alpha, \beta, t) + R_{\beta,n}(f)P_n(\beta, \alpha, t)\right] ,$$

where the series converges uniformly with the rate N^{x-s}, $\sqrt{2q} < x < s$, *and*

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$$P_n(\alpha,\ \beta,\ t) = \sum_{j|n} \mu\left(\frac{n}{j}\right) \frac{\sin 2\pi j(t-\beta)}{\sin 2\pi j(\alpha-\beta)} .$$

Results analogous to Theorem 2 *(ii)* in the case that f is the boundary value of an analytic function were announced in [1] and established in [5].

THEOREM 3. *Let* α *and* β *be any two real numbers in* $[0, 1]$ *such that* $\alpha - \beta$ *is algebraic and let* f *be in* B^s *for any* $s > 0$, *then*

$$R_{\alpha,m}(f') = S_{\alpha,m}(f') = 2\pi m \sum_{k=1}^{\infty} \sum_{l|k} l\mu\left(\frac{k}{l}\right)\left[R_{\alpha,km}(f)\cot 2\pi lm(x-\beta) - R_{\beta,km}(f)\csc 2\pi lm(\alpha-\beta)\right] ,$$

$$R_{\alpha,m}(f'') = -4\pi^2m^2 \sum_{k=1}^{\infty} R_{\alpha,km}(f)\left[\sum_{l|k} \mu\left(\frac{k}{l}\right)l^2\right] .$$

As f is uniquely determined by the averages $\{R_{\alpha,n}(f)\}$ and $\{R_{\beta,n}(f)\}$, consequently $R_{\alpha,m}(f')$ and $R_{\beta,m}(f')$ depend also on these averages. However it is not clear from the definition that $R_{\alpha,m}(f')$ and $R_{\beta,m}(f')$ depend only on those $R_{\alpha,k}(f)$ and $R_{\beta,k}(f)$ such that k is a multiple of m .

We also remark that $P_n(\alpha,\ \beta;\ t)$ has the property that $S_{\beta,m}[P_n(\alpha,\ \beta;\ t)] = 0$ and $S_{\alpha,m}[P_n(\alpha,\ \beta;\ t)] = \delta_{m,n}$ for all positive integers m and n . The assumption on $\alpha - \beta$ is not satisfactory, yet we guess that the series (1) will diverge if $\alpha - \beta$ is a Liouville number, (*cf.* [7]). Now we give an application of the above theorems to a boundary value problem.

Let α and β be any two real numbers such that $\alpha - \beta$ is an algebraic number of degree $q \geq 2$. Let $u(t) \in B^s$, $s > \sqrt{2q} - 1$, be the solution of the following boundary value problem:

$$au'' + bu' + cu = f\ ;\quad f \in B^p\ ,\quad p > 1\ ;$$

$$u(0) = u(1)\ ,$$

$$u'(0) = u'(1)\ ,$$

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where a, b, c are constants satisfying

$$c + 2\pi i n b - 4\pi^2 n^2 a \neq 0$$

for all $n = 0, \pm 1, \ldots$. Then

$$(2) \quad R_{\alpha,\infty}(u) = R_{\beta,\infty}(u) = \frac{1}{c} R_{\alpha,\infty}(f) ,$$

$$R_{\alpha,n}(u) = \sum_{k=1}^{\infty} R_{\alpha,kn}(f) \left| \sum_{l|k} \mu\left(\frac{k}{l}\right) \frac{(c-4\pi^2 l^2 n^2 a) - 2\pi l n b \cot 2\pi l n(\alpha-\beta)}{c^2 - 8\pi^2 l^2 n^2 a + 4\pi^2 l^2 n^2 b^2 + 16\pi^4 l^4 n^4 a^2} \right|$$

$$+ \sum_{k=1}^{\infty} R_{\beta,kn}(f) \left| \sum_{l|k} \mu\left(\frac{k}{l}\right) \frac{2\pi l n b \csc 2\pi l n(\alpha-\beta)}{c^2 - 8\pi^2 l^2 n^2 a + 4\pi^2 l^2 n^2 b^2 + 16\pi^4 l^4 n^4 a^2} \right|$$

It follows from the above formula that if for fixed j , $\{R_{\alpha,jn}(f)\}$ and $\{R_{\beta,jn}(f)\}$ are known for all $n = 1, 2, \ldots$, then $\{R_{\alpha,jn}(u)\}$ and $\{R_{\beta,jn}(u)\}$ can be computed from (2). However we can see from Theorem 2 *(ii)* that if $j \neq 1$, then $\{R_{\alpha,jn}(f)\}$ and $\{R_{\beta,jn}(f)\}$ do not determine f uniquely. In other words we can find some averages of the output from a portion of the averages of the input. The details of the proof of the theorems and other related results will appear elsewhere.

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