R. C. Gunning, "Connections for a Class of Pseudo-group Structures."

H. Hironaka, "A Fundamental Lemma on Point Modifications."

H. Rohrl, "Transmission Problems for Holomorphic Fiber Bundles."

H. Rossi, "Attaching Analytic Spaces to an Analytic Space Along a Pseudoconcave Boundary."

A. Morimoto, "Non-compact Complex Lie Groups without Non-constant Holomorphic Functions."

E. Bishop, "Uniform Algebras."

B. Maskit, "Construction of Kleinian Groups."

L. V. Ahlfors, "The Modular Function and Geometric Properties of Quasiconformal Mappings."

E. Kallin, "Polynomial Convexity: The Three Spheres Problem."

The volume ends with a list of twenty-six problems posed by members of the Conference.

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<u>Number Theory</u>, by Z.I. Borevich and I.R. Shafarevich. Academic Press, New York, 1966. x + 435 pages. $12.95.

<u>Theorie des nombres</u>, par Z.I. Borevitch et I.R. Chafarevitch. Traduit par M. et J.-L. Verley; Monographies internationales de mathematiques modernes, sous la direction de S. Mandelbrojt. Gauthier-Villars, Paris, 1967. vi + 489 pages. Price: 78 F.

With the following notation and terminology of the authors' it is possible to state the aims of this book fairly simply. Let k denote the rational field and let K be any finite extension of k of degree n. Let $\mu_1, \mu_2, \ldots, \mu_m$ $(m \leq n)$ be linearly independent elements of K over k with conjugates μ_i^j $(1 \leq i \leq m,\ 1 \leq j \leq n)$ over k and define a form $F(x_1, \ldots, x_m)$ of degree n over $k[x_1, \ldots, x_m]$ by taking the norm:

$$N(x_1\mu_1 + \ldots + x_m\mu_m) = \prod_{j=1}^{n} (x_1\mu_1^j + \ldots + x_m\mu_m^j) \tag{1}$$

of $x_1\mu_1 + \ldots + x_m\mu_m$. Then the number-theoretic questions are mainly concerned with the rational integral solutions, for given $a \in k$, of the diophantine equation

$$F(x_1, \ldots, x_m) = a\ , \tag{2}$$

the existence or non-existence of such solutions and, generally, the structure of the set of solutions. We may distinguish two categories in the treatment of (2), namely:

$$\text{(I)}\quad m = n\ (a \neq 0)\quad ,\qquad \text{(II)}\quad m < n\ .$$

Briefly, in (I) the existence of a solution is shown to lead to an infinity of solutions (and to a characterization of the set of solutions) by means of the celebrated theorem of Dirichlet on the structure of the group of units* (i.e., integers $\xi \in K$ with $N(\xi) = \pm 1$) of K and the treatment here follows the usual lines. However, in (II), where Fermat's equation

$$x_1^{\ell} + x_2^{\ell} - x_3^{\ell} = 0$$

resides, the apparatus required for a modern treatment is elaborate and the results are by no means as complete as in category (I).

Firstly, let us note that, in either category, the solvability of (2) is clearly impossible unless it is solvable as a congruence to every modulus and the first chapter is devoted to an account of polynomial congruences (in several variables) and to the closely related question of p-adic integral solutions of (2), where the notion of a discrete valuation is first broached.

The main problem is investigated by means of the properties of the module M belonging to F, that is, the Z-module defined as the set of numbers $\{x_1\mu_1 + \ldots + x_m\mu_m\}$, $x_i \in Z$ in K. M (and F) is said to be "full" if $m = n$ and "non-full" otherwise. A full module M which contains the unit 1 and has ring structure is said to be an order D of the field K. In particular, the coefficient ring (i.e., take all $\alpha \in K$ with $\alpha M \subset M$) is an example of an order of K and the resolution of the problem in (I) takes the following form:

If K has s real and $2t$ complex isomorphisms into the field of complex numbers, $(n = s + 2t)$, then there exist $r = s+t-1$ units $\varepsilon_1, \ldots, \varepsilon_r$ in the order D and there exists a finite set of elements $\nu_1, \ldots, \nu_\lambda$ of M with norm a such that

$$\mu \in M,\ N(\mu) = a \Rightarrow \mu = \begin{cases} \nu_i\, \varepsilon_1^{u_1} \ldots \varepsilon_r^{u_r}, & n \text{ odd}, \\ \nu_i\, \xi\, \varepsilon_1^{u_1} \ldots \varepsilon_r^{u_r}, & n \text{ even}. \end{cases} \qquad u_i \in Z, \xi \text{ a root of } 1.$$

The special case $n = 2(r=1)$ is then considered and a classification of "similar" modules defined to bring the development into parity with the usual theory of representation of numbers by binary quadratic forms.

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* except when K is a complex quadratic field when the group of units is finite.

