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## Cohomology of quiver moduli, functional equations, and integrality of Donaldson–Thomas type invariants

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# Cohomology of quiver moduli, functional equations, and integrality of Donaldson–Thomas type invariants

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## ABSTRACT

A system of functional equations relating the Euler characteristics of moduli spaces of stable representations of quivers and the Euler characteristics of (Hilbert-scheme-type) framed versions of quiver moduli is derived. This is applied to wall-crossing formulas for the Donaldson–Thomas type invariants of M. Kontsevich and Y. Soibelman, in particular confirming their integrality.

## 1. Introduction

In [KS08], a framework for the definition of Donaldson–Thomas type invariants for Calabi–Yau categories endowed with a stability structure is developed. One of the key features of this setup is a wall-crossing formula for these invariants, describing their behavior under a change of stability structure in terms of a factorization formula for automorphisms of certain Poisson algebras defined using the Euler form of the category.

In [Rei10], such factorization formulas are interpreted using quiver representations, their moduli spaces, and Hall algebras. The main result of [Rei10] interprets the factorization formula in terms of generating series of the Euler characteristic of the smooth models of [ER09], which can be viewed as Hilbert schemes in the setup of quiver moduli.

In the general framework of [KR00, LeB99], series of moduli spaces of stable representations of quivers are viewed as the commutative ‘approximations’ to a fictitious non-commutative geometry of (the path algebras of) quivers. In this framework, the smooth models can be viewed as Hilbert schemes of points of this non-commutative geometry (for example, in the case of moduli spaces of semisimple representations of quivers, the smooth models parameterize finite codimensional left ideals in the path algebra of the quiver, in the same way as the Hilbert schemes of points of an affine variety parameterize finite codimensional ideals in the coordinate ring of the variety; see [ER09, §6]). Since path algebras of quivers are of global dimension one, this setup thus describes aspects of a one-dimensional non-commutative geometry.

The first aim of this paper (after reviewing some facts on quiver moduli in §2) is to develop a (one-dimensional, non-commutative) analog of the result [Che96] calculating the generating series of Euler characteristics of Hilbert schemes of points of a threefold  $X$  as the  $\chi(X)$ th power of the MacMahon series (see [BF08, Theorem 4.12]; see also [LP09] and [MNOP06, Conjecture 1] for the corresponding statement for Donaldson–Thomas invariants). Namely, we relate the (generating series of) Euler characteristics of moduli spaces of stable quiver representations

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and Euler characteristics of their smooth models by a coupled system of functional equations; see Theorem 3.4 and Corollary 3.5. This is achieved using a detailed analysis of a Hilbert–Chow type morphism from a smooth model to a moduli space of semistable representations, whose fibres are non-commutative Hilbert schemes (see § 3). The explicit cell decompositions for the latter, constructed in [ER09], yield functional equations for the Euler characteristic.

The second aim is to prove the integrality conjecture [KS08, Conjecture 1] (relative integrality) for the Donaldson–Thomas type invariants appearing in the wall-crossing formula of [KS08]; see § 5. These numbers arise by a factorization of the generating series of Euler characteristics as an Euler product (this process can thus be interpreted as fitting a genuinely non-commutative (one-dimensional) object into a commutative (three-dimensional) framework). Using the functional equations mentioned above, we can interpret this process as passing to the compositional inverse of an Euler product, and elementary number-theoretic considerations in § 4 yield the desired integrality property (it should be noted that a similar process appears in [Sti06] in relating modular forms and instanton expansions). We also confirm a conjectural formula of [KS08] for diagonal Donaldson–Thomas type invariants using results of [Wei09].

An independent proof of a refined version of the integrality conjecture [KS08, Conjecture 1] was announced in [KS09].

## 2. Recollections on quiver moduli

In this section, we fix some notation related to quivers and their representations and collect information on moduli spaces of stable representations of quivers and some of their variants, like Hilbert schemes of path algebras and the smooth models of [ER09]. See [Rei08] for an overview of these moduli spaces and the techniques used to prove some of the results cited below.

### 2.1 Quivers and their representations

Let  $Q$  be a finite quiver, with set of vertices  $I$ , and arrows written as  $\alpha : i \rightarrow j$  for  $i, j \in I$ . Denote by  $r_{i,j}$  the number of arrows from  $i \in I$  to  $j \in I$  in  $Q$ . A *path*  $\omega$  of length  $s$  in  $Q$  starting in a vertex  $i \in I$  and ending in  $j \in I$  is a sequence  $i = i_0 \xrightarrow{\alpha_1} i_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_s} i_s = j$  of arrows. For each  $i \in I$ , we formally include an *empty* path of length 0 starting and ending in  $i$ ; furthermore, arrows are viewed as paths of length one. A path of positive length starting and ending in the same vertex is called an *oriented cycle* in  $Q$ . For a path  $\omega$  ending in  $i$  and a path  $\omega'$  starting in  $i$ , the *concatenation*  $\omega'\omega$  is defined.

