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FATOU-JULIA THEORY ON TRANSCENDENTAL SEMIGROUPS

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In this paper, we shall study the dynamics on transcendental semigroups. Several properties of Fatou and Julia sets of transcendental semigroups will be explored. Moreover, we shall investigate some properties of Abelian transcendental semigroups and wandering domains of transcendental semigroups.

1. INTRODUCTION

The development of complex dynamics took nearly a century, from the work of Fatou [**7, 8**] and Julia [**12, 13**] appearing in 1918-1922 to the recent studies on the dynamics of transcendental meromorphic functions [**2, 3, 4**]. Nowadays, complex dynamics has a new significant development with the explosion of popular interest in the beautiful fractal objects that form the subject matter of the theory. The computer-generated images of Julia and Mandelbrot sets [**14**] bombard mathematicians to investigate the nature of both the Fatou and Julia sets of a given complex function.

Complex dynamics is the subject that studies the behaviour of an iteration sequence $\{f^n(z_0)\}$, where $z_0 \in \mathbb{C}$ and $f^n = f^{n-1} \circ f$ with $f^0(z) \equiv z$. For a given analytic function f, the set $F(f)$ is defined to be the maximum open set of z where the family $\{f^n(z)\}$ is normal and $J(f) = \mathbb{C} \backslash F(f)$. They are called the Fatou and Julia sets of f respectively. The theory is a study of the properties of the sets $F(f)$ and $J(f)$. Although the theory has been developed for nearly 80 years, there are many open problems which have not yet been resolved. In the past few decades, there were some significant breakthroughs in the subject. Sullivan [**16**] obtained the solution to the important problem concerning the non-existence of a wandering domain of the Fatou components of rational functions. By further classifying the periodic domains, together with the theorems of Siegel and Arnold concerning the existence of rotation domains [**15**], a rather complete description of the dynamics for rational functions has been derived. For the dynamics of transcendental entire functions, although the basic properties of the Fatou and Julia sets are similar to the rational functions, they actually possess some rather different dynamical behaviours.

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In contrast to the rational functions case, recently Baker [1] has given an example to show that certain transcendental entire functions can have wandering domains. Due to the existence of an essential singularity at infinity, transcendental entire functions may have a Baker domain in which the iteration sequence $\{f^n\}$ converges to the point of infinity.

A natural generalisation of complex dynamics is to investigate the dynamics of a sequence of different functions by means of composition. There are two main streams of study. One is by Chinese mathematicians Zhou and Ren (see [**18, 19**] for details), who published a series of papers on random iterations. The other one is by A. Hinkkanen and G. J. Martin [**10, 11**] concerning semigroups of rational functions. Our work follows the idea of Hinkkanen and extends the study to transcendental semigroups.

2. Definitions and basic properties on transcendental semigroups

Before going to the details, we begin with some definitions and notations on transcendental entire functions. Given a transcendental entire function f, we say that

1. a is a critical value (algebraic singularity) of f if there exists a $z_0 \in \mathbb{C}$ such that $f(z_0) = a$ and $f'(z_0) = 0$. In this case, z_0 is called a critical point of f.
2. a is an asymptotic value (transcendental singularity) of f if there exists a curve Γ tending to infinity such that $f(z) \to a$ as $z \to \infty$ along Γ.

Notation. Denote the set of singular points of f^{-1} by $\operatorname{Sing}(f^{-1})$, which is the set of all critical values and asymptotic values of f, and all its finite limit points.

Definition: A transcendental entire function f is said to be of *finite type* if the set $\operatorname{Sing}(f^{-1})$ has only finitely many elements, that is, f has only finitely many critical and asymptotic values.

Now, we give the definition of transcendental semigroup.

Definition: A transcendental semigroup G is a semigroup generated by a family of transcendental entire functions $\{f_1, \dots, f_n, \dots\}$ with the semigroup operation being the composition of functions. We denote the semigroup by:

$$G = \langle f_1, f_2, \dots \rangle.$$

In this way, $h \in G$ is a transcendental entire function and G is close under composition. Hence h can be written as

$$h = f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_k}.$$

A semigroup $G = \langle f_1, \dots, f_n \rangle$ generated by finitely many functions is called finitely generated. From now on, transcendental semigroup mean non-empty finitely generated transcendental semigroup unless otherwise specified.

Based on the Fatou-Julia theory of a complex function, we can define the set of normality of a transcendental semigroup G.

DEFINITION: Let G be a transcendental semigroup. The *set of normality* or the *Fatou set* $F(G)$ of G is the largest open subset of $\mathbb{C}$ on which the family of transcendental entire functions in G is normal. Thus $F(G)$ consists of those z_0 that have a neighbourhood U such that $\{g : g \in G\}$ is a normal family on U. The *Julia set* $J(G)$ of G is the complement of $F(G)$, that is $J(G) = \overline{\mathbb{C}} \backslash F(G)$.