We enter category (II) in Chapter 3 which is devoted to divisibility properties in an arbitrary integral domain (where, subsequently, K is to be regarded as its quotient field). Following historical precedent, this is illustrated by a discussion of the usual two cases of Fermat's theorem, where the equation is now expressed in multiplicative form:

$$\prod_{1 \leq j \leq \ell} (x_1 + \zeta^j x_2) = x_3^{\ell}$$

over $k = k(\zeta)$, ζ being a primitive ℓ^{th} root of 1. In the minority of cases where ℓ is prime and the (maximal) order $D = \{1, \zeta, \zeta^2, \ldots, \zeta^{\ell-2}\}$ has the unique factorization property (u.f.p.) Fermat's theorem presents no insurmountable difficulty and the main body of this chapter is concerned with restoring the deficiency in the u.f.p. (whether ℓ is prime or not) as far as possible. The method consists, in essentials, in setting up a "theory of divisors" with the u.f.p., on a modified "copy" of D. More precisely, starting with an arbitrary integral domain D and denoting by D^* the multiplicative semi-group of all non-zero elements of D, a theory of divisors for D is defined if there exists a homomorphism $\alpha \to (\alpha)$ of D^* into $\mathfrak{D}$ with the properties*

(1) $\beta | \alpha$ in $D^* \Longleftrightarrow (\beta) | (\alpha)$ in $\mathfrak{D}$,

(2) $\mathfrak{a} | (\alpha)$, $\mathfrak{a} | (\beta), \mathfrak{a} \in \mathfrak{D}$, $\alpha \in D^*$, $\beta \in D^*$ $\Longrightarrow$ $|(\alpha \pm \beta)$ in $\mathfrak{D}$,

(3) Given $\mathfrak{a} \in \mathfrak{D}$, $\mathfrak{b} \in \mathfrak{D}$ and the set of all $\alpha \in D$ such that $\mathfrak{a} | (\alpha)$ in $\mathfrak{D}$ coincides with the set of all $\beta \in D$ such that $\mathfrak{b} | (\beta)$ in $\mathfrak{D}$. Then $\mathfrak{a} = \mathfrak{b}$. The elements of $\mathfrak{D}$ are called divisors of D and divisors of the form (α), where $\alpha \in D^*$ are called principal divisors (unit element of $\mathfrak{D}$ is called the unit divisor). Also, it is conventional to write $\mathfrak{a} | \alpha$ for $\mathfrak{a} | (\alpha)$.

It is important to observe that a theory of divisors is not asserted. However, it is easily established that if one such exists then, up to isomorphisms, it is unique and that D has the u.f.p. if, and only if, D has a theory of divisors in which every divisor is principal. This approach leads naturally to, and provides motivation for a discussion of discrete valuations for D by assigning to each $\alpha \in D^*$ the integer $\nu_{\mathfrak{p}}(\alpha)$ corresponding to the power to which the prime divisor $\mathfrak{p}$ enters in the (unique) factorization of the principal divisor (α) into prime factors in $\mathfrak{D}$, i.e.,

$$(\alpha) = \prod_i \mathfrak{p}_i^{\nu_{\mathfrak{p}_i}(\alpha)};$$

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* It is important to note that condition (3) ensures that no element of $\mathfrak{D}$ is spurious in the sense that for each $\mathfrak{a} \in \mathfrak{D}$ there is an element $\alpha \in D$ such that $\mathfrak{a} | (\alpha)$.

the corresponding extension to the quotient field K of D being obtained by putting

$$\nu_{\mathfrak{p}}(\xi) = \nu_{\mathfrak{p}}(\alpha) - \nu_{\mathfrak{p}}(\beta) ,$$

where $\xi = \alpha/\beta \in K$, $\alpha \in D$, $\beta \in D^*$. Moreover, the introduction of valuations provides a constructive method of determining whether a given D has a theory of divisors. Thus, for example, it is shown that if D is the maximal order of K then there exists a theory of divisors for D that is induced by the set of all (discrete) valuations of the field K (and more generally the same holds for any Dedekind ring D). Then, following a brief discussion of divisor classes (where two divisors of K are said to belong to the same class if, and only if, they differ by a factor which is a principal divisor), the application to Fermat's equation is apparent - the so-called first case of Fermat's theorem holds for all prime exponents ℓ which do not divide the number of divisor classes of $K = k(\zeta)$. Further examples illustrate the practical problem of finding the prime divisors of numbers in specific algebraic member fields (e.g., $k(\theta)$, where $\theta^5 = 2$ and $\theta^3 - 9\theta - 6 = 0$) and quadratic fields are fully treated.

