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GENERALIZED SPECTRAL THEORY AND SECOND ORDER ORDINARY DIFFERENTIAL OPERATORS

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1. Introduction. This paper continues the study, begun in [**7**], of the spectral theory of non-self-adjoint second order ordinary differential operators on a half-line. The case of a "very small" potential was studied in [**4**; **5**; **6**]. The case considered in [**7**], and in the present paper, is that where the potential is not so small. Precisely, the potential is assumed to be of the form

$$p(t) = \frac{\alpha}{t} + q(t) \tag{1}$$

where α is a (not necessarily real) constant, and where q is "small". We shall consider the spectral properties of the operators defined in $L_2[1, \infty)$ by the formal differential operator

$$\tau = -\frac{d^2}{dt^2} + p(t) \tag{2}$$

(we work in the half-line $[1, \infty)$ because we are not interested in the problems arising from the singularity of α/t at $t = 0$).

One sees easily that, in order to obtain nontrivial spectral properties, it is necessary to impose one boundary condition (for the precise definitions concerning differential operators and boundary conditions, cf. [**3**, Chapter XIII]) at the fixed endpoint.

In [**7**], we showed that, in general, the operators obtained in this way are not spectral (in Dunford's sense, cf. [**2**] and [**4**]). The present study is concerned with the applicability to these operators of generalized spectral theory. The basic idea of this theory is that, by analogy with L. Schwartz's theory of distributions, the notion of a "spectral measure" can be extended, to study more general entities called "spectral distributions". This makes it possible to consider "generalized scalar" operators (i.e. operators that can be written as the "integral" of a spectral distribution, in the same way as an operator of the scalar type is the integral of a spectral measure), and "generalized

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spectral" operators (i.e. sums $S + Q$, where S is generalized scalar, Q is quasi-nilpotent, and $SQ = QS$). A detailed presentation of generalized spectral theory for bounded operators is given in [**1**]. Here, we shall need two extensions of the theory, namely:

(a) the extension to the unbounded case, and

(b) the study of "Ω-generalized scalar operators", where Ω is an arbitrary open subset of the complex plane (plus the point at infinity).

Precisely, generalized scalar operators are operators that have an operational calculus for functions that are C^∞ in a neighbourhood of the spectrum. We want to study operators that have an operational calculus for functions that are C^∞ in a neighbourhood of the spectrum, and analytic in a neighbourhood of a fixed closed subset K of the spectrum. If Ω s the complement of K, such an operator will be called "Ω-generalized scalar".

The relevant definitions are given in sections 2 and 3. In section 4 we study a particular situation in which the problem of Ω-generalized scalarity can be reduced to the study of the growth of the resolvent near the spectrum.

Section 5 deals with the application of the theory to the second order operator. The main result is contained in Theorem 1.3, which says that the operators studied here are Ω-generalized scalar, where Ω is the complement of the origin (this result is the best possible, as will be shown elsewhere; cf. also section 6).

Section 5 should be read together with [**7**]. The other sections rely heavily on the results and methods of [**1**], but are independent of [**7**]. (The following two misprints of [**7**] should be corrected: (a) In the statement of Lemma 2.3, the expression "as $(t, \mu) \to (\infty, \infty)$" must be replaced by "as $|t\mu| \to \infty$", and (b) in the second integral of formula (3.11), "$g(t)$" should be replaced by "$\overline{g(t)}$".)

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2. Preliminaries. All the operators considered here are assumed to be closed, densely defined operators on a Hilbert space H. The set of all bounded operators on H is denoted by $\mathscr{B}(H)$. We consider $\mathscr{B}(H)$ as a Banach algebra, with the norm

$$\|B\| = \sup\{\|Bx\| : \|x\| = 1\}. \tag{3}$$

The identity operator is denoted by I.

The set of complex numbers is denoted by $\mathbf{C}$, and the Riemann sphere (i.e. the one point compactification of $\mathbf{C}$) by $\mathbf{C}^*$. Thus $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$. The Riemann sphere is, in a well-known way, a compact analytic manifold.

If T is an operator, then the resolvent $\rho(T)$ and the spectrum $\sigma(T)$ will be viewed as subsets of $\mathbf{C}^*$. By definition, $\rho(T)$ is the set of all $\lambda \in \mathbf{C}$ such that $\lambda I - T$ is one-to-one and maps the domain $\mathscr{D}(T)$ of T onto H. In addition, we let $\infty \in \rho(T)$ if and only if T is bounded. If $\mu \in \rho(T)$ is finite then $\mu I - T$ has an inverse, called the *resolvent* of T at μ, and denoted by $R(\mu, T)$. If T is bounded,

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we let $R(\infty, T) = 0$. With these definitions, it is clear that $\rho(T)$ is an open subset of $\mathbf{C}^*$, and that the $\mathscr{B}(H)$-valued function $\mu \to R(\mu, T)$ is analytic in $\rho(T)$. The *spectrum* $\sigma(T)$ is, by definition, the complement of $\rho(T)$ in $\mathbf{C}^*$. Thus $\sigma(T)$ is always compact, and $\infty \in \sigma(T)$ if and only if T is unbounded. We shall only consider operators T that satisfy

(n.r.) *The set $\rho(T)$ is nonempty.*

An operator T is said to have the *single-valued extension property* (henceforth abbreviated as s.v.e.p.) if

(i) T satisfies condition (n. r.), and

(ii) for every open set Ω of complex numbers, if f is an H-valued analytic function on Ω that satisfies

$$(4) \qquad (\lambda I - T)f(\lambda) = 0 \qquad \text{for all } \lambda \in \Omega$$

(i.e., for every $\lambda \in \Omega$, $f(\lambda)$ belongs to $\mathscr{D}(T)$ and (4) holds) then, necessarily, $f(\lambda) = 0$ for all $\lambda \in \Omega$.

It is easy to see that every operator whose resolvent set is dense in $\mathbf{C}^*$ has the s.v.e.p. The adjoint S^* of the shift operator S in l_2 is an example of an operator not having the s.v.e.p. (cf. [**1**, p. 10, Example 1.7]).

