BULL. AUSTRAL. MATH. SOC. 22A05, 43A40, 22B05
VOL. 52 (1995) [297–311]

FREE ABELIAN TOPOLOGICAL GROUPS AND THE PONTRYAGIN-VAN KAMPEN DUALITY

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We study the class of Tychonoff topological spaces such that the free Abelian topological group $A(X)$ is reflexive (satisfies the Pontryagin–van Kampen duality). Every such X *must be totally path-disconnected and (if it is pseudocompact) must* have a trivial first cohomotopy group $\pi^1(X)$. If X is a strongly zero-dimensional space which is either metrisable or compact, then $A(X)$ is reflexive.

INTRODUCTION

In this paper we investigate to what extent the machinery of the Pontryagin–van Kampen duality is applicable to the study of Markov free topological groups [**14**] on completely regular topological spaces. A topological group is called *reflexive* if it is canonically isomorphic to the second character group, and the Pontryagin–van Kampen duality provides a fine tool for analysing the structure of such groups. Among the best known classes of reflexive groups are locally compact Abelian groups, their direct products [**12**], and additive groups of Banach spaces [**27**]. However, free Abelian topological groups are not to be found in any of these classes – unless X is discrete, $A(X)$ is neither locally compact nor metrisable (see, for example, [**1**, 4.11 and 4.14]). Therefore, the problem of singling out those spaces X for which the group $A(X)$ is reflexive is of some interest.

It is known that for a vast class of spaces X the group $A(X)$ is non-reflexive – this is the case if at least one path component of a k_ω-space X contains more than one point. (A *k_ω-space*, X, is a union of countably many compact subspaces carrying the weak topology with respect to this family.) This result of a negative character, due to Nickolas [**18**], remained for some years the only known bit of information about the reflexivity of the groups $A(X)$.

In the present paper the character group $A(X)^\wedge$ is computed and certain necessary or sufficient conditions for the group $A(X)$ to be reflexive are analysed. In particular, it is shown that total path-disconnectedness forms such a necessary condition for *every* space X. For the first time the results in a positive direction are obtained: it is shown in Section 5 that $A(X)$ is reflexive for a rather wide class of spaces X, containing,

Received 6th December 1994.

in particular, all zero-dimensional compact spaces and all strongly zero-dimensional metrisable spaces.

The present paper was originially written (in Russian) in early 1983, while the author was a PhD student at Moscow State University, and this research had been stimulated by Morris's book [**16**] (which had just become available in Russian translation) and Noble's paper [**19**]; the author's thanks go to his PhD advisor, Prof. A.V. Arhangel'skiĭ, for bringing Noble's work to his attention. The results of the paper were not included in the author's 1983 PhD thesis, and only a fraction of them, practically without proofs, were announced in a minuscule 1985 note [**22**] in a rather obscure journal, not readily available beyond (ex)USSR's borders. For a number of years that followed the author deemed the manuscript lost.

The decision to publish the results now, more than a decade later, was motivated by an unceasing, yet moderate, flow of research on Pontryagin–van Kampen duality (see [**3**] or the recent elegant notes [**27**] and [**15**]). In general terms, dual objects, of which the character groups form a simple commutative speciman, are of paramount importance in quantum group theory and noncommutative analysis and geometry; on the other hand, various close relatives of free Abelian topological groups seem to gain in significance as well (see our recent survey [**23**]).

0. Preliminaries

All topological spaces are assumed to be completely regular T_1-spaces. For a topological group G we denote by $G^\wedge$ the group of characters of G (continuous homomorphisms to the circle group $\mathbb{T}$). A subset A of a topological group G is *precompact* if it can be covered by finitely many translations of an arbitrary neighbourhood of the identity of G. For our purpose it is more convenient, following Raĭkov [**24**], to equip $G^\wedge$ with the topology of uniform convergence on all precompact subsets of G (instead of the ordinary compact-open topology; however, we shall observe soon that for a very broad class of topological groups the two resulting dualities are identical.) By $\nu: G \to G^{\wedge\wedge}$ we denote the canonical homomorphism from G to the second character group: $\nu(g)(\chi) = \chi(g)$ for all $g \in G$ and $\chi \in G^\wedge$. A group G is called *reflexive* if ν is a topological isomorphism onto.

By $\widehat{G}$ we denote the completion of a Hausdorff topological group (with respect to the two-sided uniformity, see [**25, 20**]).

An arbitrary space X sits as a closed topological subspace in the (Markov) free Abelian topological group $A(X)$ on X; the group $A(X)$ is algebraically free on X and every continuous mapping $f: X \to G$, where G is any Abelian topological group, extends in a unique fashion to a continuous homomorphism $\overline{f}: A(X) \to G$ (see [**14, 10, 1**]. A solid survey on the subject is [**17**]; a number of fine results on the structure of

$A(X)$ are contained in [**29**]).

We denote by $L(X)$ the free locally convex space on X [**14, 6, 7, 26, 30**]. It contains X as a closed topological subspace and as a vector basis in such a way that every continuous mapping $f: X \to E$, where E is any locally convex space, extends in a unique way to a continuous linear operator $\overline{f}: L(X) \to E$. The identity map Id_X gives rise to a continuous group homomorphism γ from $A(X)$ to the additive group of $L(X)$; Theorem 3 in [**29**] states that γ is a topological isomorphism of $A(X)$ onto a closed subgroup of $L(X)$.

By θX we denote the Dieudonné completion of a topological space X, that is, the completion of X with respect to the finest compatible uniform structure $\mathcal{U}_X$ [**5**, 8.5.13].