REMARKS.

1. It is obvious that $J(G)$ is a closed set for any transcendental semigroup G.
2. If G is generated by only one transcendental function f, then G is cyclic and $F(f) = F(G) = F(\langle f \rangle)$; $J(f) = J(G) = J(\langle f \rangle)$.
3. For each $g \in G$, $F(G) \subset F(g)$ and $J(g) \subset J(G)$.

DEFINITION: For a given transcendental semigroup G, a set $A \subset \mathbb{C}$ is said to be *forward invariant* if for all $f \in G$, $f(A) \subset A$. It is called *backward invariant* if for all $f \in G$,

$$f^{-1}(A) = \{z \in \mathbb{C} : f(z) \in A\} \subset A.$$

Similar to rational semigroups [**10**], we have the following theorem.

THEOREM 2.1. *Given a transcendental semigroup G, the Fatou set of G is forward invariant, while the Julia set of G is backward invariant.*

It is well known that for transcendental entire functions, very frequently their Julia set equals the whole complex plane. For example, it was shown by Devaney [**6**] that if $f(z) = e^{\lambda z}$, if $\lambda \geqslant e^{-1}$, then $J(f) = \mathbb{C}$. On the contrary, it is rare that the Julia set of a given rational function is the whole complex plane. Since $F(G) \subset F(f)$ for any $f \in G$, one may ask whether there exists a non-trivial transcendental semigroup G such that $F(G)$ is non-empty. Non-trivial here means that G has more than one generator. An example is given below.

EXAMPLE 2.1. Let $G = \langle e^{\lambda z}, e^{\lambda z} + 2\pi i/\lambda \rangle$, where $0 < \lambda < e^{-1}$. We can check that

$$F(e^{\lambda z}) = F(e^{\lambda z} + 2\pi i/\lambda).$$

Let $g = e^{\lambda z}$, then for any $f \in G$, f has either the form

$$f = g^m,$$

or

$$f = g^n + \frac{2\pi i}{\lambda},$$

where $g^n = g \circ \ldots \circ g$, n times. In both cases, $F(f) = F(g)$. Hence $F(G) = F(e^{\lambda z}) \neq \emptyset$.

PROOF OF THEOREM 2.1: For any $f \in G$, $G \circ f = \{g \circ f : g \in G\} \subset G$. $G \circ f$ is normal on $F(G)$ by definition. Thus G is normal on $f(F(G))$, hence $f(F(G)) \subset F(G)$.

Moreover, if $z \in J(G)$, we can find $w \in \mathbb{C}$ such that $f(w) = z$; then w necessary belongs to $J(G)$, for otherwise, we have $z \in F(G)$ by the previous arguments. Hence $J(G)$ is backward invariant. ∎

3. Exceptional sets of transcendental semigroups

Given a transcendental semigroup G, we define the set

$$O^-(z) = \{w \in \mathbb{C} : \text{ there exists } g \in G \text{ such that } g(w) = z\}$$

and the *exceptional set* of G is defined by

$$E(G) = \{z \in \mathbb{C} : O^-(z) \text{ is finite}\}.$$

REMARKS. It is a fundamental result that if f is a transcendental entire function, then f takes every complex value infinitely often except possible at most one value, called the Picard exceptional value of f. A value $c \in \mathbb{C}$ is called a *Fatou exceptional value* of f if $f(z) = c$ has no solution or the only solution of $f(z) = c$ is c itself. From the definition, a Fatou exceptional value is a Picard exceptional value, but the converse need not be true. In our case, $E(G)$ contains at most one value in $\mathbb{C}$.

4. Julia sets of transcendental semigroups

In analogy to complex functions [**5**] and rational semigroups [**10**], we have the following theorem.

THEOREM 4.1. *Suppose G is a transcendental semigroup, then $J(G)$ is a perfect set.*

Based on Nevanlinna theory of transcendental functions and a remarkable result on the theory of normal families due to Zalcman [**17**], we obtain the following theorem.

THEOREM 4.2. *Given a transcendental semigroup G,*

$$J(G) = \overline{\bigcup_{g \in G} J(g)}.$$

In order to prove Theorem 4.2, we need the following lemma.

LEMMA 4.1. [**17**] *A family F of transcendental entire functions is not normal at $w_0 \in \mathbb{C}$ if and only if there exists a sequence $\{f_j\} \in F$, a sequence $z_j \to w_0$, a positive sequence $\rho_j \to 0$ and an entire function f on the plane with $f' \not\equiv 0$, such that*

$$f_j(z_j + \rho_j z) \to f(z)$$

spherically uniformly on all compact subsets of $\mathbb{C}$.