Define  $\Lambda = \Lambda(Q) = \mathbf{Z}I$ , with elements written in the form  $d = \sum_{i \in I} d_i i$ , and define  $\Lambda^+ = \Lambda^+(Q) = \mathbf{N}I \subset \Lambda$ ; elements of  $\Lambda^+$  are called dimension vectors for  $Q$ . Occasionally (namely, for the tree quiver of § 2.4 and for the big local quiver of § 3.2), we will also use *locally finite* quivers, for which the set of vertices is possibly infinite, but with only finitely many arrows between each pair of vertices. For locally finite quivers, dimension vectors are assumed to be supported on a finite subquiver.

Introduce a non-symmetric bilinear form  $\langle -, - \rangle$ , the *Euler form*, on  $\Lambda$  by

$$\langle d, e \rangle = \sum_{i \in I} d_i e_i - \sum_{\alpha: i \rightarrow j} d_i e_j$$

for  $d, e \in \Lambda$ ; we thus have  $\langle i, j \rangle = \delta_{i,j} - r_{i,j}$ . For a functional  $\Theta \in \Lambda^* = \text{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z})$ , called a *stability*, define the *slope of*  $d \in \Lambda^+ \setminus 0$  as the rational number  $\mu(d) = \Theta(d)/\dim d$ , where

$\dim d = \sum_{i \in I} d_i$ . For  $\mu \in \mathbf{Q}$ , define

$$\Lambda_\mu^+ = \{d \in \Lambda^+ \setminus 0, \mu(d) = \mu\} \cup \{0\}$$

(a subsemigroup of  $\Lambda^+$ ) and  $\Lambda_\mu^+ = \Lambda_\mu^+ \setminus 0$ .

We consider complex finite dimensional representations  $M$  of  $Q$ , consisting of a tuple of complex vector spaces  $M_i$  for  $i \in I$  and a tuple of  $\mathbf{C}$ -linear maps  $M_\alpha : M_i \rightarrow M_j$  indexed by the arrows  $\alpha : i \rightarrow j$  of  $Q$ . The *dimension vector*  $\underline{\dim} M \in \Lambda^+$  is defined by  $(\underline{\dim} M)_i = \dim_{\mathbf{C}} M_i$ . The abelian  $\mathbf{C}$ -linear category of all such representations is denoted by  $\text{mod}_{\mathbf{C}} Q$ .

Define the slope of a non-zero representation  $M$  of  $Q$  as the slope of its dimension vector; thus,  $\mu(M) = \mu(\underline{\dim} M)$ . Call  $M$   $\Theta$ -semistable if  $\mu(U) \leq \mu(M)$  for all non-zero subrepresentations  $U$  of  $M$ , and call  $M$   $\Theta$ -stable if  $\mu(U) < \mu(M)$  for all proper non-zero subrepresentations  $U$  of  $M$ . Finally, call  $M$   $\Theta$ -polystable if it is isomorphic to a direct sum of stable representations of the same slope. The full subcategory  $\text{mod}_{\mathbf{C}}^\mu Q$  of all  $\Theta$ -semistable representations of slope  $\mu \in \mathbf{Q}$  is an abelian subcategory, that is, it is closed under extensions, kernels, and cokernels. Its simple (respectively semisimple) objects are precisely the  $\Theta$ -stable (respectively  $\Theta$ -polystable) representations of  $Q$  of slope  $\mu$ .

Note that in the case  $\Theta = 0$ , all representations are semistable, and the stable (respectively polystable) ones are just the simple ones (respectively semisimple ones).

### 2.2 Moduli spaces of representations

By [Kin94], for every  $d \in \Lambda^+$ , there exists a (typically singular) complex variety  $M_d^{\Theta\text{-pst}}(Q)$  whose points parameterize the isomorphism classes of  $\Theta$ -polystable representations of  $Q$  of dimension vector  $d$ . It contains a smooth open subset  $M_d^{\Theta\text{-st}}(Q)$  whose points parameterize the isomorphism classes of  $\Theta$ -stable representations of  $Q$  of dimension vector  $d$ .

In the case  $\Theta = 0$ , the variety  $M_d^{\Theta\text{-pst}}(Q)$  is affine, parameterizing isomorphism classes of semisimple representations of  $Q$  of dimension vector  $d$ ; it will be denoted by  $M_d^{\text{ssimp}}(Q)$ . This variety always contains a special point  $0$  corresponding to the semisimple representations  $\bigoplus_{i \in I} E_i^{d_i}$ , where  $E_i$  denotes the one-dimensional representation of  $Q$  concentrated at a vertex  $i \in I$ , and with all arrows represented by zero maps. Note that all  $M_d^{\text{ssimp}}(Q)$  reduce to the single point  $0$  if  $Q$  has no oriented cycles. There exists a projective morphism from  $M_d^{\Theta\text{-pst}}(Q)$  to  $M_d^{\text{ssimp}}(Q)$ .