Recall that a topological space X is *strongly zero-dimensional* [**5**, 6.2] if every finite cover of X consisting of functionally open sets has a finite disjoint open refinement. (A set is functionally open if its completement is the locus of zeros of a continuous real-valued function on X.) If X is normal, "functionally" becomes redundant. A space X is strongly zero-dimensional if and only if βX is.

A subset A of a space X is termed *bounded in* X if the restriction to A of every continuous real-valued function f on X is bounded. By μX we denote the smallest subspace of the Stone-Čech compactification βX, containing X and such that every bounded closed subset of μX is compact. We say that X is a μ*-space* if $X = \mu X$. Always $\mu X \subseteq \theta X$ and, in particular, every Dieudonné complete space is a μ-space. (The above definitions and results can be found in [**4**].)

A topological group G is called a k*-group* [**19**] if an arbitrary homomorphism h from G to an arbitrary topological group is continuous whenever the restrictions of h to all compact subspaces of G are continuous. The most important examples of k-groups are locally compact groups, metrisable groups, and, more generally, all topological groups which are k-spaces. Recall also that a space X is called a k_f*-space* [**2**] if an arbitrary real-valued function on X (equivalently: an arbitrary mapping from X to a completely regular space) is continuous as soon as its restriction to every compact subset of X is continuous.

1. The group $A(X)^\wedge$

Proposition 1. *If X is a k_f-space, then $A(X)$ is a k-group.*

Proof: Let f be a homomorphism from $A(X)$ to a topological group G, and assume that the restriction of f to any compact subset of G is continuous. In particular, the restriction of the map $f' \equiv f|_X$ to any compact subset of X is continuous; since X is a k-space, f' is continuous, and so is its unique homomorphic extension to $A(X)$, that is, f. ∎

https://doi.org/10.1017/S0004972700014726 Published online by Cambridge University Press

We say that a topological group G is a *pk-group* if any homomorphism f from G to a topological group is continuous as soon as the restriction of f to every precompact subset of G is continuous. Following [**30**], we call a topological space X a *b_f-space* if an arbitrary real-valued function on X is continuous whenever its restriction to every bounded subset of X is continuous.

LEMMA 1. *A subset H of a space X is bounded in X if and only if H is precompact in $A(X)$.*

PROOF: The closure of X in the completion of $A(X)$ coincides with θX. (See [**21**]; the proof of the Theorem and all results presented in [**21**] for free topological groups are valid *verbatim* for free Abelian topological groups.) Boundedness of H is equivalent to compactness of $\mathrm{cl}_{\theta X} H$ (θX is a μ-space), and precompactness of H in $A(X)$ means exactly that the closure of H in the completion of $A(X)$ (and, in fact, in θX) is compact. ∎

PROPOSITION 2. *If X is a b_f-space then $A(X)$ is a pk-group.*

PROOF: This follows from Lemma 1 and is similar to the proof of Proposition 1. ∎

PROPOSITION 3. *If G is a pk-group then the group $G^\wedge$ is complete.*

PROOF: This follows from the fact that $G^\wedge$ in the case where G is a pk-group can be thought of as the group of *all* homomorphisms $G \to \mathbb{T}$ with continuous restrictions to all precompact subsets of G, endowed with the topology of uniform convergence on precompact subsets; but such an Abelian group is obviously complete. ∎

We denote by G^* the character group of G, endowed with the compact-open topology.

PROPOSITION 4. *If G is a complete group, then $G^\wedge$ and G^* are (canonically) topologically isomorphic.*

PROOF: In a complete group all closed precompact subsets are compact. ∎

Propositions 3 and 4 imply:

PROPOSITION 5. *If G is a complete k-group then $G^{\wedge\wedge}$ and G^{**} are (canonically) topologically isomorphic.*

For an $x \in A(X)$ we denote by $\mathrm{supp}\, x$ the set $\{x_1, \dots, x_n\} \subseteq X$, where $x = \sum_{i=1}^{n} \varepsilon_i x_i$, $\varepsilon_i \in \mathbb{Z} \setminus \{0\}$, $x_i \neq x_j$ for $i \neq j$ is an irreducible representation of x in the alphabet X. If $B \subseteq A(X)$, we set $\mathrm{supp}\, B = \bigcup\{\mathrm{supp}\, x : x \in B\}$.

PROPOSITION 6. (Arhangel'skiĭ) *If $C \subseteq A(X)$ and C is precompact, then* $\mathrm{supp}\, C$ *is bounded in X.*

PROOF: This follows by applying Proposition 2 in [**2**] to the spaces X and $A(X)$, respectively, and to the correspondence $f \mapsto \overline{f}$, where $f: X \to \mathbb{R}$ and $\overline{f}$ is a homomorphism from $A(X)$ to $\mathbb{R}$, viewed as a linear map $C(X) \to C(A(X))$. □

We denote by r the restriction map $f \mapsto f|_X$ from $A(X)^\wedge$ to the group $C_b(X, \mathbb{T})$ (the subscript "b" denotes the topology of uniform convergence on bounded subsets). For an $H \subseteq X$, $F \subseteq G$, and and an open subset $U \subseteq \mathbb{T}$ we set

$$M(F, U) \stackrel{def}{=} \{\chi \in G^\wedge: \chi(F) \subseteq U\}$$

(basic sets for the topology of $G^\wedge$) and

$$N(H, U) \stackrel{def}{=} \{f \in C(X, \mathbb{T}): f(H) \subseteq U\}$$

(basic sets for the topology of $C_b(X, \mathbb{T})$).