In Chapter 4 we remain with the general problem in category II and investigate the class of non-full forms (1) for which there are but a finite (possible zero) number of solutions of the type under consideration. That this class is non-void is known from the classical work of Thue on irreducible binary forms $f(x_1, x_2)$ of degree $n \geq 3$. However, Thue's method, which is based on rational approximations to a given algebraic number, in this case a root of $f(x, 1) = 0$, is limited in applications and the authors' have preferred (i) to apply the valuation theory of Chapter 3 to finite extensions of complete fields and (ii) couple it to the pioneering work of Skolem on local methods for diophantine equations. So far as (i) is concerned, the supplementary material required relates to the completions of the algebraic number field K (or, at no extra cost, any field possessing a theory of divisors). Thus if $\mathfrak{p}$ is any prime divisor of K and $\nu = \nu_{\mathfrak{p}}$ is the corresponding valuation and we fix upon a real number ρ with $0 < \rho < 1$, a metric (*) $\phi = \phi_{\mathfrak{p}}$ on K is induced by putting

$$\phi(x) = \rho^{\nu(x)} , \quad x \in K$$

and $\overline{K} = K_{\mathfrak{p}}$ (i.e., the $\mathfrak{p}$-adic completion of K) is simply the completion of K with respect to this metric. In particular, the valuation ν on K extends uniquely to a valuation $\overline{\nu}$ on $\overline{K}$ with $\overline{\nu}(x) = \nu(x)$ for $x \in K$. Now $\overline{K}$ has algebraic extensions of all degrees and so, for fixed $n \geq 2$, we can consider such a field K' with $[K':\overline{K}] = n$. By the work of Chapter 3, we know that there exists an extension of the valuation $\overline{\nu}$ of $\overline{K}$ to a valuation ν' of K', but now, under the additional hypothesis

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* In this translation, valuations are discrete with value group Z and the term "metric" is reserved for similar mappings into the reals.

that $\bar{K}$ is complete, it follows that ν' is unique and that K' is complete. If $\mathfrak{P}'$ is the prime divisor corresponding to ν', the set of elements $\alpha' \varepsilon K'$ with $\nu'(\alpha') \geq 0$ form a ring (the ring of the valuation) $D_{\nu'}$ and the elements are the $\mathfrak{P}'$-adic integers of K'. With these preliminary remarks about (i) we can briefly sketch the coupling (ii) with Skolem's method:

Regard the module $M = \{x_1\mu_1 + \ldots + x_m\mu_m\}$ of the form F (where now $m < n$) as a submodule of a suitable chosen full module of the shape

$$\bar{M} = \{x_1\mu_1 + \ldots + x_m\mu_m + x_{m+1}\mu_{m+1} + \ldots + x_n\mu_n\}$$

and use the technique of Chapter 2 (c.f. (3) above) to find all rational integral solutions $x_1, \ldots, x_n$ of

$$N(\mu) = a\ , \quad \mu \ \varepsilon\ \bar{M}\ .$$

Then it is sufficient to find a method of picking out those solutions of the form $\mu = x_1\mu_1 + \ldots + x_n\mu_n$ for which the coefficients of $x_{m+1}, \ldots, x_n$ are simultaneously zero. Now, in terms of the dual basis $\mu_1^*, \ldots, \mu_n^*$, where we have the usual relations

$$T(\mu_j\mu_i^*) = \delta_{ij}\ , \quad 1 \leq i \leq n\ , \quad 1 \leq j \leq n$$

for the trace, the numbers of $\bar{M}$ which belong to the submodule M are completely characterized by the conditions

$$T(\mu\mu_i^*) = 0 \qquad (i = m+1, \ldots, n)\ .$$

But, on reference to (3), we also know that μ has the form

$$\mu = \nu_i\, \zeta\, \varepsilon_1^{u_1} \ldots \varepsilon_r^{u_r}\ .$$

Hence the problem amounts to showing that, under suitable conditions, the system of equations in the rational integral variables $u_1, \ldots, u_r$:

$$(4) \qquad T(\nu\mu_i^*\varepsilon_1^{u_1} \ldots \varepsilon_r^{u_r}) = 0\ , \quad (i = m+1, \ldots, n)$$

where ν takes one of at most finitely many values in $\bar{M}$ with norm a, does not have an infinity of solutions. Since we have already introduced numbers composed from the fields conjugate to K, e.g. μ_i^*, we now fix upon an algebraic number field K' containing all fields conjugate to K. If $\sigma_1, \ldots, \sigma_n$ is the set of isomorphisms of K into K', then

$T(\xi) = \sigma_1(\xi) + \ldots + \sigma_n(\xi)$ for all $\xi \varepsilon K'$ and the system (4) reduces to

$$(5) \qquad \sum_{j=1}^{n} \sigma_j(\nu\mu_i^*) \sigma_j(\varepsilon_1)^{u_1} \ldots \sigma_j(\varepsilon_r)^{u_r} = 0 , \quad (i = m+1, \ldots, n) .$$