Let  $n \in \Lambda^+$  be another dimension vector, and fix complex vector spaces  $V_i$  of dimension  $n_i$  for  $i \in I$ . A pair  $(M, f)$  consisting of a semistable representation  $M$  of  $Q$  of dimension vector  $d$  and a tuple  $f = (f_i : V_i \rightarrow M_i)$  of  $\mathbf{C}$ -linear maps is called stable in [ER09] if the following condition holds: if  $U$  is a proper subrepresentation of  $M$  containing the image of  $f$  (in the sense that  $f_i(V_i) \subset U_i$  for all  $i \in I$ ), then  $\mu(U) < \mu(M)$ . Two such pairs  $(M, f), (M', f')$  are called equivalent if there exists an isomorphism  $\varphi : M \rightarrow M'$  intertwining the additional maps, that is, such that  $f'_i = \varphi_i \circ f_i$  for all  $i \in I$ .

By [ER09], there exists a smooth complex variety  $M_{d,n}^\Theta(Q)$ , called a *smooth model for*  $M_d^{\Theta\text{-pst}}(Q)$ , whose points parameterize equivalence classes of stable pairs  $(M, f)$  as above. It admits a projective morphism  $\pi_d : M_{d,n}^\Theta(Q) \rightarrow M_d^{\Theta\text{-pst}}(Q)$ .

In the case of trivial stability, the smooth model (a *Hilbert scheme for the path algebra of*  $Q$ )  $\text{Hilb}_{d,n}(Q) := M_{d,n}^0(Q)$  parameterizes arbitrary representations  $M$  of  $Q$  of dimension vector  $d$ , together with maps  $f_i : V_i \rightarrow M_i$  whose images generate the representation  $M$ . We have

a projective morphism  $\pi_d : \text{Hilb}_{d,n}(Q) \rightarrow M_d^{\text{ssimp}}(Q)$ . We denote by  $\text{Hilb}_{d,n}^{\text{nilp}}(Q)$  the inverse image under  $\pi_d$  of the special point  $0 \in M_d^{\text{ssimp}}(Q)$ ; it parameterizes pairs  $(M, f)$  as above, with  $M$  being a *nilpotent* representation, in the sense that all maps  $M_{\alpha_n} \circ \dots \circ M_{\alpha_1}$  representing oriented cycles  $i_1 \xrightarrow{\alpha_1} i_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} i_1$  in  $Q$  are nilpotent.

We summarize the different moduli spaces in the following diagram.

$$\begin{array}{ccccc}
 & M_{d,n}^{\ominus}(Q) & & \text{Hilb}_{d,n}(Q) & \longleftarrow & \text{Hilb}_{d,n}^{\text{nilp}}(Q) \\
 & \downarrow & & \downarrow & & \\
 M_d^{\ominus\text{-st}}(Q) & \hookrightarrow & M_d^{\ominus\text{-pst}}(Q) & \longrightarrow & M_d^{\text{ssimp}}(Q) & 
 \end{array}$$

### 2.3 Stratifications of moduli spaces

The variety  $M_d^{\ominus\text{-pst}}(Q)$  admits the following Luna-type stratification (that is, a finite decomposition into locally closed subsets) induced by the decomposition types of polystable representations: let  $\xi = ((d^1, \dots, d^s), (m_1, \dots, m_s))$  be a pair consisting of a tuple of dimension vectors of the same slope as  $d$  and a tuple of non-negative integers, such that  $d = \sum_{k=1}^s m_k d^k$ . We call such  $\xi$  a *polystable type for  $d$* . Analogously to [LP87] in the case of trivial stability, the set of all polystable representations  $M$  such that  $M = \bigoplus_{k=1}^s W_k^{m_k}$  for pairwise non-isomorphic stable representations  $W_k$  of dimension vector  $d^k$  forms a locally closed subset of  $M_d^{\ominus\text{-pst}}(Q)$ , denoted by  $M_d^{\ominus\text{-pst}}(Q)_\xi$  and called a *Luna stratum* of  $M_d^{\ominus\text{-pst}}(Q)$ . Note that the open stratum  $M_d^{\ominus\text{-pst}}(Q)_{((d),(1))}$ , corresponding to the polystable type  $((d), (1))$ , is precisely  $M_d^{\ominus\text{-st}}(Q)$ .

Following [AL03], for any polystable type  $\xi$  for  $d$ , introduce *local quiver data*  $Q_\xi, d_\xi, n_\xi$  as follows: the quiver  $Q_\xi$  has vertices  $i_1, \dots, i_s$  with  $\delta_{k,l} - \langle d^k, d^l \rangle$  arrows from  $i_k$  to  $i_l$  for  $k, l = 1, \dots, s$ . The dimension vector  $d_\xi$  is defined by  $(d_\xi)_{i_k} = m_k$  for  $k = 1, \dots, s$ , and the dimension vector  $n_\xi$  is defined by  $(n_\xi)_{i_k} = n \cdot d^k$ , where  $n \cdot e = \sum_i n_i e_i$  for  $n, e \in \Lambda$ . With this notation, we have the following result (see [ER09]).