Proposition 7. *The group $A(X)^\wedge$ is topologically isomorphic to the group $C_b(X, \mathbb{T})$ under the restriction mapping r.*

Proof: The fact that r is an algebraic isomorphism is obvious. Let C be a precompact subset of $A(X)$. It is well known that for some $n \in \mathbb{N}$, $C \subseteq A(X)_n$, where the last symbol stands for the set of all $x \in A(X)$ having reduced length $\leqslant n$ over X [1]. If U is an arbitrary neighbourhood of the identity e in $\mathbb{T}$, choose a symmetric neighbourhood V of e in $\mathbb{T}$ such that $V^n \subseteq U$. Proposition 6 implies now that $r^{-1}[M(\operatorname{supp} C, V)] \subseteq N(C, U)$, which yields that r is open. Conversely, if H is a bounded subset of X, then H is precompact in $A(X)$ (Lemma 1) and $r^{-1}[M(C, U)] = N(C, U)$; this means that r is continuous. □

In what follows we identify the topological groups $A(X)^\wedge$ and $C_b(X, \mathbb{T})$.

2. The evaluation map $\nu: A(X) \to A(X)^{\wedge\wedge}$

The reflexivity of a topological group G is equivalent to the following properties of the evaluation map $\nu: G \to G^{\wedge\wedge}$.

1. ν is one-to-one.
2. ν is continuous.
3. ν is relatively open. (That is, ν is open as a map from G onto its image in $G^{\wedge\wedge}$.)
4. ν is onto.

In this section we shall investigate the first three properties for the evaluation map $\nu: A(X) \to A(X)^{\wedge\wedge}$. The last property is examined in three separate sections, Section 3 and Section 4 (necessary conditions) and Section 5 (sufficient conditions).

1. Injectivity of ν.

Proposition 8. *For every topological space X the mapping $\nu: A(X) \to A(X)^{\wedge\wedge}$ is a monomorphism.*

Proof: This is just a reformulation of a well-known and rather simple fact: continuous characters separate points in free Abelian topological groups (see, for example, [**9**]). □

2. Continuity of ν.

Theorem 1. *If G is a pk-group then the mapping $\nu: G \to G^{\wedge\wedge}$ is continuous.*

Proof: The proof is modelled on Noble's proof of Theorem 2.3 in [**19**]. Denote by t' the topology on G, induced by the embedding

$$G \ni x \mapsto (x, \nu(x)) \in G \times G^{\wedge\wedge}$$

It is the smallest topology which (a) is finer than the original topology t on G, (b) is translation-invariant, and (c) contains all sets of the form $\bigcap_{\chi \in K} \chi^{-1}(U)$, where K is an arbitrary precompact subset of $G^{\wedge}$ and $e_{\mathbb{T}} \in U \subseteq \mathbb{T}$ is open. The Ascoli Theorem in a somewhat generalised form (see problem 379 in [**13**]) implies that any precompact subset $K \subseteq G^{\wedge}$ is equicontinuous with respect to the finest topology t'' on G inducing the originial topology on every precompact subset of G. Since t'' is clearly translation-invariant, one concludes: $t'' \subseteq t' \subseteq t$. The last observation implies that t' is the finest group topology on G inducing the originial topology on every precompact subset of G. The property of G being a *pk*-group means exactly that t has the same property. Therefore, $t = t'$, and ν is continuous. □

The following is an immediate consequence of Theorem 1 and Proposition 2.

Corollary 1. *If X is a b_f-space then the map $\nu: A(X) \to A(X)^{\wedge\wedge}$ is continuous.*

3. Relative openness of ν.

Let F, G, H be sets, let $A \subseteq G^F$, $B \subseteq H^G$. Set $B \circ A = \{f \circ g : f \in B,\ g \in A\}$.

Lemma 2. *Let F, G, H be topological groups, and let $A \subseteq G^F$, $B \subseteq H^G$. If the sets A and B are equicontinuous, then $B \circ A$ is equicontinuous.*

Proof: This follows from the fact that for every neighbourhood U of the identity in H the set $(B \circ A)^{-1}(U) \equiv A^{-1}(B^{-1}(H))$ is open in F. □

Theorem 2. *For every X the topology of $A(X)$ is the topology of uniform convergence on all equicontinuous subsets of $C(X, \mathbb{T})$.*

Proof: Let U be an arbitrary neighbourhood of zero in $A(X)$. We shall find an open neighbourhood W of the identity in $\mathbb{T}$ and an equicontinuous subset $H \subseteq C(X, \mathbb{T})$ such that $\bigcap\{\bar{h}^{-1}(W) : h \in H\} \subseteq U$.

https://doi.org/10.1017/S0004972700014726 Published online by Cambridge University Press

Let V be a neighbourhood of zero in the free locally convex space $L(X)$ such that $\gamma(U) = V \bigcap \gamma(A(X))$. According to Theorem 3′ in [26], there exists an equicontinuous subset $\Phi \subseteq C(X,\mathbb{R})$ such that $\bigcap\{\overline{\phi}^{-1}(\mathcal{I}) : \phi \in \Phi\} \subseteq V$, where $\mathcal{I} = (-1,1) \subset \mathbb{R}$. Because of the reflexivity of the additive group of $\mathbb{R}$, there exist an equicontinuous set $\Psi \subseteq \mathbb{R}^\wedge = C(\mathbb{R},\mathbb{T})$ and a neighbourhood of the identity in $\mathbb{T}$ such that $\bigcap\{\psi^{-1}(W) : \psi \in \Psi\} \subseteq \mathcal{I}$. The subset $H = \Psi \circ \Phi$ of $C(X,\mathbb{T})$ is equcontinuous by Lemma 2. Finally,

$$\bigcap\{\overline{h}^{-1}(W) : h \in H\} \subseteq \bigcap\{\overline{\phi}^{-1}\Big(\bigcap\{\psi^{-1}(W) : \psi \in \Psi\}\Big) : \phi \in \Phi\} \subseteq V.$$

□

In the sequel the subscript "c" stands for the compact-open topology.