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PROOF OF THEOREM 4.1: By assumption, G contains at least one transcendental entire function f. Since $J(f)$ is perfect, $J(G)$ is non-empty and it contains at least three points as $J(f)$ is contained in $J(G)$. Suppose $\beta \in J(G)$ is an isolated point. Choose a neighbourhood U of β so that $U\backslash\{\beta\} \subset F(G)$. Since $g(F(G)) \subset F(G)$, each $g \in G$ omits $J(G)$ on $V = U\backslash\{\beta\}$ which implies every element in G is normal on U, which gives a contradiction. ∎

PROOF OF THEOREM 4.2: Since $J(g) \subset J(G)$ for all $g \in G$ and $J(G)$ is closed,

$$\overline{\bigcup_{g\in G} J(g)} \subset J(G).$$

Conversely, we choose an element $f \in G$. Hence f is transcendental entire and we let A be the set

$$A = \left\{w \in \mathbb{C} : \Theta(w,f) \geqslant \frac{1}{2}\right\},$$

where

$$\Theta(w,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,w)}{T(r,f)}$$

is an index of multiplicity of f. The set A has at most two elements and for $w \notin A$, $f(z) = w$ has infinitely many simple solutions. (See [9] for more details.) Now, we choose $w_0 \in J(G)\backslash A$. We want to prove that w_0 is a limit point of some sequences of repelling periodic points of the elements in G.

By definition, G is not normal on w_0, hence by Lemma 4.1, we can find a sequence $\{f_j\} \in G$, a sequence $z_j \to w_0$, a positive sequence $\rho_j \to 0$ and an analytic function h with $h' \not\equiv 0$ such that

$$f_j(z_j + \rho_j z) \to h(z)$$

spherically uniformly on all compact subsets of $\mathbb{C}$. Therefore,

$$f \circ f_j(z_j + \rho_j z) \to f \circ h(z)$$

as $j \to \infty$. Since $w_0 \notin A$, there exists infinitely many simple solutions of $f(z) = w_0$. We denote the solutions by $\{\alpha_n\}$, that is, $f(\alpha_n) = w_0$ and $f'(\alpha_n) \neq 0$ for $n \in \mathbb{N}$. If h is transcendental, we have $\Theta(\alpha_j, h) < 1/2$ for some $\alpha_1, \alpha_2, \alpha_3$. Without loss of generality, we assume

$$h(\beta) = \alpha_1, \quad h'(\beta) \neq 0,$$

for some $\beta \in \mathbb{C}$. If h is a polynomial, one of the equations $h(z) = \alpha_i$, $i \in \mathbb{N}$ has a simple root, say β, since $h'(z)$ has only finitely many zeros. In both case, we can find $\beta \in \mathbb{C}$ such that

$$f \circ h(\beta) = w_0, \quad (f \circ h)'(\beta) \neq 0.$$

We obtain

$$f \circ f_j(z_j + \rho_j z) - z_j + \rho_j z \to f \circ h(z) - w_0.$$

$f \circ h(z) = w_0$ has a solution β, and it is non-constant. Hurwitz's theorem implies that

$$f \circ f_j(z_j + \rho_j z) = z_j + \rho_j z$$

has a solution z_j^* for sufficiently large j and $z_j^* \to \beta$. $z_j + \rho_j z_j^*$ is a fixed point of $f \circ f_j$ and $z_j + \rho_j z_j^* \to w_0$. It is repelling because

$$\left|(f \circ f_j)'(z_j + \rho_j z_j^*)\right| = \left|\frac{(f \circ f_j)(z_j + \rho_j z)'(z_j^*)}{\rho_j}\right| \to \left|\frac{(f \circ h)'(\beta)}{\rho_j}\right| > 1$$

for sufficiently large j. Hence the result follows. $\square$

5. Abelian transcendental semigroups

DEFINITION: $G = \langle f_1, f_2, \dots, f_n \rangle$ is called an Abelian transcendental semigroup if all the generators in G are permutable, that is $f_i \circ f_j = f_j \circ f_i$ for all $1 \leqslant i, j \leqslant n$.

THEOREM 5.1. *Suppose $G = \langle f_1, \dots, f_n \rangle$ is an Abelian transcendental semigroup in which each f_i is of finite type for $i = 1, \dots, n$. Then for any $f \in G$, $F(G) = F(f)$.*

Before proving the above theorem, we need the following two lemmas.