**THEOREM 2.1.** *The variety  $M_{d,n}^{\ominus}(Q)$  admits a stratification (in the sense defined above) by the locally closed subsets  $M_{d,n}^{\ominus}(Q)_\xi = \pi_d^{-1}(M_d^{\ominus\text{-pst}}(Q)_\xi)$ . Each  $M_{d,n}^{\ominus}(Q)_\xi$  admits a fibration (that is, an étale locally trivial surjection) over the corresponding Luna stratum  $M_d^{\ominus\text{-pst}}(Q)_\xi$ , whose fibre is isomorphic to the nilpotent part of the Hilbert scheme for the local quiver  $\text{Hilb}_{d_\xi, n_\xi}^{\text{nilp}}(Q_\xi)$ .*

### 2.4 Cell decomposition of Hilbert schemes

By a cell decomposition of a variety  $X$ , we mean a filtration  $\emptyset = X_0 \subset X_1 \subset \dots \subset X_s = X$  by closed subvarieties such that the complements  $X_k \setminus X_{k-1}$  are isomorphic to affine spaces.

For every vertex  $i \in I$ , we construct a (locally finite) tree quiver  $Q_i$  as follows: the vertices  $i_\omega$  of  $Q_i$  are indexed by the paths  $\omega$  in  $Q$  starting in  $i$  (including the empty path starting and ending in  $i$ ); there is an arrow  $i_\omega \rightarrow i_{\alpha\omega}$  to the vertex corresponding to the concatenation  $\alpha\omega$  for every path  $\omega$  from  $i$  to  $j$  and every arrow  $\alpha : j \rightarrow k$ . Note that  $Q_i$  has a unique source, corresponding to the empty path. By a subtree  $T$  of  $Q_i$ , we mean a finite full subquiver which is closed under taking predecessors. The dimension vector  $\underline{\dim} T \in \Lambda^+(Q)$  is defined by setting  $(\underline{\dim} T)_j$  as the number of vertices  $i_\omega$  in  $T$  corresponding to paths  $\omega$  ending in  $j \in I$ . By an  $n$ -forest, we mean a tuple  $T_* = (T_{i,k})_{i \in I, k=1, \dots, n_i}$  of subtrees  $T_{i,k}$  of  $Q_i$ ; its dimension vector is defined as  $\underline{\dim} T_* = \sum_{i \in I} \sum_{k=1}^{n_i} \underline{\dim} T_{i,k}$ . The following theorem is proved in [ER09].

**THEOREM 2.2.** *For all  $d$  and  $n$ , the Hilbert scheme  $\text{Hilb}_{d,n}(Q)$  admits a cell decomposition, whose cells are parameterized by the  $n$ -forests of dimension vector  $d$ .*

It follows immediately from this theorem that the Euler characteristic (all Euler characteristics refer to singular cohomology with compact support) of the Hilbert scheme  $\text{Hilb}_{d,n}(Q)$  can be computed as the number of  $n$ -forests of dimension vector  $d$ . This allows us to characterize the generating function of these Euler characteristics by a functional equation. For all  $n \in \Lambda^+$ , we write

$$F^{(n)}(t) = \sum_{d \in \Lambda^+} \chi(\text{Hilb}_{d,n}(Q)) t^d \in \mathbf{Z}[[\Lambda^+]],$$

where the generating series is considered as an element of the formal power series ring  $\mathbf{Z}[[\Lambda^+]] = \mathbf{Z}[[t_i : i \in I]]$  and monomials in the generators  $t_i$  are written as  $t^d = \prod_{i \in I} t_i^{d_i}$  for  $d \in \Lambda^+$ .

**PROPOSITION 2.3.** *The series  $F^{(n)}(t)$  are the uniquely determined elements of  $\mathbf{Z}[[\Lambda^+]]$  satisfying the following functional equations:*

- (1) for all  $n \in \Lambda^+$ , we have  $F^{(n)}(t) = \prod_{i \in I} F^{(i)}(t)^{n_i}$ ;
- (2) for all  $i \in I$ , we have  $F^{(i)}(t) = 1 + t_i \prod_{j \in I} F^{(j)}(t)^{r_{i,j}}$ .

*Proof.* Comparing coefficients of  $t^d$  on both sides of the first identity, we see that the first claim reduces to the definition of  $n$ -forests. With the same method, the second identity reduces to the existence of a bijection between subtrees of  $Q_i$  of dimension vector  $d$  and tuples  $(T_{j,k})_{j \in I, k=1, \dots, r_{i,j}}$  of subtrees  $T_{j,k}$  of  $Q_j$  such that  $\sum_{j \in I} \sum_{k=1}^{r_{i,j}} \dim T_{j,k} = d - i$ . Such a bijection is provided, by definition of the trees  $Q_i$ , by grafting the subtrees  $T_{j,k}$  to a common root  $i$  to obtain any subtree of  $Q_i$  exactly once. □

*Remark.* In the special case of a quiver with a single vertex and a number of loops, this functional equation is derived in [Rei05].

### 3. Functional equation for $\chi(M_{d,n}^\Theta(Q))$ and the big local quiver

#### 3.1 Preparations

We prepare the derivation of functional equations by two preliminary results.