Lemma 3. *The natural restriction mapping $r : C_c(\theta X,\mathbb{T}) \to C_b(X,\mathbb{T})$ is continuous.*

Proof: If $C \subseteq X$ is a bounded subset and V is a neighbourhood of the identity in $\mathbb{T}$, then $\mathrm{cl}_{\beta X} C \subseteq \mu X \subseteq \theta X$ and $r[N(\mathrm{cl}_{\beta X} C, V)] \subseteq N(C,V)$. □

Lemma 4. *If a set $H \subseteq C(X,\mathbb{T})$ is equicontinuous, then the set $\overline{H} = r^{-1}(H) \subseteq C(\theta X,\mathbb{T})$ is equicontinuous.*

Proof: For every entourage of the diagonal V from the (unique) uniformity of the compact group $\mathbb{T}$ we denote $\overline{H}^{-1}(V) = \{(x,y) \in (\theta X)^2 : (h(x),h(y)) \in V$ for all $h \in H\}$. One can assume that V is closed. Since $H_1 = \overline{H}^{-1}(V) \cap X^2$ belongs to the finest compatible uniform structure $\mathcal{U}_X$ on X because of the equicontinuity of H, and the trace of $\mathcal{U}_{\theta X}$ on X is $\mathcal{U}_X$, the set $\overline{H}^{-1}(V)$ is the closure of H_1 in $(\theta X)^2$ and therefore $\overline{H}^{-1}(V) \in \mathcal{U}_{\theta X}$.

(Here we used the following easy fact: a family of mappings H from a topological space X to a uniform space Y is equicontinuous if and only if H is equicontinuous as a family of maps from the uniform space $(X,\mathcal{U}_X)$ to Y.) □

Lemma 5. *The restriction mapping $r : C(\theta X,\mathbb{T}) \to C(X,\mathbb{T})$ is onto.*

Proof: Every continuous $f : X \to \mathbb{T}$ extends over βX, and $\theta X \subseteq \beta X$. □

Theorem 3. *For every topological space X the map $\nu : A(X) \to A(X)^{\wedge\wedge}$ is relatively open.*

Proof: Let H be an arbitrary equicontinuous subset of $C(X,\mathbb{T})$; the desired statement will follow immediately from Theorem 2 if we show that H is precompact in $C_b(X,\mathbb{T})$. The set $H_1 = r^{-1}(H)$ is equicontinuous in $C(\theta X,\mathbb{T})$ (Lemma 4). The closure H_2 of H_1 in the topology of pointwise convergence in $\mathbb{T}^{\theta X}$ is equicontinuous [**16**, Proposition 27], and therefore $H_2 \subseteq C(\theta X,\mathbb{T})$. Obviously, H_2 is closed in $C_c(\theta X,\mathbb{T})$ and for each $x \in \theta X$ the set $\mathrm{cl}_{\mathbb{T}}\{h(x) : h \in H_2\}$ is compact. By virtue of Ascoli's theorem [16, Theorem 9] H_2 is compact in $C_c(\theta X, \mathbb{T})$. Finally, Lemma 3 implies that $r(H_2)$ is compact in $C_b(X, \mathbb{T})$, and due to Lemma 5, $H \subseteq r(H_2)$. ∎

One deduces from Proposition 8, Corollary 1, and Theorem 3 the following result.

COROLLARY 2. *Let X be a b_f-space. Then the evaluation mapping ν is a topological isomorphism of the group $A(X)$ with a subgroup of $A(X)^{\wedge\wedge}$.*

3. SURJECTIVITY OF THE EVALUATION MAP AND THE GROUP $\pi^1(X)$

The first cohomotopy group, $\pi^1(X)$, of a topological space X is the collection of all homotopy classes of continuous maps $X \to \mathbb{T}$, equipped with the following group operations: the product $\alpha\beta$ of two elements $\alpha, \beta \in \pi^1(X)$ is the class of the pointwise product of representatives from α and β; the element inverse to an $\alpha \in \pi^1(X)$ is the homotopy class of a map pointwise inverse to a representative of α [11].

We shall denote by ζ a group homomorphism from $C(X, \mathbb{T})$ to $\pi^1(X)$ assigning to every mapping its homotopy class. Let us denote by $C^0(X, \mathbb{T})$ the kernel of ζ. We shall assume that $\pi^1(X)$ is topologised as a topological factor-group of $C_b(X, \mathbb{T})$ by its subgroup $C^0(X, \mathbb{T})$. Let $\xi: A(X)^{\wedge\wedge} \to C_b^0(X, \mathbb{T})^{\wedge}$ be a map dual to the embedding $C_b^0(X, \mathbb{T}) \hookrightarrow C_b(X, \mathbb{T})$.