If T has the s.v.e.p. then for every $x \in H$ there exists a unique pair (Ω, f) such that

(a) Ω is an open subset of $\mathbf{C}^*$,

(b) f is an analytic H-valued function defined in Ω,

(c) if $\lambda \in \Omega$, $\lambda \neq \infty$, then $f(\lambda) \in \mathscr{D}(T)$ and $(\lambda I - T)f(\lambda) = x$, and

(d) if (Ω', f') is any other pair that satisfies (a), (b) and (c),

then $\Omega' \subset \Omega$ and f' is the restriction of f to Ω. If (Ω, f) satisfies (a), (b), (c) and (d) above, then the set Ω will be denoted by $\rho_T(x)$, and the function f by $\bar{x}$. The complement of $\rho_T(x)$ in $\mathbf{C}^*$ is denoted by $\sigma_T(x)$. Clearly $\sigma_T(x)$ is compact and contained in $\sigma(T)$. The following two lemmas are trivial.

LEMMA 1. *If T has the s.v.e.p. and if $A \in \mathscr{B}(H)$ commutes with T (i.e., A maps $\mathscr{D}(T)$ into $\mathscr{D}(T)$, and $TAx = ATx$ for every $x \in \mathscr{D}(T)$) then $\sigma_T(Ax) \subset \sigma_T(x)$.*

LEMMA 2.

(i) *If $\mu \in \mathbf{C}$, $\mu \neq 0$, and $x \in H$, then $\sigma_T(\mu x) = \sigma_T(x)$.*

(ii) *If $x, y \in H$, then $\sigma_T(x + y) \subset \sigma_T(x) \cup \sigma_T(y)$.*

(iii) $\sigma_T(x) = \emptyset \Leftrightarrow x = 0$.

If T is an operator with the s.v.e.p. and if F is a compact subset of $\mathbf{C}^*$, then we will write $\chi_T(F)$ to denote the set of all $x \in H$ such that $\sigma_T(x) \subset F$. The following corollary is an immediate consequence of Lemmas 1 and 2.

COROLLARY 3.

(i) *$\chi_T(F)$ is a linear subspace of H.*

(ii) $\chi_T(F)$ *is ultrainvariant for* T(i.e., $\chi_T(F)$ *is invariant under every bounded operator that commutes with* T).

The space $\chi_T(F)$ need not be closed (cf. [**1**, p. 25, Example 3.9]). However, we have the following result.

PROPOSITION 4. *If* $\chi_T(F)$ *is closed, then* $\chi_T(F)$ *satisfies*
(a) $\sigma(T|\chi_T(F)) \subset \sigma(T) \cap F$, *and*
(b) $\chi_T(F)$ *is a spectral maximal space of* T.

Here the symbol | denotes restriction, and the definition of a spectral maximal space is as follows: a closed subspace K of H is a spectral maximal space of T if K is invariant under T and if, for every closed subspace L of N which is invariant under T, the inclusion $\sigma(T|L) \subset \sigma(T|K)$ implies that $L \subset K$.

For a proof of Proposition 4 see [**1**, p. 23, Proposition 3.8] (the result is proved there for bounded T, but the proof is valid as well in the unbounded case).

If Ω is an open subset of $\mathbf{C}^*$, we shall use $\mathscr{U}(\Omega)$ to denote the set of all complex-valued functions f, defined on $\mathbf{C}^*$, with the property that
(i) f is C^∞ on $\mathbf{C}^*$, and
(ii) f is analytic on a neighbourhood of $\mathbf{C}^*\backslash\Omega$ (the symbol "\" denotes difference of sets).

The set $\mathscr{U}(\Omega)$ is clearly an algebra, and we shall always consider $\mathscr{U}(\Omega)$ as a topological algebra, with the topology such that a sequence $\{\theta_n\}$ converges to θ if and only if:
(a) $\theta_n \to \theta$ in the topology of $C^\infty(\mathbf{C}^*)$ (i.e., $\theta_n \to \theta$ uniformly, together with all the partial derivatives of all orders), and
(b) there exists a fixed open neighbourhood U of $\mathbf{C}^*\backslash\Omega$ such that all the functions θ_n are analytic in U.

If Ω is the resolvent set $\rho(T)$ of an operator T, then $\mathscr{U}(\Omega)$ is the set of all C^∞ functions on $\mathbf{C}^*$ that are analytic in a neighbourhood of $\sigma(T)$. Dunford's functional calculus gives a continuous homomorphism $F \to F(T)$ from $\mathscr{U}(\rho(T))$ into $\mathscr{B}(H)$. We will use D_T to denote this homomorphism (for the definition and basic properties of D_T, cf. Dunford and Schwartz [**3**]). The following proposition summarizes some properties of D_T.

PROPOSITION 5.
(a) $D_T(1) = I$ (*here* "1" *denotes the constant function* $f(\lambda) \equiv 1$).
(b1) *If* T *is bounded then* $D_T(\Lambda) = T$ (*here* Λ *is the function defined by* $\Lambda(\lambda) = \lambda$ *for* $\lambda \neq \infty$, cf. *Remark below*).
(b2) *If* T *is unbounded and if* $F \in \mathscr{U}(\rho(T))$ *vanishes at* ∞ (*so that* $\Lambda F \in \mathscr{U}(\rho(T))$, *then* $D_T(F)$ *maps* H *into* $\mathscr{D}(T)$ *and* $D_T(\Lambda F) = TD_T(F)$.
(c) *If* $\mu \in \rho(T)$ *then* $D_T((\mu - \Lambda)^{-1}) = R(\mu, T)$ (*here* $(\mu - \Lambda)^{-1}$ *is the function* $\lambda \to (\mu - \lambda)^{-1}$, *cf. Remark below*).

(d) *If* supp $\theta \subset \rho(T)$ *then* $D_T(\theta) = 0$.
(e) D_T *is completely characterized by conditions* (a), (b) *and* (d).

Proof. (a), (b), (c) and (d) are trivial. In addition, it is clear that (c) follows from (b). To prove (e), let D_T' be another continuous homomorphism from $\mathscr{U}(\rho(T))$ into $\mathscr{B}(H)$ that satisfies (a), (b) and (d) (and therefore (c)). Let f be analytic in a neighbourhood U of $\sigma(T)$, and let Γ be a curve contained in U that surrounds $\sigma(T)$. For each partition $P = \{\xi_1, \ldots, \xi_m\}$ of Γ, let

$$f_P(\mu) = \frac{1}{2\pi i}\sum_{k=1}^{m} f(\xi_k)\left(\frac{\xi_{k+1} - \xi_k}{\xi_k - \mu}\right).$$

(where $\xi_{m+1} = \xi_1$). Then f_P is an analytic function on a neighbourhood of $\sigma(T)$, and

$$D_T(f_P) = D_T'(f_P) = \frac{1}{2\pi i}\sum_{k=1}^{m} f(\xi_k)R(\xi_k, T)(\xi_{k+1} - \xi_k).$$

If $\{P_n\}$ is a sequence of partitions whose mesh goes to zero, then

$$\lim_{n\to\infty} f_{P_n}(\mu) = f(\mu)$$

(by Cauchy's formula), uniformly with respect to μ, for μ in some neighbourhood of $\sigma(T)$. Thus $D_T f = D_T' f$, and the proof is complete.