**PROPOSITION 3.1.** *For all  $d, n \in \Lambda^+$ , we have  $\chi(\text{Hilb}_{d,n}^{\text{nilp}}(Q)) = \chi(\text{Hilb}_{d,n}(Q))$ .*

*Proof.* We adopt an argument used in [CV04]. There is a natural  $\mathbf{C}^*$ -action on representations of  $Q$  by rescaling the maps representing the arrows by a common factor. This action induces actions on  $\text{Hilb}_{d,n}(Q)$  and  $M_d^{\text{ssimp}}(Q)$ , for which the map  $\pi_d : \text{Hilb}_{d,n}(Q) \rightarrow M_d^{\text{ssimp}}(Q)$  is equivariant. Moreover, there exists a unique fixed point for the action of  $\mathbf{C}^*$  on  $M_d^{\text{ssimp}}(Q)$ , namely the point 0, to which all points of  $M_d^{\text{ssimp}}(Q)$  attract, in the sense that  $\lim_{t \rightarrow 0} t \cdot M = 0$  for all  $M \in M_d^{\text{ssimp}}(Q)$ . Therefore, all points of  $\text{Hilb}_{d,n}(Q)$  admit a well-defined limit in the projective variety  $\pi_d^{-1}(0) = \text{Hilb}_{d,n}^{\text{nilp}}(Q)$ . For each connected component  $C$  of  $\text{Hilb}_{d,n}^{\text{nilp}}(Q)$ , we have its attractor  $A_C$  consisting of all points of  $\text{Hilb}_{d,n}(Q)$  whose limit belongs to  $C$ . By the Bialynicki-Birula theorem [Bia73], the attractors  $A_C$  are affine fibrations over the components  $C$ . Consequently, the Euler characteristics of  $\text{Hilb}_{d,n}(Q)$  and of  $\text{Hilb}_{d,n}^{\text{nilp}}(Q)$  coincide. □

Next, we calculate the Euler characteristics of certain strata of symmetric products of a variety. Denote by  $\mathcal{P}$  the set of all partitions. For  $\lambda$  in  $\mathcal{P}$ , denote by  $m_k(\lambda)$  the multiplicity of  $k$  in  $\lambda$ , that is, the number of indices  $l$  such that  $\lambda_l = k$ . For a variety  $X$ , we denote by  $S^n(X)$  its  $n$ th symmetric power, that is, the quotient of  $X^n$  by the natural action of the symmetric group  $\Sigma_n$ . The product variety  $X^n$  admits a stratification by strata  $X_I^n$ , where  $I = (I_1, \dots, I_s)$  is a decomposition of  $\{1, \dots, n\}$  into pairwise disjoint subsets. Namely,  $X_I^n$  is defined as the set of ordered tuples  $(x_1, \dots, x_n)$  such that  $x_k = x_l$  if and only if  $k, l$  belong to the same subset  $I_t$ . Obviously,  $X_I^n$  is isomorphic to  $\check{X}^k$ , the open subset of  $X^k$  of ordered  $k$ -tuples of pairwise different points in  $X$ .

Any  $I$  as above induces a partition  $\lambda(I)$  of  $n$ , with parts being the cardinalities of the subsets  $I_t$  forming  $I$ . The image of  $X_I^n$  under the quotient map  $q: X^n \rightarrow S^n(X)$  depends only on the partition  $\lambda = \lambda(I)$  and is denoted by  $S_\lambda^n(X)$ . The inverse image under  $q$  of  $S_\lambda^n(X)$  is precisely the union of the strata  $X_I^n$  such that  $\lambda(I) = \lambda$ . Moreover, the fibre of  $q$  over a point in  $S_\lambda^n(X)$  is finite of cardinality  $n!/\lambda_1! \dots \lambda_s!$ . The number of decompositions  $I$  such that  $\lambda(I) = \lambda$  equals

$$\frac{n!}{\lambda_1! \cdot \dots \cdot \lambda_s!} \cdot \frac{1}{\prod_k (m_k(\lambda)!)}.$$

An easy induction shows that the Euler characteristic of  $\check{X}^n$  equals

$$\chi(X)(\chi(X) - 1) \cdots (\chi(X) - n + 1) = n! \binom{\chi(X)}{n}.$$

We have thus proved the following lemma.

LEMMA 3.2. *For all partitions  $\lambda$  of  $n$  of length  $s$ , we have*

$$\chi(S_\lambda^n(X)) = \frac{1}{\prod_k m_k(\lambda)!} s! \binom{\chi(X)}{s}.$$

### 3.2 The big local quiver

Now we fix quiver data  $Q, \Theta, \mu, n$  as before, and associate to it a locally finite quiver, called the *big local quiver*  $\tilde{Q}$ , as follows: the vertices  $i_{(d,k)}$  of  $\tilde{Q}$  are indexed by pairs  $(d, k)$  in  $\Lambda_\mu^+ \times \mathbf{N}$ . The number of arrows from the vertex  $i_{(d,k)}$  to  $i_{(d',k')}$  is given as  $\delta_{d,d'} \cdot \delta_{k,k'} - \langle d, d' \rangle$ . For a function  $l: \Lambda_\mu^+ \rightarrow \mathbf{N}$ , we define  $\tilde{Q}_l$  as the full subquiver of  $\tilde{Q}$  supported on the set of vertices  $i_{(d,k)}$  for  $d \in \Lambda_\mu^+$  and  $1 \leq k \leq l(d)$ .