LEMMA 5.1. *Suppose f and g are transcendental entire functions. If both f and g are of finite type, then $f \circ g$ is of finite type. In particular, f^2 is of finite type.*

LEMMA 5.2. *Suppose f and g are permutable transcendental entire functions of finite type. Then $F(f) = F(g)$.*

PROOF OF LEMMA 5.1: Suppose $a \in \operatorname{Sing}(f \circ g)^{-1}$ is an algebraic singularity, then $a = f \circ g(b)$ and $(f \circ g)'(b) = f'\big(g(b)\big)g'(b) = 0$. Hence either $f'\big(g(b)\big) = 0$ or $g'(b) = 0$. Thus $g(b)$ is a critical point of f or b is a critical point of g. In the former case, a is a critical value of f. For the other case, we can find a c which is a critical value of g such that $g(b) = c$, and so $a = f(c)$. Hence the number of algebraic singularities of $f \circ g$ is at most equal to the sum of the number of algebraic singularities of f and g.

Finally, if $a \in \operatorname{Sing}(f \circ g)^{-1}$ is a transcendental singularity, then there exists a path Γ approaching infinity such that $f \circ g(z) \to a$ as z transverses along Γ. If $g(\Gamma)$ is unbounded, then $g(\Gamma)$ is an asymptotic path of f and hence a is an asymptotic value of f. If $g(\Gamma)$ is bounded, it must terminate at one of the solutions $f(z) = a$. Since a-points of f are isolated, we have a $c \in \operatorname{Sing}(g^{-1})$ is a transcendental singularity of g such that $f(c) = a$. Hence the number of transcendental singularities of $f \circ g$ is at most equal to the sum of the number of transcendental singularities of f and g and the lemma follows. $\square$

PROOF OF LEMMA 5.2: Since both f and g are of finite type, f and g have no wandering domains or Baker domains. (See [5].) This will ensure that any subsequence $\{f^{n_j}\}$ and $\{g^{n_j}\}$ will not tend to infinity for z belonging to their Fatou sets respectively.

In order to reach our conclusion, we only need to show that $F(f) \subset F(g)$ since by symmetry we can also conclude $F(g) \subset F(f)$. Indeed, it is sufficient to prove $g(F(f)) \subset F(f)$. Take $\alpha \in F(f)$ and consider a neighbourhood $U_\alpha \subset F(f)$ which contains α. Consider any sequence f^{n_j} which is convergent on U_α. Then $\{f^{n_j}\}$ cannot converge to infinity. Hence it converges to a holomorphic function $h : U_\alpha \to \mathbb{C}$. In this case, f^{n_j} converges to $g \circ h$ on $g(U_\alpha)$ by $f^n \circ g = g \circ f^n$. Therefore we have $g(U_\alpha) \subset F(f)$. Hence the lemma follows. $\square$

PROOF OF THEOREM 5.1: Since for any $1 \leqslant i \leqslant j \leqslant n$, f_i and f_j are permutable, by Lemma 5.2, we can conclude that $F(f_i) = F(f_j)$. Now, for any $f \in G$, by the permutability of each f_i, f can be represented by:

$$f = f_1^{m_1} \circ f_2^{m_2} \circ \ldots \circ f_n^{m_n}$$

Applying Lemma 5.1 repeatedly, $f_1^{m_1}, \ldots , f_n^{m_n}$ are of finite type and so is $f_1^{m_1} \circ \ldots \circ f_n^{m_n}$. Hence f is of finite type. Observe that f and f_i are permutable for $1 \leqslant i \leqslant n$. By Lemma 5.2 again, $F(f) = F(f_i)$ for all $i = 1, \ldots , n$. In conclusion, $F(G) = F(f)$ for all $f \in G$. $\square$

6. Wandering domains on $F(G)$

DEFINITION: A component U of $F(G)$ is called a wandering domain of G if there are infinitely many distinct components U_j of $F(G)$ and elements f_j of G such that $f_j(U) \subset U_j$. In this case, $\{f_j\}$ is called a sequence of wandering functions on U.

THEOREM 6.1. *Let G be a transcendental semigroup. If U is a wandering domain of G and $\{f_j\}$ is a sequence of functions with wandering domains U, then any limit function of $\{f_j\}$ on U must be constant and belong to the Julia set of G.*

PROOF OF THEOREM 6.1: Suppose $\{f_j\}$ tends to a limit function c locally uniformly on U. The limit function must be constant, for otherwise, the image under f_j of a close disc contained in U contains a fixed disc for all large j, so that the U_j coincide for sufficiently large j, which is a contradiction. Similarly, we can conclude $c \in J(G)$. $\square$

REMARKS. In analogy to the case of complex functions, we can also defined the periodic components in $F(G)$ as it is defined in the rational semigroups case, see [**10**].

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