From the preceding proof it follows that

$$D_T f = \frac{1}{2\pi i}\int_\Gamma f(\xi)R(\xi, T)d\xi.$$

If Ω is an open subset of $\mathbf{C}^*$ such that $\rho(T) \subset \Omega$, we define an Ω-spectral distribution for T (or an $\Omega - C^\infty$ operational calculus) to be a continuous homomorphism D from $\mathscr{U}(\Omega)$ into $\mathscr{B}(H)$ whose restriction to $\mathscr{U}(\rho(T))$ is D_T.

If an operator T has an $\Omega - C^\infty$ operational calculus, we will say that T is an Ω-*generalized scalar operator*. If T is $\mathbf{C}^*$-generalized scalar, we will say that T is *generalized scalar*.

Remark. If $D : \mathscr{U}(\Omega) \to \mathscr{B}(H)$ is an Ω-spectral distribution, and if $U \subset \Omega$ is an open set such that $D(\theta) = 0$ for every $\theta \in \mathscr{U}(\Omega)$ whose support is contained in U, then $D(\psi)$ can be defined, in a "natural" way, for every function ψ which is C^∞ in some neighbourhood V of $\mathbf{C}^*\backslash U$, and analytic in some neighbourhood of $\mathbf{C}^*\backslash\Omega$ (take $\theta \in C^\infty(\mathbf{C}^*)$ such that $\theta = 1$ in a neighbourhood of $C^*\backslash U$ and that supp $\theta \subset V$, and let $D(\psi) = D(\theta\psi)$). If T is bounded, we can take U to be a neighbourhood of ∞; thus $D(\Lambda)$ makes sense, even though Λ is not C^∞ at ∞. In a similar way, if T is bounded or unbounded, and if $\mu \in \rho(T)$, then $D((\mu - \Lambda)^{-1})$ is also defined.

3. Elementary properties of Ω-generalized scalar operators. Let $\rho(T) \subset \Omega$, and let D be an Ω-spectral distribution for T. We assume throughout that T has the s.v.e.p.

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LEMMA 6.

(a) $D(\theta) = 0$ *for every* θ *that vanishes in a neighbourhood of* $\sigma(T)$.

(b) $D(1) = 1$.

(c) *If* T *is bounded, then* $D(\Lambda) = T$.

(d) *If* T *is bounded or unbounded, and if* $\theta \in \mathscr{U}(\Omega)$ *is such that* $\Lambda\theta \in \mathscr{U}(\Omega)$, *then* $D(\theta)$ *maps* H *into* $\mathscr{D}(T)$ *and* $D(\Lambda\theta) = TD(\theta)$.

(e) *If* $\mu \in \rho(T)$, *then* $D((\mu - \Lambda)^{-1}) = R(\mu, T)$.

Proof. (a), (b), (c) and (e) follow from the fact that the restriction of D to $\mathscr{U}(\rho(T))$ is D_T. To prove (d), let $\mu \in \rho(T)$. Then $(\mu - \Lambda)^{-1} \in \mathscr{U}(\Omega)$ and $D((\mu - \Lambda)^{-1}) = R(\mu, T)$. It follows that $D((\mu - \Lambda)^{-1}\theta)$ equals $R(\mu, T)D(\theta)$ and is, therefore, a mapping into $\mathscr{D}(T)$. Applying this with θ replaced by $\Lambda\theta$, we conclude that $D((\mu - \Lambda)^{-1}\Lambda\theta)$ maps H into $\mathscr{D}(T)$. Also, $D(\mu(\mu - \Lambda)^{-1}\theta)$ maps H into $\mathscr{D}(T)$. Subtracting, we get that $D(\theta)$ maps H into $\mathscr{D}(T)$. Moreover

$$R(\mu, T)(\mu I - T)D(\theta) = D(\theta) = R(\mu, T)D((\mu - \Lambda)\theta) = R(\mu, T)(\mu D(\theta) - D(\Lambda\theta)).$$

Since $R(\mu, T)$ is one-to-one, we conclude that $TD(\theta) = D(\Lambda\theta)$, and the proof is complete.

LEMMA 7. *If* $x \in H$, $\theta \in \mathscr{U}(\Omega)$, *then* $\sigma_T(D(\theta)(x)) \subset \operatorname{supp} \theta$.

Proof. Let $\theta_\mu(\lambda) = (\mu - \lambda)^{-1}\theta(\lambda)$. If $\mu \notin \operatorname{supp} \theta$, then θ_μ is well defined and belongs to $\mathscr{U}(\Omega)$. One sees easily that $D(\theta_\mu)(x)$ depends analytically on μ. The function $\Lambda\theta_\mu$ clearly belongs to $\mathscr{U}(\Omega)$. Therefore $D(\theta_\mu)(x)$ belongs to $\mathscr{D}(T)$, and $D(\Lambda\theta_\mu)(x) = TD(\theta_\mu)(x)$ (cf. Lemma 5). Since $D(\mu\theta_\mu)(x) = \mu D(\theta_\mu)(x)$, we conclude that

$$(\mu I - T)D(\theta_\mu)(x) = D(\theta)(x)$$

for all μ in the complement of the support of θ. Thus $\mathbf{C}^*\backslash\operatorname{supp} \theta \subset \rho_T(x)$ and the desired conclusion follows.

LEMMA 8. *If* $\rho(T) \subset \Omega' \subset \Omega$, *then* T *is* Ω'*-generalized scalar. If* $\overline{\Omega'} \subset \Omega$ (*the closure being taken in* $\mathbf{C}^*$), *then the distribution* D *has finite order on* Ω', *i.e. the restriction of* D *to* $C_0^\infty(\Omega')$ (*the space of* C^∞ *functions whose support is contained in* Ω') *is continuous in the* C_0^m *topology for some* $m \geqq 0$. (*A sequence* $\{\theta_n\}$ *of functions in* $C_0^\infty(\Omega')$ *converges to* θ *in the* C_0^m *topology if:*

(i) $\theta_n \to \theta$ *uniformly as well as all the partial derivatives of order* $\leqq m$, *and*

(ii) *there is a compact* $K \subset \Omega'$ *such that* $\operatorname{supp} \theta_n \subset K$ *for every* n.)