We define dimension vectors  $\tilde{n}$  for the various quivers  $\tilde{Q}_l$  by  $\tilde{n}_{i_{(d,k)}} = n \cdot d$ . The product  $\Sigma = \prod_{d \in \Lambda_\mu^+} \Sigma_\infty$  of infinite symmetric groups acts on the vertices of  $\tilde{Q}$  by permutation  $(\sigma_e)_{e \in \Lambda_\mu^+} i_{(d,k)} = i_{(d, \sigma_d(k))}$ ; this restricts to an action of the subgroup  $\Sigma_l = \prod_{d \in \Lambda_\mu^+} \Sigma_{l(d)}$  of  $\Sigma$  on  $\tilde{Q}_l$ .

For a polystable type  $\xi = ((d^1, \dots, d^s), (m_1, \dots, m_s))$  as above, we can view the local quiver  $Q_\xi$  as the quiver  $\tilde{Q}_{l_\xi}$  just defined, where the function  $l_\xi$  is given by defining  $l_\xi(d)$  as the number of indices  $1 \leq t \leq s$  such that  $d = d^t$ . The dimension vector  $d_\xi$  for  $Q_\xi$  can then be viewed as a dimension vector  $\tilde{d}_\xi$  for  $\tilde{Q}_{l_\xi}$ . This dimension vector can be made unique by assuming that its entries  $(\tilde{d}_\xi)_{i_{(d,k)}}$ , for every fixed  $d \in \Lambda_\mu^+$ , form a partition, that is,  $(\tilde{d}_\xi)_{i_{(d,1)}} \geq \dots \geq (\tilde{d}_\xi)_{i_{(d,l_\xi(d))}}$ . Therefore, we call dimension vectors  $\tilde{d}$  of  $\tilde{Q}_l$  *partitive* if  $\tilde{d}_{i_{(d,1)}} \geq \dots \geq \tilde{d}_{i_{(d,l(d))}}$  for all  $d \in \Lambda_\mu^+$ ; the set of all partitive dimension vectors for  $\tilde{Q}$  (respectively  $\tilde{Q}_l$ ) is denoted by  $\Lambda(\tilde{Q})^\geq$  (respectively  $\Lambda(\tilde{Q}_l)^\geq$ ).

We have a natural specialization map  $\nu : \Lambda(\tilde{Q}_l)^+ \rightarrow \Lambda_\mu^+$  given by  $\nu(i_{(d,k)}) = d$ . It is well-defined, since, by assumption, dimension vectors for the (possibly infinite) quiver  $\tilde{Q}_l$  have finite support.

We consider the generating function

$$R_l^{(n)}(t) = \sum_{\tilde{d} \in \Lambda^+(\tilde{Q}_l)} \chi(\text{Hilb}_{\tilde{d}, \tilde{n}}(\tilde{Q}_l)) t^{\nu(\tilde{d})} \in \mathbf{Z}[[\Lambda_\mu^+]],$$

the specialization of the generating function  $F^{\tilde{n}}(t) \in \mathbf{Z}[[\Lambda(\tilde{Q}_l)^+]]$  for the quiver  $\tilde{Q}_l$  with respect to the map  $\nu$ . By the natural  $\Sigma_l$ -symmetry of  $\tilde{Q}_l$ , we have  $R_l^{(i_{(d,k)})}(t) = R_l^{(i_{(d,k')})}(t)$  for all  $d \in \Lambda_\mu^+$  and all  $1 \leq k, k' \leq l(d)$ . We denote this series by  $R_l^{[d]}(t)$ . Applying Proposition 2.3 and the definition of  $\tilde{Q}_l$ , we get

$$R_l^{(n)}(t) = \prod_{d \in \Lambda_\mu^+} R_l^{[d]}(t)^{l(d) \cdot (n \cdot d)}$$

and

$$R_l^{[d]}(t) = 1 + t^d \cdot R_l^{[d]}(t) \cdot \prod_{e \in \Lambda_\mu^+} R_l^{[e]}(t)^{-(d,e) \cdot l(e)}. \tag{1}$$

For a function  $l : \Lambda_\mu^+ \rightarrow \mathbf{N}$  with finite support, we call a dimension vector for  $\tilde{Q}_l$  *faithful* if all its entries are non-zero, and we denote by  $\Lambda(\tilde{Q}_l)^{++}$  the set of all such dimension vectors. Define

$${}'R_l^{(n)}(t) = \sum_{\tilde{d} \in \Lambda^+(\tilde{Q}_l)^{++}} \chi(\text{Hilb}_{\tilde{d}, \tilde{n}}(\tilde{Q}_l)) t^{\nu(\tilde{d})} \in \mathbf{Z}[[\Lambda_\mu^+]].$$

Using again the symmetry of  $\tilde{Q}_l$ , we see that

$$R_l^{(n)}(t) = \sum_{l' : \Lambda_\mu^+ \rightarrow \mathbf{N}} \prod_{d \in \Lambda_\mu^+} \binom{l(d)}{l'(d)} \cdot {}'R_{l'}^{(n)}(t), \tag{2}$$

where the sum ranges over all functions  $l'$  with finite support.