PROPOSITION 9. *The homomorphism $\xi \circ \nu: A(X) \to C_b^0(X, \mathbb{T})$ is mono.*

PROOF: Let $x \in A(X)$; we are looking for an $f \in C_b^0(X, \mathbb{T})$ such that $\overline{f}(x) \neq e_{\mathbb{T}}$. There exists a $g \in L(X)'$ with $\alpha =_{def} g(\gamma(x)) \neq 0$. (Here $\gamma: A(X) \to L(X)$ is a canonical embedding.) Let $\theta: \mathbb{R} \to \mathbb{T}$ be a homomorphism given by the rule $x \mapsto \exp\left(i\alpha^{-1}x\right)$. Clearly, $f =_{def} \theta \circ g|_X$ has the desired properties. ∎

PROPOSITION 10. *If X is a b_f-space, then $\xi \circ \nu$ is a topological isomorphism of $A(X)$ onto a subgroup of $C_b^0(X, \mathbb{T})^{\wedge}$.*

PROOF: Since every precompact subset of $C_b^0(X, \mathbb{T})$ remains so in $C_b(X, \mathbb{T})$ as well, the mapping ξ is continuous and therefore, in view of Corollary 1, the map $\xi \circ \nu$ is continuous. On the other hand, it follows from the proof of Theorem 2 that the topology of $A(X)$ is the topology of uniform convergence on equicontinuous subsets of $C^0(X, \mathbb{T})$, because the subset H constructed in the process of the proof is in the latter group. It follows from the proof of Theorem 3 that every such set is precompact. Therefore, $\xi \circ \nu$ is relatively open. Proposition 9 completes the proof. ∎

PROPOSITION 11. *The group $\pi^1(X)^{\wedge}$ canonically embeds in $A(X)^{\wedge\wedge}$ as a topological subgroup in such a fashion that $\nu(A(X)) \cap \pi^1(X)^{\wedge} = \{0\}$.*

PROOF: A desired canonical embedding is the map $\zeta^{\wedge}$ dual to $\zeta: C_b(X, \mathbb{T}) \to \pi^1(X)$. If $x \in A(X)$, then for some $f \in C_b^0(X, \mathbb{T})$ one has $f(x) \neq 0$, while for all $\chi \in \pi^1(X)^{\wedge}$ one has $\chi(f) = 0$. ∎

Corollary 3. *If $A(X)$ is reflexive, then $\pi^1(X)^\wedge = (0)$.*

Apparently, in the general case computation of the character group of the topologised $\pi^1(X)$ is not easy. But in a particular case below (Corollary 4) the result becomes quite meaningful.

Lemma 6. *If X is pseudocompact then $C_b^0(X,\mathbb{T})$ is open in $C_b(X,\mathbb{T})$.*

Proof: The group $C_b^0(X,\mathbb{T})$ contains an open subset $M(X,V)$, where $V = \{x \in \mathbb{T}: |x - e_{\mathbb{T}}|_{\mathbb{C}} < \sqrt{3}\}$. Indeed, if $f \in M(X,V)$, then f is contractible to a constant function on X; the contraction at any point $x \in X$ is performed along a geodesic in $\mathbb{T}$ joining $f(x)$ and e. ∎

Corollary 4. *If X is pseudocompact and $A(X)$ is reflexive, then $\pi^1(X) = (0)$.*

Proof: It follows from Lemma 6 that $\pi^1(X)$ is a discrete group, and therefore the nontriviality of this group implies the nontriviality of $\pi^1(X)^\wedge$. ∎

We shall see below (Section 4), that even the condition $\pi^1(X) = (0)$ does not ensure the reflexivity of $A(X)$.

Below we denote by G_d a group G endowed with the discrete topology, and $G_d^\wedge$ stands for $(G_d)^\wedge$.

Proposition 12. *If X is pseudocompact, then $A(X)^{\wedge\wedge}$ is topologically isomorphic to $C_b^0(X,\mathbb{T})^\wedge \oplus \pi^1(X)_d^\wedge$.*

Proof: The group $C_b^0(X,\mathbb{T})$ is open in $C(X,\mathbb{T})$, and it is divisible (unlike, in general, $C(X,\mathbb{T})$) – indeed, for X pseudocompact, it is algebraically generated by a subset $M(X,V)$, where $V = \{x \in \mathbb{T}: |x - e_{\mathbb{T}}|_{\mathbb{C}} < \sqrt{3}\}$, and for every $f \in M(X,V)$ and each $n \in \mathbb{N}$ there is a $g \in M(X,V)$ with $ng = f$.

According to [**10**, 6.22.b], the group $C_b(X,\mathbb{T})$ is topologically isomorphic to $C_b^0(X,\mathbb{T}) \oplus \pi^1(X)$. Theorem 13 in [**16**] finishes the proof. (Clearly, it remains valid for all – not just locally compact – topological groups, and also in the case where the character group is endowed with our "precompact-open" topology.) ∎

4. Surjectivity of the evaluation map and path-connectedness of X

Let X be a topological space. Denote by θ a map of the linear space $C(X,\mathbb{R})$ to the group $C(X,\mathbb{T})$, given by the rule $\theta(f)(x) = \exp(2\pi i f(x))$. The image of $C(X,\mathbb{R})$ under θ is contained in $C^0(X,\mathbb{T})$ and θ is an additive group homomorphism.

If $x_0 \in X$, denote

$$C(X,x_0,\mathbb{R}) = \{f \in C(X,\mathbb{R}): f(x_0) = 0\},\ C(X,x_0,\mathbb{T}) = \{f \in C(X,\mathbb{T}): f(x_0) = e_{\mathbb{T}}\},$$

$$C^0(X,x_0,\mathbb{T}) = \{f \in C^0(X,\mathbb{T}): f(x_0) = e_{\mathbb{T}}\} \equiv C^0(X,\mathbb{T}) \cap C(X,x_0,\mathbb{T}).$$

Obviously, θ maps $C(X,x_0,\mathbb{R})$ to $C^0(X,x_0,\mathbb{T})$. Denote by θ_0 the restriction of θ to $C(X,x_0,\mathbb{R})$.