We will not prove Lemma 8. The first statement is trivial. The second is proved in exactly the same way as the corresponding result for scalar distributions.

LEMMA 9.

(a) *If* F *is a compact subset of* Ω, *then* $\chi_T(F)$ *is closed and the restriction of* T *to* $\chi_T(F)$ *is a generalized scalar operator.*

(b) *If* F *is compact and* $\mathbf{C}^*\backslash\Omega \subset F$, *then* $\chi_T(F)$ *is closed.*

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Proof. We show that, if F satisfies the conditions of (a) or (b) then $\chi_T(F)$ is the set of all $x \in H$ with the property that $D(\theta)(x) = 0$ for every $\theta \in \mathscr{U}(\Omega)$ such that $\theta = 0$ in a neighbourhood of F. This will clearly imply that $\chi_T(F)$ is the intersection of a family of closed subspaces, and hence closed. If $x \in \chi_T(F)$, and if $\theta \in \mathscr{U}(\Omega)$ vanishes in a neighbourhood of F, then $\sigma_T(D(\theta)x)$ is contained in F (by Lemma 1) and in supp θ (by Lemma 7). Therefore $\sigma_T(D(\theta)x) = \emptyset$. By Lemma 2 (iii), $D(\theta)(x) = 0$. Conversely, let $D(\theta)(x) = 0$ for all $\theta \in \mathscr{U}(\Omega)$ that vanish in a neighbourhood of F, and let $\mu \notin F$. Then there exists $\theta \in \mathscr{U}(\Omega)$ such that $\theta = 0$ in a neighbourhood of F and $\theta = 1$ in a neighbourhood of μ. (Let $K = \mathbf{C}^*\backslash\Omega$. In case (a), take $\theta \in C^\infty(\mathbf{C}^*)$, $\theta = 0$ near F, and $\theta = 1$ near $K \cup \{\mu\}$; then θ is analytic near K, and hence $\theta \in \mathscr{U}(\Omega)$. In case (b), if $\theta = 0$ near F, then θ is analytic near K.) Then $D(\theta)(x) = 0$ and $D(1 - \theta)(x) = x$. It follows that $\sigma_T(x)$ is contained in the support of $1 - \theta$, and therefore $\mu \notin \sigma_T(x)$.

It remains to be shown that, if $F \cap K = \emptyset$, then the restriction $T|\chi_T(F)$ of T to $\chi_T(F)$ is generalized scalar. Let $\theta = 1$ on a neighbourhood of F, $\theta = 0$ near K. Define $D(\psi) = D(\psi\theta)|\chi_T(F)$ for $\psi \in C^\infty(\mathbf{C}^*)$. One shows easily that D is a $\mathbf{C}^*$-spectral distribution for $T|\chi_T(F)$. The proof is now complete.

4. Operators whose spectrum is contained in a union of analytic arcs. We shall now study a situation in which the Ω-generalized scalarity of an operator can be characterized in terms of a growth property of the resolvent. We let T be an operator on H that has the s.v.e.p., and we assume that Ω is an open subset of $\mathbf{C}^*$, and that $\rho(T) \subset \Omega$. Moreover, we assume that $\Omega \cap \sigma(T)$ is contained in a (not necessarily connected) one-dimensional closed immersed analytic submanifold M of Ω (here "immersed" means that M is a topological subspace of Ω).

We first assume that T is a bounded operator. We shall say that the growth condition (G_n) holds at $\lambda_0 (\in \Omega)$ if

$$||R(\lambda, T)|| = O(d(\lambda, M)^{-n}) \text{ as } \lambda \to \lambda_0, \lambda \notin M.$$

We have

THEOREM 10. *Let T, Ω, M be as above, and assume that T is bounded. Then T is Ω-generalized scalar if and only if a growth condition $(G_{n(\lambda)})$ is satisfied at every $\lambda \in \Omega \cap \sigma(T)$.*

Proof. Assume that the growth condition is satisfied at $\lambda_0 \in \sigma(T)$. Clearly, there is a neighbourhood U of λ_0 such that $U \cap \sigma(T)$ is contained in an analytic arc γ. Precisely, there is an analytic mapping γ defined on an interval (a, b) such that $\gamma'(t) \neq 0$ for $a < t < b$, and that $U \cap \sigma(T) \subset \gamma(a, b)$. We can assume that γ is the restriction of a conformal mapping Γ defined in the set $S = \{z : a < \text{Re}\, z < b, |\text{Im}\, z| < \epsilon\}$, and that U is precisely $\Gamma(S)$. We can also assume that the inequality

$$||R(\lambda, T)|| \leqq C_0 d(\lambda, M)^{-n}$$

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holds for all $\lambda \in U\backslash\gamma(a, b)$. Here $C_0 > 0$ is a constant, and $n = n(\lambda_0)$. Moreover, it is not hard to show that the above inequality implies (after replacing U by a "smaller" strip) that

$$||R(\Gamma(t + is), T)|| \leqq C|s|^{-n}$$

for $a < t < b$, $0 < |s| < \epsilon$, and for some fixed $C > 0$.

We let

$$S_- = \{z : a < \operatorname{Re} z < b, -\epsilon < \operatorname{Im} z < 0\}$$

and

$$S_+ = \{z : a < \operatorname{Re} z < b, 0 < \operatorname{Im} z < \epsilon\}.$$

Let $F(z) = R(\Gamma(z), T)$, for $z \in S_- \cup S_+$. Then F is an analytic $\mathscr{B}(H)$-valued function on $S_- \cup S_+$. Let F_k be an analytic function on $S_- \cup S_+$ whose kth derivative is F. One shows easily that F_{n+1} is bounded on $S_- \cup S_+$. Therefore $F_{n+2}(z)$ has limits as z approaches (a, b) from below and from above. Let these limits be denoted by $F_{n+2}^-(t)$ and $F_{n+2}^+(t)$, respectively.