Let  $c : \Lambda_\mu^+ \rightarrow \mathbf{Z}$  be a function with arbitrary integer values (in contrast to the function  $l$  considered so far). We can extend the definition of the series  $R_l^{(n)}(t)$  to this case using the previous identity. We thus define a formal series  $R_c^{(n)}(t)$  by

$$R_c^{(n)}(t) = \sum_{l' : \Lambda_\mu^+ \rightarrow \mathbf{N}} \prod_{d \in \Lambda_\mu^+} \binom{c(d)}{l'(d)} \cdot {}'R_{l'}^{(n)}(t),$$

the sum running again over functions with finite support.

Similarly to the above, we have series  $R_c^{[d]}(t)$  as special cases of the series  $R_c^{(n)}(t)$ .

LEMMA 3.3. *The series  $R_c^{(n)}(t)$  are given by the functional equations*

$$R_c^{(n)}(t) = \prod_{d \in \Lambda_\mu^+} R_c^{[d]}(t)^{c(d) \cdot (n \cdot d)}$$

and

$$R_c^{[d]}(t) = 1 + t^d \cdot R_c^{[d]}(t) \cdot \prod_{e \in \Lambda_\mu^+} R_c^{[e]}(t)^{-(d,e) \cdot c(e)}.$$



*Proof.* For a function  $c$  as above, it is easy to see that there exist unique series  $P_c^{[d]}(t)$  for all  $d \in \Lambda_\mu^+$  fulfilling the equations

$$P_c^{[d]}(t) = 1 + t^d \cdot P_c^{[d]}(t) \cdot \prod_{e \in \Lambda_\mu^+} P_c^{[e]}(t)^{-\langle d, e \rangle \cdot c(e)},$$

since these functional equations define recursions determining the coefficients of the series. These coefficients depend polynomially on the values  $c(d) \in \mathbf{Z}$ . The same holds for the coefficients of the series  $R_c^{[d]}(t)$  by definition. Now the equality  $P_c^{[d]}(t) = R_c^{[d]}(t)$  holds for all functions  $c$  with values in  $\mathbf{N}$  by (1) and (2); thus, it has to hold for arbitrary  $c$ . The same argument applies to the series  $P_c^{(n)}(t)$  defined by

$$P_c^{(n)}(t) = \prod_{d \in \Lambda_\mu^+} R_c^{[d]}(t)^{c(d) \cdot (n \cdot d)},$$

proving the lemma. □

### 3.3 Functional equation for $\chi(M_{d,n}^\Theta(Q))$

We can now consider the generating function of the Euler characteristics of arbitrary smooth models, using the big local quiver notation of § 3.2.

In particular, to a polystable type  $\xi$  of a dimension vector  $d \in \Lambda^+$ , we have associated a partitive dimension vector  $p$  for  $\tilde{Q}$  (respectively for a large enough  $\tilde{Q}_l$ ); we denote the stratum by  $M_d^{\Theta-\text{pst}}(Q)_\xi$  by  $M_d^{\Theta-\text{pst}}(Q)_p$ . With the above notation, we have

$$M_d^{\Theta-\text{pst}}(Q)_p \simeq \prod_{d \in \Lambda_\mu^+} S_{p(d)}^{|p(d)|}(M_d^{\Theta-\text{st}}(Q)),$$

by definition of  $M_d^{\Theta-\text{pst}}(Q)_\xi$ . Theorem 2.1 can now be rephrased as stating that  $M_{d,n}^\Theta(Q)$  admits a stratification indexed by partitive dimension vectors  $p \in \Lambda(\tilde{Q})^+$  such that  $\nu(p) = d$ . Each stratum is a locally trivial fibration over  $M_d^{\Theta-\text{pst}}(Q)_p$ , with fibre isomorphic to  $\text{Hilb}_{p,\tilde{n}}^{\text{nilp}}(\tilde{Q})$ . We thus have, using Lemma 3.2 for the second equality,

$$\begin{aligned} \chi(M_{d,n}^\Theta(Q)) &= \sum_p \chi(M_d^{\Theta-\text{pst}}(Q)_p) \cdot \chi(\text{Hilb}_{p,\tilde{n}}^{\text{nilp}}(\tilde{Q})) \\ &= \sum_p \prod_{d \in \Lambda_\mu^+} \frac{1}{\prod_k m_k(p(d))!} l(p(d))! \binom{\chi(M_d^{\Theta-\text{st}}(Q))}{l(p(d))} \cdot \chi(\text{Hilb}_{p,\tilde{n}}^{\text{nilp}}(\tilde{Q})), \end{aligned}$$

the sum running over all partitive dimension vectors  $p$  for  $\tilde{Q}$  such that  $\nu(p) = d$ .