PROPOSITION 13. *Let X be a path-connected space and let $x_0 \in X$. Then the map $\theta_0: C(X,x_0,\mathbb{R}) \to C^0(X,x_0,\mathbb{T})$ is an algebraic isomorphism. If in addition X is a pseudocompact space, then θ_0 is a topological isomorphism between $C_b(X,x_0,\mathbb{R})$ and $C_b^0(X,x_0,\mathbb{T})$.*

PROOF: The first statement follows from the covering mapping theorem [**8**, 5.6.19, **11**]. The continuity of θ_0 is obvious. Now let $0 < \varepsilon < \sqrt{3}$. Set $V_\varepsilon = \{\exp(ix): |x| < \varepsilon\} \subseteq \mathbb{T}$. We shall show now that $\theta_0^{-1}(N(X,V_\varepsilon)) \subseteq N(X,(-\varepsilon,\varepsilon))$ (the notation is obvious). If $f \in N(X,V_\varepsilon)$, then $\theta_0^{-1}(f)(x_0) = 0$ and, by virtue of the path-connectedness of X, $\theta_0^{-1}(f)(X)$ is in the same path component of $\theta_0^{-1}(V_\varepsilon) = \bigcap\{(n-\varepsilon, n+\varepsilon): n \in \mathbb{N}\}$ as 0 is, that is, in $(-\varepsilon,\varepsilon)$. ∎

LEMMA 7. *For every element $x_0 \in X$ the groups $C_b^0(X,\mathbb{T})$ and $C_b^0(X,x_0,\mathbb{T}) \oplus \mathbb{T}$ are topologically isomorphic under the mapping $f \mapsto \left(f \cdot f(x_0)^{-1}, f(x_0)\right)$.*

For a compact space X denote by $M(X)$ the Banach space of all regular Borel measures on X, that is, the strong dual to the space $C_c(X,\mathbb{R})$.

THEOREM 4. *If $\nu: A(X) \to A(X)^{\wedge\wedge}$ is onto then X is totally path-disconnected, that is, every path-component of X is a singleton.*

PROOF: Assume that ν is onto and at the same time there exists a path-component of X containing more than one point. Denote by C a non-trivial image of the closed interval $[0,1]$ in X. Let r stand for the restriction mapping from $C_b(X,\mathbb{T})$ to $C_b(C,\mathbb{T})$. Since every compact subset of a completely regular space is C^*-embedded, r is onto. By virtue of the continuity of r, the dual map $r^\wedge: C_b(X,\mathbb{T}) \to C_b(X,T)^\wedge \equiv A(X)^{\wedge\wedge}$ is defined correctly. Since r is onto, $r^\wedge$ is a monomorphism.

Let $m \in M(C)$, $m: C_b(C,\mathbb{R}) \to \mathbb{R}$. Fix $c_0 \in C$. The mapping $\chi'_m : C_b^0(C,c_0,\mathbb{T}) \to \mathbb{T}$, given by the rule

$$f \mapsto \exp[2\pi i m(\theta_0^{-1}(f))],$$

is a continuous homomorphism according to Proposition 13. Proposition 12 and Lemma 7 imply that χ'_m extends to a continuous homomorphism $\chi_m: C_b(C,\mathbb{T}) \to \mathbb{T}$. Since $\chi_m \in A(X)^{\wedge\wedge} \equiv \nu(A(X))$ and $\nu^{-1}\chi_m \in A(X)$, one can find a finite subset $X' = \{x_1,\dots,x_m\} \subseteq X$ such that for every $f \in C(C,\mathbb{T})$ the property $f(X') \subseteq \{e_\mathbb{T}\}$ implies $\chi_m(f) = e_\mathbb{T}$.

If now $\phi \in C(X,x_0,\mathbb{R})$ and in addition $\phi(X') \subseteq \{0\}$, then the function $\psi: x \mapsto \exp[2\pi i\phi(x)]$ belongs to $C^0(X,x_0,\mathbb{T})$ and $\psi(X') \subseteq \{e_\mathbb{T}\}$, so that $\chi_m(\psi) = e_\mathbb{T}$. At the same time, $\chi_m(\psi) = \exp[2\pi i m(\theta_0^{-1}(\psi))] = \exp[2\pi i m(\phi)]$, and therefore $m(\phi) \in \mathbb{Z}$. The same applies to the function $\phi_1 = \pi\phi$, and we conclude: $m(\phi) \in \mathbb{Z} \cap \pi^{-1}\mathbb{Z} = \{0\}$.

This means that the support of m is finite. We have arrived at a contradiction, because C is an infinite compact set and m is an arbitrary measure on it. ∎

In [18] the statement of Theorem 4 was obtained in the case where X is a k_ω-space.

5. A class of reflexive groups $A(X)$

For a topological space X we denote

$$E(X) = \bigcup\{\mathrm{cl}_{\beta X} B : B \in \mathcal{B}\},$$

where $\mathcal{B}$ is the family of all bounded subsets of X. Clearly, $X \subseteq E(X) \subseteq \mu X$. However, $E(X)$ need not coincide with $\mu(X)$ [4], though if $X = E(X)$ then X is a μ-space.