Let $\theta \in C_0^\infty(U)$. We let

$$I_\delta(\theta) = \frac{1}{2\pi i}\int_{\gamma_\delta} \theta(\lambda)R(\lambda, T)d\lambda,$$

where γ_δ is the arc $t \to \Gamma(t + i\delta)$. Clearly, $I_\delta(\theta)$ exists for $0 < |\delta| < \epsilon$, and

$$I_\delta(\theta) = \frac{1}{2\pi i}\int_a^b \theta(\Gamma(t + i\delta))F(t + i\delta)\Gamma'(t + i\delta)dt.$$

Using integration by parts, we can rewrite this as

$$I_\delta(\theta) = \frac{(-1)^k}{2\pi i}\int_a^b \frac{\partial^k}{\partial t^k}(\theta(\Gamma(t + i\delta))\Gamma'(t + i\delta))F_k(t + i\delta)dt.$$

It follows that the limits

$$I^-(\theta) = \lim_{\delta \to 0, \delta < 0} I_\delta(\theta)$$

and

$$I^+(\theta) = \lim_{\delta \to 0, \delta > 0} I_\delta(\theta)$$

exist, and are given by

$$I^\pm(\theta) = \frac{(-1)^{n+2}}{2\pi i}\int_a^b \frac{\partial^{n+2}}{\partial t^{n+2}}(\theta(\gamma(t))\gamma'(t))F_{n+2}^\pm(t)dt.$$

We define

$$D(\theta) = I^-(\theta) - I^+(\theta).$$

Clearly, D is a continuous linear mapping from $C_0^\infty(U)$ into $\mathscr{B}(H)$. Moreover, D has the following important property, which will be used later:

(A) If $D(\theta)(x) = 0$ for all $\theta \in C_0^\infty(U)$, then $U \subset \rho_T(x)$.

To prove (A), assume that $D(\theta)(x) = 0$ for all $\theta \in C_0^\infty(U)$. Then clearly,

$$\int_a^b \frac{d^{n+2}}{dt^{n+2}}(\varphi(t))[F_{n+2}^-(t) - F_{n+2}^+(t)](x)dt = 0$$

for every $\varphi \in C_0^\infty(a, b)$. It is not hard to show that $[F_{n+2}^-(t) - F_{n+2}^+(t)](x)$ must be a polynomial $P(t)$ of degree not greater than $n + 1$, with coefficients in H. Redefine $F_{n+2}(z)(x)$ by subtracting $P(z)$ on S_- (this does not change the $2 + n$th derivative of $F_{n+2}(x)$). Now $F_{n+2}(z)(x)$ is an analytic function of z on $S_- \cup S_+$, and is continuous on S. Therefore $F_{n+2}(z)(x)$ is analytic on S, and so are all its derivatives. In particular, the function $z \to R(\Gamma(z), T)(x)$ is analytic on S. Thus $R(\lambda, T)(x)$ is an analytic function of λ on U, and $U \subset \rho_T(x)$. This completes the proof of (A).

We also observe that $D(\theta)$ depends only on the restriction of θ to $\gamma(a, b)$. If $\varphi \in C_0^\infty(M)$ is such that supp $\varphi \subset \gamma(a, b)$, we can define $D^*(\varphi) = D(\theta)$, where θ is any function in $C_0^\infty(U)$ whose restriction to $\gamma(a, b)$ is φ.

We now show that $D : C_0^\infty(U) \to \mathscr{B}(H)$ is a homomorphism. Clearly, this is equivalent to showing that D^* is a homomorphism. Notice that D^* is defined for all $\theta \in C_0^{n+2}(M)$ whose support is contained in $\gamma(a, b)$, and that D^* is continuous in the C_0^{n+2} topology. To prove that D^* is a homomorphism, we first observe that it is sufficient to show that $D^*(\theta_1\theta_2) = D^*(\theta_1)D^*(\theta_2)$ when, for $j = 1, 2$, θ_j is of the form

$$\theta_j(\lambda) = 0 \text{ for } \lambda \in M, \lambda \notin \gamma[a_j, b_j],$$
$$\theta_j(\gamma(t)) = f_j(\gamma(t)) \text{ for } a_j \leqq t \leqq b_j,$$

where f_j is an analytic function of λ in U which has a zero of order $n + 2$ at $\gamma(a_j)$ and at $\gamma(b_j)$ (to see this, notice that every $\theta \in C_0^{n+2}(M)$ whose support is contained in $\gamma(a, b)$ can be approximated in the C_0^{n+2} topology by functions of the above type). We can also assume that $a_2 < a_1 < b_1 < b_2$. Let C_1, C_2 be contours as in figure 1.
Then, clearly:

$$D^*(\theta_j) = \frac{1}{2\pi i}\int_{C_j} f_j(\lambda)R(\lambda, T)d\lambda \text{ for } j = 1, 2.$$

Therefore

$$D^*(\theta_1)D^*(\theta_2) = -\frac{1}{4\pi^2}\left[\int_{C_1} f_1(\lambda)R(\lambda, T)d\lambda\right]\left[\int_{C_2} f_2(\mu)R(\mu, T)d\mu\right].$$

From the resolvent equation

$$R(\lambda, T)R(\mu, T) = (\mu - \lambda)^{-1}(R(\lambda, T) - R(\mu, T))$$

we get

$$D^*(\theta_1)D^*(\theta_2) =$$
$$-\frac{1}{4\pi^2}\int_{C_1} f_1(\lambda)\left[\int_{C_2}(\mu - \lambda)^{-1}f_2(\mu)(R(\lambda, T) - R(\mu, T))d\mu\right]d\lambda.$$