This is a system of n-m equations in r variables and on empirical grounds we might expect that, if $n-m \geq r$, there are at most finitely many solutions. However, this is not always the case (when e.g., M contains a full module of a subfield of K which is neither the rational field nor an imaginary quadratic field) and, at present, certain technical difficulties present a barrier to results of any generality. Skolem's approach, which is successful in some cases, is to extend the domain of the variables $u_1, \ldots, u_r$ by allowing them to take integral values in the completion $K'_{\mathfrak{P}'}$, where $\mathfrak{P}'$ is a prime divisor of K'. More precisely, the system (5) when expressed in logarithmic form:

$$(6) \qquad \sum_{j=1}^{n} A_{ij} L_j(u_1, \ldots, u_r) = 0 , \quad (i = m+1, \ldots, n)$$

where $L_j(u_1, \ldots, u_r) = \sum_{k=1}^{r} u_k \log \sigma_j(\varepsilon_k)$, $A_{ij} = \sigma_j(\nu\mu_i^*)$ and where the left side of (6) is a formal power series (converging for all $\mathfrak{P}'$-adic integral u_i), is regarded as defining, for each ν, a local analytic manifold in the r-dimensional space of points $(u_1, \ldots, u_r)$, where each component u_i lies in $K'_{\mathfrak{P}'}$ or in some finite extension of $K'_{\mathfrak{P}'}$. The success of this method hinges upon showing that (6) has no solution in formal power series $w_i(t)$ of the type $u_i = w_i(t)$, $(i = 1, 2, \ldots, r)$. The procedure is illustrated by taking Thue's example, where $m = 2$ and the condition $n - m \geq r$ reduces to $t \geq 1$, i.e., $f(x, 1) = 0$ has at least one non-real root.

The final Chapter (ch. 5) has a distinctly different flavour from the preceding ones in the sense that, on introducing analytical methods into a discussion of properties of certain arithmetical constants of an algebraic number field K, there is a much closer affinity to the works of the classical writers, e.g. of Kummer, Dirichlet and Dedekind. This is particularly true in the derivation of formulae for the number h of divisor classes (a positive integer!), where the Dedekind Zeta function, Dirichlet's principle and the Dirichlet unit theorem not only play their classical role but provide the only known general way of expressing h in terms of certain simpler arithmetic constants of K. The special case $K = k(\zeta)$, where $\zeta^\ell = 1$, of cyclotomic fields is discussed at length and h is expressed as the product of two positive integers h_o and h^* which have special significance in the proof of the second case of Fermat's theorem (for regular primes ℓ). Analogous formulae for the case of quadratic fields are also given.

Apart from the chapters in the book there is an excellent appendix supplying relevant background material on algebraic topics: equivalence of quadratic forms (Witt's theorem), algebraic extensions, finite fields and characters of finite abelian groups. Among the chapters and sometimes augmenting the text of the various sections comprising a chapter, there is also a very fine collection of examples. References to the literature and discussions of the latest developments on the various topics are particularly welcome and one would have wished that the authors had been more systematic in this one respect, especially as the work of Russian authors may occasionally be overlooked. The book concludes with a number of tables of arithmetical constants taken from various published sources, e.g. class numbers for quadratic and cubic fields, fundamental units for quadratic fields, irregular primes and discriminants of maximal and non-maximal orders. In total, this makes a really significant and distinguished addition to the literature on Number-theory and in recording our indebtedness to the authors, I also thank the translator who has managed to convey the mathematical sense of the original while permitting the style of the authors to permeate the translation.

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Scales and Weights, by Bruno Kisch. McGill University Press, 295 pages + 98 plates. $15.00.

This interesting book is devoted to the history of weighing, from the earliest times to the introduction of the metric system. Paintings in Egyptian tombs, excellently reproduced in the book, show clearly that metrology was already an old art, or perhaps an old science, more than 3000 years ago. The Egyptian paintings show beautifully constructed scales sometimes weighing materials and sometimes used by Egyptian gods to weigh peoples' souls before passing divine judgement. This concept of spiritual weighing was, of course, a familiar one in biblical times as is shown by the famous story of Belshazzer's feast.

The book deals faithfully with the various instruments used in weighing, such as scales with a variety of weights, bismars with no weights and steelyards which use only one weight. Many of the 98 plates give clear pictures of such apparatus dating from the earliest times down to the modern chemical balance. A bismar consists of a beam carrying at one end a scale pan and having a variable point of support. The point of support is moved until a balance is reached and a mark at the position of balance then shows the weight in the pan. The steelyard is somewhat similar but the point of support is fixed on the beam while a weight can be moved along the beam until a balance is reached.

Comprehensive tables are given of various systems of weights used from biblical times to our own time, a feature which should be of great use to all students of metrology. The number and variety of these systems is amazing.

The history of the metric decimal scale is given in chapter 3. It