If γ is a disjoint cover of X, we call a mapping $f: X \to Y$ γ*-labelled* if the restriction of f to each member of γ is a constant map.

Denote by Γ_X the totality of all finite disjoint open covers of the space X.

Lemma 8. *Let* $\dim X = 0$. *The subset of all mappings which are* γ*-labelled for some* $\gamma \in \Gamma_X$ *(depending on a mapping) is uniformly dense in* $C(X, \mathbb{T})$.

Proof: Let $f \in C(X, \mathbb{T})$ and let V be a neighbourhood of $e_{\mathbb{T}}$ in $\mathbb{T}$ (we assume that $|V|_{\mathbb{C}} \subseteq (-\sqrt{3}, \sqrt{3})$). Let W be open in $\mathbb{T}$ and such that $W \cdot W^{-1} \subseteq V$. Choose a finite set $A = \{a_1, \dots, a_n\} \subset \mathbb{T}$ with $A \cdot W = \mathbb{T}$. The family $\{f^{-1}(a_i W): i = 1, \dots, n\}$ forms a functionally open cover of X; refine it to a finite disjoint open cover γ. For each $U \in \gamma$ fix an $x_U \in U$ and set $\phi(U) = \{f(x_U)\}$. The resulting map ϕ is γ-labelled and for every $x \in X$ one has $\phi(x)f(x)^{-1} = f(x_U)f(x)^{-1}$, if $x \in U \in \gamma$. For a suitable $i = 1, 2, \dots, n$ one has $U \subseteq f^{-1}(a_i W)$, therefore $f(x_U)f(x)^{-1} \in f(U)f(U)^{-1} \subseteq a_i W (a_i W)^{-1} = WW^{-1} \subseteq V$. ∎

Theorem 5. *Let X be a μ-space and let* $\dim X = 0$. *Then* $\nu: A(X) \to A(X)^{\wedge\wedge}$ *is onto.*

Proof: Let $\chi \in A(X)^{\wedge\wedge}$, $\chi: C_b(X, \mathbb{T}) \to \mathbb{T}$. We shall define an integer-valued function i_χ on the set $\mathcal{V}$ of all open-and-closed subsets of X as follows. Let $V \in \mathcal{V}$. Denote by G_V the subgroup of $C(X, \mathbb{T})$ formed by all mappings f such that $f(V)$ is a singleton and $f(X \setminus V) \subseteq \{e_{\mathbb{T}}\}$. Clearly, G_V is topologically isomorphic to $\mathbb{T}$ under the mapping $f \mapsto f(V)$. Set $i_\chi(V)$ to be equal to k, if the restriction of χ to G_V is of the form $\exp(2\pi i x) \mapsto \exp(2\pi k i x)$ under the above identification $G_V \cong \mathbb{T}$; in other words, $i_\chi(V)$ is simply the degree of the mapping $\chi|_{G_V}: G_V \to \mathbb{T}$.

The set Γ_X is naturally directed: $\gamma \prec \gamma'$ if γ is a refinement of γ'. Any family of sets of the form $\{V_\gamma : \gamma \in \Gamma_X\}$, where $V_\gamma \in \gamma$ and $V_\gamma \subseteq V_{\gamma'}$ whenever $\gamma \prec \gamma'$, forms a basis for a Cauchy filter with respect to the finest totally bounded compatible uniformity on X, which we denote by $C^*(X)$. Since the completion of the uniform space $(X, C^*(X))$ is βX [5, 8.3.18], each family of the form $\{V_\gamma : \gamma \in \Gamma_X\}$ with the above properties converges to some element of βX. Moreover, each element of βX can be obtained in this way.

Denote by S_χ the collection of all $x \in \beta X$ which are limits of prefilters of the form $\{V_\gamma : \gamma \in \Gamma_X\}$, where $V_\gamma \in \gamma$ and $V_\gamma \subseteq V_{\gamma'}$ if $\gamma \prec \gamma'$, with the additional property that for every neighbourhood $U \ni x$ there is a $\gamma \in \Gamma_X$ such that $V_\gamma \subseteq U$ and $i_\chi(V_\gamma) \neq 0$.

Claim 1. S_χ is a finite set.

◁ Assuming the contrary, one can easily construct by induction a countable relatively discrete (that is, discrete in itself) set $\{s_n : n \in \mathbb{N}\} \subseteq S_\chi$, and a disjoint family $\{V_n : \in \mathbb{N}\}$ of open-and-closed subsets of X such that for all $n \in \mathbb{N}$ one has $s_n \in \operatorname{cl}_{\beta X} V_n$ and $i_\chi(V_n) = i_n \neq 0$. For $n \in \mathbb{N}$ define the function $f_n \in G_{V_n}$ by letting $f_n(x) = e_{\mathbb{T}}$ for $x \in X \setminus V_n$ and $f_n(x) = \exp\left(2\pi i n^{-1} i_n^{-1}\right)$ for $x \in V_n$. Since for every neighbourhood W of the identity in $\mathbb{T}$ one has $f_n(X) \subseteq W$ for all n starting from some natural number, the function $f = \sum_{n\in\mathbb{N}} f_n$ is continuous on X. Furthermore, f is a uniform limit of the sequence of functions $\left(\sum_{n=1}^{m} f_n\right)$ as $m \to \infty$, and therefore one should have $\chi(f) = \prod_{n\in\mathbb{N}} \chi(f_n) = \prod_{n\in\mathbb{N}} \exp\left(2\pi i n^{-1}\right)$. However, the latter infinite product diverges in $\mathbb{T}$. ▷

Since every function from $C(X, \mathbb{T})$ extends in a unique way to a continuous $\mathbb{T}$-valued mapping on βX, the group $C_b(X, \mathbb{T})$ can be canonically identified with the group $C(\beta X, \mathbb{T})$, equipped with the topology of uniform convergence on bounded subsets of X. Every $f \in C(X, \mathbb{T})$ can be thought of as a mapping $\beta X \to \mathbb{T}$.