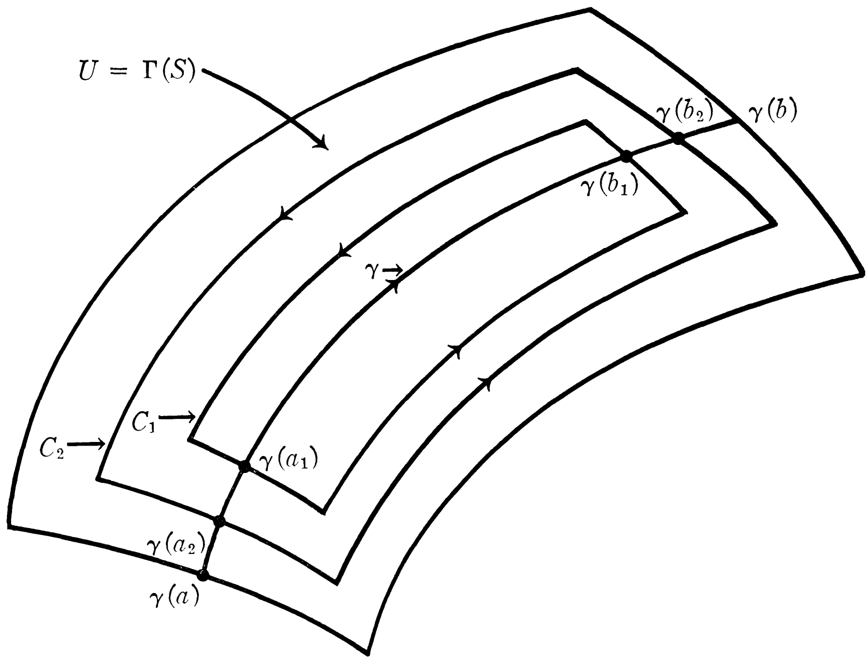

FIGURE 1

Now, the integral of $(\mu - \lambda)^{-1}f_2(\mu)d\mu$ along C_2 is $2\pi i f_2(\lambda)$, for $\lambda \in C_1$. Also, the integral of $f_1(\lambda)(\mu - \lambda)^{-1}d\lambda$ along C_1 is 0, for $\mu \in C_2$. Therefore we get

$$D^*(\theta_1)D^*(\theta_2) = \frac{1}{2\pi i}\int_{C_1} f_1(\lambda)f_2(\lambda)R(\lambda, T)d\lambda = D^*(\theta_1\theta_2).$$

So far, D has been defined and shown to be a homomorphism "locally", i.e. for functions $\theta \in C_0^\infty(U)$ whose support is "small" enough. Notice, however, that if λ_0 and λ_1 are two points of M, and if D_0 and D_1 are the corresponding homomorphisms, then D_0 and D_1 coincide in the intersection of their domains of definition. This enables us to define $D : C_0^\infty(U) \to \mathscr{B}(H)$ (using partitions of unity), and D is clearly a continuous homomorphism. We now define $D : C_0^\infty(\Omega) \to \mathscr{B}(H)$ by letting $D(\theta) = D^*(\theta^*)$, where θ^* is the restriction of θ to M. Then D is a continuous homomorphism.

Our aim is to extend D to the algebra $\mathscr{U}(\Omega)$. In order to define this extension, we shall use the following two facts:

(I) if $\theta \in C_0^\infty(\Omega)$, and $x \in H$, then $\sigma_T(D(\theta)(x)) \subset \operatorname{supp} \theta$ and

(II) if $V \subset \Omega$ is an open set, and if $x \in H$ is such that $D(\theta)(x) = 0$ for every $\theta \in C_0^\infty(V)$, then $\sigma_T(x) \cap V = \emptyset$.

The proof of (I) offers no difficulty. One can show, using the definition of D, that $D(\Lambda\theta) = TD(\theta)$ for every $\theta \in C_0^\infty(\Omega)$. From this it follows that, if $\mu \notin \operatorname{supp} \theta$, and if $\theta_\mu(\lambda) = \theta(\lambda)(\mu - \lambda)^{-1}$, then $(\mu I - T)D(\theta_\mu)(x) = D(\theta)(x)$. This gives the desired analytic extension of $R(\mu, T)D(\theta)(x)$ to the complement of supp θ, and completes the proof of (I). Assertion (II) is a trivial consequence of statement (A), which was proved before.

Using (I) and (II), we shall define D on $\mathscr{U}(\Omega)$. Let $f \in \mathscr{U}(\Omega)$. Let f be analytic on a neighbourhood U of $\mathbf{C}^*\backslash\Omega$. Take $x \in H$. Let $\theta \in C_0^\infty(\Omega)$ be such that $\theta = 1$ on a neighbourhood V of the complement of U.

Let $y = x - D(\theta)(x)$. If $\psi \in C_0^\infty(V)$, then $\psi\theta = \psi$, and $D(\psi)(y) = D(\psi)(x) - D(\psi)(x) = 0$. By (II), we conclude that $\sigma_T(y) \cap V = \emptyset$. Thus $\sigma_T(y)$ is contained in U, and f is analytic in U. We can therefore define $D(f)(y)$ by a formula similar to that of Dunford's functional calculus: Let Δ be a contour that surrounds $\sigma_T(y)$ and is contained in U, and take

$$D(f)(y) = \frac{1}{2\pi i}\int_\Delta f(\lambda)\bar{y}(\lambda)d\lambda.$$

Finally, define

$$D(f)(x) = D(f)(y) + D(f\theta)(x).$$

It is easily checked that the above definition of $D(f)$ is in fact independent of the choice of θ, and that D is an Ω-spectral distribution for T.

The proof of the converse, namely, that if T is Ω-generalized scalar then the growth condition is satisfied, will be briefly sketched. Assume that T is Ω-generalized scalar. Take $\lambda_0 \in \Omega$, and let F be a compact neighbourhood of λ_0, contained in Ω, and such that $F \cap \sigma(T)$ is an analytic arc $\gamma([a, b])$. By Lemma 9, the subspace $\chi_T(F)$ is closed, and the restriction T' of T to this subspace is a generalized scalar operator. Let D be an Ω-spectral distribution for T, and θ a C^∞ function which is equal to 1 in a neighbourhood U of λ_0 and vanishes outside of F. Then

$$R(\mu, T) = R(\mu, T)D(\theta) + R(\mu, T)D(1 - \theta).$$

Since $\sigma_T(D(1 - \theta)(x)) \cap U = \emptyset$ for every $x \in H$, it follows that the function $\mu \rightarrow R(\mu, T)D(1 - \theta)(x)$ is analytic, and hence bounded, in a neighbourhood of λ_0. By the uniform boundedness theorem, $||R(\mu, T)D(1 - \theta)||$ is bounded near λ_0. Now, clearly,

$$||R(\mu, T)D(\theta)|| \leqq ||R(\mu, T')|| \, ||D(\theta)||.$$

This shows that it is sufficient to prove that the growth condition holds for T'. In other words, we can assume that T is generalized scalar, and that $\sigma(T)$ is contained in an analytic arc. The proof for this case is, with some obvious modifications, identical to the proof of Theorem 4.5 of Colojoara and Foias [**1**, p, 160].

Remark. From our proof of Theorem 10, and from a detailed analysis of the proof of the above mentioned theorem in Colojoara and Foias [**1**], it follows that T is Ω-generalized scalar of finite order if and only if there is an n, independent of λ, such that (G_n) holds at every $\lambda \in \Omega$. Moreover, if this is the case, then the particular Ω-spectral distribution D constructed in the proof of Theorem 10 has finite order in Ω.

We now want to derive a similar result for the unbounded case. The following lemma is trivial.

LEMMA 11. *Let T be an operator, and let $\mu \in \rho(T)$. Let F be the function $F(\lambda) = (\mu - \lambda)^{-1}$, so that $F(\rho(T)) = \rho(R(\mu, T))$. Let Ω be open in* $\mathbf{C}^*$, *and let $\rho(T) \subset \Omega$. Then T is Ω-generalized scalar if and only if $R(\mu, T)$ is $F(\Omega)$-generalized scalar.*

Using Lemma 11, we can extend Theorem 10 to the unbounded case. For simplicity, we shall only consider the following situation: we assume that $\rho(T)$ contains a real number μ, and that the intersection of $\sigma(T)$ with the set $\{\lambda : |\lambda| > C\}$ is contained in the real axis (plus the point at infinity), for some $C > 0$. If Ω does not contain the point at ∞, then Theorem 10 holds without any modification. If, on the other hand, $\infty \in \Omega$, we seek a condition that will replace the growth condition (G_n). Clearly, what we need is that the resolvent $R(\mu, T)$ satisfy a growth condition at $F(\infty)$ (where F is the function of Lemma 11). We have

$$R(\lambda, R(\mu, T)) = \lambda^{-1}(\mu I - T)R(\mu - \lambda^{-1}, T).$$

Now $F(\infty) = 0$. Therefore, the growth condition (G_n) for $R(\mu, T)$ at $\lambda_0 = 0$ becomes

$$||(\mu I - T)R(\mu - \lambda^{-1}, T)|| = O(|\lambda| \cdot |\mathrm{Im}\, \lambda|^{-n}) \quad \text{as } \lambda \to 0,\ \mathrm{Im}\, \lambda \neq 0.$$

Letting $\lambda = (\mu - \zeta)^{-1}$, we get

$$||(\mu I - T)R(\zeta, T)|| = O(|\zeta|^{2n-1}|\,\mathrm{Im}\, \zeta|^{-n}) \quad \text{as } \zeta \to \infty,\ \mathrm{Im}\, \zeta \neq 0.$$

Since $TR(\zeta, T) = \zeta R(\zeta, T) - I$, the previous formula is equivalent (if $n \geqq 2$) to

$$(G_n^{\infty}) \ ||R(\zeta, T)|| = O(|\zeta|^{2n-2}|\,\mathrm{Im}\, \zeta|^{-n}), \quad \text{as } \zeta \to \infty,\ \mathrm{Im}\, \zeta \neq 0.$$

Summarizing, we have

THEOREM 12. *Let T be an operator with the s.v.e.p., and let Ω be open in* $\mathbf{C}^*$, *$\rho(T) \subset \Omega$. Assume that $\Omega \cap \sigma(T)$ is contained in a one-dimensional closed immersed analytic submanifold M of Ω, and that the intersection of $\rho(T)$ with the real axis is nonempty. Assume, moreover, that the intersection of $\sigma(T)$ with the set $\{\lambda : |\lambda| > C\}$ is contained in the real axis for a sufficiently large C. Then T is Ω-generalized scalar if and only if a growth condition (G_n) is satisfied near every finite $\lambda_0 \in M$ and if, in addition, the condition (G_n^{∞}) holds for some n when $\infty \in \Omega$.*

5. The second order differential operator. We shall now apply the theory of the preceding sections to the operator T of [**7**]. We recall that T is the (unbounded) operator defined in $L_2[1, \infty)$ by the formal differential operator

$$\tau = -\frac{d^2}{dt^2} + \frac{\alpha}{t} + q(t) \tag{5}$$

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with the boundary condition

$$(6) \qquad k_1 f(1) + k_2 f'(1) = 0.$$

Precisely, the domain of T is the set of all functions $f \in L_2[1, \infty)$ such that τf exists and belongs to $L_2[1, \infty)$, and such that f satisfies (6). The constant α is not required to be real. The boundary condition (6) is assumed to be non-trivial, i.e., $(k_1, k_2) \neq (0, 0)$. As for the function q, we assume that it satisfies, for some $r > 1$, the condition H_r of [**7**, p. 820].

It was proved in [**7**] that the spectrum of T consists of the halfline $[0, \infty]$, plus a set of eigenvalues which is a discrete subset of the complement of $\{0\}$ in $\mathbf{C}$. Moreover, T is not a spectral operator if $\alpha \neq 0$. Here we shall prove:

THEOREM 13. *The operator T is $\mathbf{C}^*\backslash\{0\}$-generalized scalar.*

Proof. We shall use Theorem 12. From the estimate of Lemma 3.5 of [**7**], and from (ii) of Corollary 2.4 of [**7**], it follows that T satisfies a growth condition (G_n) near every λ_0 such that $0 < \lambda_0 < \infty$. If λ_0 is an eigenvalue of T such that $\lambda_0 \notin [0, \infty)$, then it is also true that T satisfies a growth condition near λ_0. Indeed, formula (3.7) of [**7**] implies that $R(\lambda, T)$ has a pole at $\lambda = \lambda_0$.

It remains to be shown that the growth condition (G_n^∞) is satisfied for some n. It is easy to see that the proof of Lemma 2.5 of [**7**] actually yields an estimate which is slightly stronger than formula (2.17) of [**7**]. Precisely, the exponent $\operatorname{Re} \mu(t + 1)/2$ of [**7**, (2.17)] can be replaced by $\operatorname{Re} \mu(t - 1)/2$. From this it follows that the estimate (3.4) of [**7**, Lemma 3.2] is also valid if $K(\mu)$ is replaced by $K^*(\mu)$, where

$$(7) \qquad K^*(\mu) = \varphi(\mu) H(-\operatorname{Re} \alpha/\mu, \tfrac{1}{2} \operatorname{Re} \mu) e^{-\operatorname{Re}\mu/2}.$$

Let $V(\eta)$ be the region defined in the statement of Lemma 3.5 of [**7**], and let

$$V'(\eta) = \{\lambda : |\lambda| > \eta^2, \lambda \notin V(\eta)\}.$$

Let $\mu(\lambda)$ be defined as in the first formula of [**7**, p. 827]. Under the mapping $\lambda \to \mu(\lambda)$, the regions $V(\eta)$ and $V'(\eta)$ correspond to

$$W(\eta) = \{\mu : |\mu| > 2\eta, 0 < \operatorname{Re} \mu < 2\},$$

and

$$W'(\eta) = \{\mu : |\mu| > 2\eta, \operatorname{Re} \mu \geqq 2\}$$

respectively.