Claim 2. If $f \in C(X, \mathbb{T})$, then $\chi(f) = \prod_{s\in S} f(s)^{n_s}$ for an appropriate collection $n_s \in \mathbb{Z}$, $s \in S$.

◁ If a γ-labelled function f equals $e_{\mathbb{T}}$ on S, then it assumes the same value on some neighbourhood U of S, which can be assumed to be open-and-closed. By the definition of S_χ, for each $x \in \beta X \setminus U$ there exist $\gamma \in \Gamma_X$ and $V_x \in \gamma$ such that the value of i_χ on every open-and-closed subset of V_x is 0. Refine the cover $\{V_x : x \in \beta X \setminus U\}$ to a finite disjoint open-and-closed subcover γ_1 of $\beta X \setminus U$. (It is possible to do this because $\beta X \setminus U$ is a strongly zero-dimensional compact space.) For every $V \in \gamma_1$, define a function f_V by

$$f_v(x) = \begin{cases} f(x), & \text{if } x \in V, \\ e_{\mathbb{T}}, & \text{otherwise.} \end{cases}$$

Clearly, for every $V \in \gamma_1$, one has $\chi(f_v) = e_{\mathbb{T}}$, because $i_\chi(V) = 0$. Since $f|_U \equiv e_{\mathbb{T}}$, one must have $f = \prod_{V\in\gamma_1} f_V$ (finitely many factors) and therefore $\chi(f) = \prod_{V\in\gamma_1} \chi(f_V) = e_{\mathbb{T}}$.

A straightforward consequence is that $\chi(f) = \chi(g)$ whenever f and g are two γ-labelled functions coinciding on S.

Now let g be an arbitrary γ-labelled function on S. Fix disjoint open subsets of βX, U_s, $s \in S$, such that $s \in U_s$ for all s and the function g is constant on each U_s. Set $n_s = i_\chi(U_s)$. Define a function $f:\beta X \to \mathbb{T}$ by letting $f \equiv 0$ outside $\bigcup\{U_s: s \in S\}$ and $f(U_s) = \{g(s)^{n_s}\}$. Obviously, $\chi(f) = \prod_{s\in S} f(s)^{n_s}$ and, by the preceding paragraph, $\chi(g) = \chi(f) = \prod_{s\in S} g(s)^{n_s}$. Lemma 8 finishes the proof of Claim 2: there is a net (f_α) of γ_α-labelled functions uniformly converges to a function $f \in C(X,\mathbb{T})$, and one can assume that $f_\alpha|_S \equiv e_{\mathbb{T}}$ for all α; therefore, $\chi(f) = \lim_\alpha \chi(f_\alpha) = \prod_{s\in S} f(s)^{n_s}$. ▷

Claim 3. $S \subseteq E(X)$.

◁ Let W be a neighbourhood of the identity in $\mathbb{T}$. There exist a bounded subset $B \subseteq X$ and a neighbourhood U of the identity in $\mathbb{T}$ such that $\chi(f) \in W$ whenever $f(B) \subseteq U$. If for some $x \in S$ one has $x \notin \mathrm{cl}_{\beta X} B$, then there exists a continuous function $f:\beta X \to \mathbb{T}$ with the properties $f(B) = \{e_{\mathbb{T}}\}$, $f(S\setminus\{x\}) \subseteq \{e_{\mathbb{T}}\}$, and $f(x)^{n_x} \notin W$. Now $\chi(f) = \prod_{s\in S} f(x)^{n_s} = f(x)^{n_x} \notin W$, a contradiction, since $f(B) = \{e_{\mathbb{T}}\} \subseteq U$. ▷

Our Theorem is proved. Indeed, since X is a μ-space, one has $E(X) = X$, therefore $S \subseteq X$, and $\chi = \nu\left(\sum_{x\in S} n_x x\right)$, where $\sum_{x\in S} n_x x \in A(X)$, S being finite. □

Combining the statements of Proposition 9 and Theorems 1, 3 and 5, we obtain our main result.

Theorem 6. *Let X be a μ-space and k_f-space, and let $\dim X = 0$. Then $A(X)$ is a reflexive topological group.*

Corollary 5. *Let X be a paracompact k-space and let $\dim X = 0$. Then $A(X)$ is a reflexive topological group.*

Corollary 6. *Let X be a zero-dimensional compact space or a strongly zero-dimensional metrisable space. Then $A(X)$ is a reflexive topological group.*

In particular, the free Abelian topological groups $A(\mathbb{Q})$ and $A(\mathbb{Z}^{\aleph_0})$ on both rational and irrational numbers provide examples of reflexive groups of a type completely unknown before.

6. Concluding remarks

1. Does any known noncommutative analogue of Pontryagin–van Kampen duality (such as Tannaka-Krein duality) work for Markov free topological groups?

2. Answering a question by Noble [**19**], Nickolas [**18**] had shown that a complete topological k-group, whose points are separated by continuous characters, need not be reflexive: his counter-example was $A[0,1]$.

However, a non-discrete free Abelian topological group is never Čech-complete [**1**]. Is every complete and Čech-complete topological group, whose points are separated by continuous characters, reflexive?

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