If η is large enough, the inequality

$$|\operatorname{Re} \alpha/\mu| < 1$$

will be valid for all $\mu \in W'(\eta)$. From the definition of $\varphi(\mu)$ (cf. [**7**, Lemma 2.5]), it follows that

(I) $\varphi(\mu)$ is bounded for $\mu \in W'(\eta)$, η sufficiently large. Also

(II) $H(-\operatorname{Re} \alpha/\mu, \frac{1}{2} \operatorname{Re} \mu)$ is bounded for $\mu \in W'(\eta)$, η sufficiently large (here H is defined as in formula (3.6) of [**7**]).

We conclude from formula (7) that

(III) $K^*(\mu)$ is bounded by a constant times $e^{-\text{Re}\mu/2}$, for $\mu \in W'(\eta)$, η sufficiently large.

We now use the modified version of [7, (3.4)], in which K is replaced by K^*. Since Re $\mu > 2$ for $\mu \in W'(\eta)$, we have

(IV) $||G(\mu)|| < Ce^{-\text{Re}\mu/2}$ for $\mu \in W'(\eta)$, η sufficiently large, and C a positive constant. From the formulas that describe the asymptotic behaviour of $A(\lambda)$ [7, Corollary 2.4 (iii)] and from formula (3.7) of [7], we conclude that

(V) If η is sufficiently large, then $||R(\lambda, T)||$ is uniformly bounded for $\lambda \in V'(\eta)$.

We now want a similar result for the region $V(\eta)$. We shall use [7, Lemma 3]. If η is sufficiently large, then: (1) $\delta(\lambda) < 1$ for all $\lambda \in V(\eta)$, and (2) $\gamma < 2$ (for the definition of $\delta(\lambda)$ and of γ, cf. the statement of Lemma 3.5 of [7]). If, in addition, we take $\eta > 1$, then the inequality

$$|\text{Im }\lambda|^{-1-\delta(\lambda)} \leqq |\lambda|\, |\text{Im }\lambda|^{-2} \tag{8}$$

is valid for all $\lambda \in V(\eta)$.

Moreover

$$1 + |\text{Re }\sqrt{\lambda}|^{\gamma} \leqq 2|\lambda| \quad \text{for all } \lambda \in V(\eta). \tag{9}$$

Finally, the formulas of [7, Corollary 2.4 (iii)], show that $A(\lambda)$ is bounded away from zero in $V(\eta)$, provided η is large enough. Therefore, the estimate (3.12) of [7] implies that

(VI) If η is large enough, then $||R(\lambda, T)||$ is bounded by a constant times $|\lambda|^2|\text{Im }\lambda|^{-2}$ for all $\lambda \in V(\eta)$.

In view of (V), it is clear that (VI) is also valid in $V'(\eta)$. Thus, we have shown that

$$||R(\lambda, T)|| = O(|\lambda|^2|\text{Im }\lambda|^{-2})$$

as $\lambda \rightarrow \infty$, $\lambda \in \rho(T)$. But this says that the growth condition (G_2^{∞}) is satisfied. The desired conclusion now follows from Theorem 12, and the proof of Theorem 13 is therefore complete.

Theorem 13 implies that the operator T has an operational calculus for functions that are C^{∞} in a neighbourhood of the spectrum of T, and analytic near the origin. A spectral distribution for T is given by the construction of the proof of Theorem 10. We shall write $\theta(T)$ to denote the operator corresponding to θ under this distribution. Thus, we have, in particular

LEMMA 14. *Let θ be a C^{∞} function of compact support in the complex plane. Assume that the intersection of the support of θ with the resolvent set of T is a subset of the open half-line $(0, \infty)$ (i.e. θ vanishes near 0 and also near every eigenvalue of T which does not belong to $[0, \infty)$). If f and g belong to $L_2[1, \infty)$, then*

$$(\theta(T)f, g) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_0^{\infty} \theta(\lambda)((R(\lambda - i\epsilon) - R(\lambda + i\epsilon))f, g)d\lambda. \tag{10}$$

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If the support of θ does not contain any of the zeroes of $A^+(\lambda)$ or $A^-(\lambda)$ (cf. [7, Corollary 2.4 (iv)]), and if the functions f and g belong to $L_2^0[1, \infty)$ (the set of all square-integrable functions in $[1, \infty)$ that vanish in the complement of a bounded interval), then we can use Corollary 3.4 of [7] (cf. also section 5 of the present paper). We get

$$(\theta(T)f, g) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\lambda}\cdot\theta(\lambda)}{A^+(\lambda)A^-(\lambda)}\, \hat{f}(\lambda)\,\overline{\hat{g}(\lambda)}\, d\lambda \tag{11}$$

where, for each $h \in L_2^0[1, \infty)$, we define

$$\hat{h}(\lambda) = \int_1^\infty \omega(t, \lambda)h(t)dt. \tag{12}$$

Clearly, formula (12) defines an analogue of the Fourier transform, and formula (11) resembles the definition of the operational calculus for self-adjoint operators.

6. Conclusion. It has been shown that the theory of operators that are generalized scalar in an open subset of $\mathbf{C}^*$ makes it possible to define an operational calculus for second order ordinary differential operators whose potential behaves like t^{-1} times a complex constant. This shows that this theory is adequate for a large class of operators for which Dunford's theory of spectral operators fails. We conjecture that the situation described here is in fact very general. If "reasonable" assumptions are made on the behaviour of the coefficients, it should be possible to extend Theorem 13 to nth order operators in arbitrary intervals, and to show that such operators are Ω-generalized scalar for some set Ω whose complement is finite (or, perhaps, countable).

A second question of interest is that of finding conditions under which it is possible to guarantee that the operator is generalized scalar, i.e. to take $\Omega = \mathbf{C}^*$. It will be shown in a forthcoming paper that, in general, it is not possible to "remove the singularities". Precisely, it will be proved that the "singularity at 0" of the operator T described in section 5 is "essential" (i.e. T is not generalized scalar), if the constant α has a nonzero imaginary part. Thus, the operational calculus that was defined here for C^∞ functions that are analytic near 0 cannot be extended to arbitrary C^∞ functions.

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