Considering the generating function  $Q_\mu^{(n)}(t) \in \mathbf{Z}[[\Lambda_\mu^+]]$  of the Euler characteristics of the smooth models  $M_{d,n}^\Theta(Q)$ , we thus have

$$\begin{aligned} Q_\mu^{(n)}(t) &= \sum_{d \in \Lambda_\mu^+} \chi(M_{d,n}^\Theta(Q)) t^d \\ &= \sum_{p \in \Lambda(\tilde{Q}) \geq} \prod_{d \in \Lambda_\mu^+} \left( \frac{1}{\prod_k m_k(p(d))!} l(p(d))! \binom{\chi(M_d^{\Theta-\text{st}}(Q))}{l(p(d))} \right) \cdot \chi(\text{Hilb}_{p,\tilde{n}}^{\text{nilp}}(\tilde{Q})) t^{\nu(p)}. \end{aligned}$$

Sorting by lengths of the partitions, this can be rewritten as

$$\sum_l \sum_{p \in \Lambda(\tilde{Q}_l) \geq} \prod_{d \in \Lambda_\mu^+} \left( \frac{1}{\prod_k m_k(p(d))!} l(d) \right)! \binom{\chi(M_d^{\Theta\text{-st}}(Q))}{l(d)} \cdot \chi(\text{Hilb}_{p, \tilde{n}}^{\text{nilp}}(\tilde{Q})) t^{\nu(p)},$$

the first sum ranging over all functions  $l : \Lambda_\mu^+ \rightarrow \mathbf{N}$ . We want to extend the range of summation in the inner sum to arbitrary dimension vectors for each  $\tilde{Q}_l$  without changing the sum. By the  $\Sigma_l$ -symmetry of  $\tilde{Q}_l$ , we can do this by incorporating a factor which counts the number of derangements of a given partitive dimension vector  $p$  into arbitrary faithful dimension vectors. This number is precisely

$$\prod_{d \in \Lambda_\mu^+} \frac{l(p(d))!}{\prod_k m_k(p(d))!},$$

this factor being already present in the previous sum. Thus, this sum equals

$$\sum_{l : \Lambda_\mu^+ \rightarrow \mathbf{N}} \sum_{\tilde{d} \in \Lambda(\tilde{Q}_l)++} \prod_{d \in \Lambda_\mu^+} \binom{\chi(M_d^{\Theta\text{-st}}(Q))}{l(d)} \cdot \chi(\text{Hilb}_{\tilde{d}, \tilde{n}}^{\text{nilp}}(\tilde{Q})) t^{\nu(\tilde{d})},$$

the inner sum now running over all faithful dimension vectors for  $\tilde{Q}_l$ . Using Proposition 3.1 and the notation of § 3.2, this equals

$$\sum_{l : \Lambda_\mu^+ \rightarrow \mathbf{N}} \prod_{d \in \Lambda_\mu^+} \binom{\chi(M_d^{\Theta\text{-st}}(Q))}{l(d)} R_l^{(n)}(t) = R_c^{(n)}(t)$$

for the function  $c : \Lambda_\mu^+ \rightarrow \mathbf{Z}$  defined by  $c(d) = \chi(M_d^{\Theta\text{-st}}(Q))$ . By Lemma 3.3, we arrive at the following result.

**THEOREM 3.4.** *Defining series  $R^{[d]}(t) \in \mathbf{Z}[[\Lambda_\mu^+]]$  by the functional equations*

$$R^{[d]}(t) = 1 + t^d \cdot R^{[d]}(t) \cdot \prod_{e \in \Lambda_\mu^+} R^{[e]}(t)^{-\langle d, e \rangle \cdot \chi(M_e^{\Theta\text{-st}}(Q))},$$

the generating function  $Q_\mu^{(n)}(t)$  of Euler characteristics of smooth models is given by

$$Q_\mu^{(n)}(t) = \sum_{d \in \Lambda_\mu^+} \chi(M_{d,n}^\Theta(Q)) t^d = \prod_{d \in \Lambda_\mu^+} R^{[d]}(t)^{\chi(M_d^{\Theta\text{-st}}(Q)) \cdot \langle n, d \rangle}.$$

To make the nature of these functional equations more transparent, we will define a slight variant of the generating functions. By the theorem, we have  $Q_\mu^{(n)}(t) = \prod_{i \in I} Q_\mu^{(i)}(t)^{n_i}$ . This suggests the definition  $Q_\mu^{(\eta)}(t) = \prod_{i \in I} Q_\mu^{(i)}(t)^{\eta(i)}$  for an arbitrary linear functional  $\eta \in \Lambda^*$ , so that  $Q_\mu^{(n \cdot)}(t) = Q_\mu^{(n)}(t)$  for all  $n \in \Lambda^+$ , where  $n \cdot$  denotes the functional with value  $n \cdot d$  on  $d$ . In particular, we consider  $S_\mu^{(d)}(t) = Q_\mu^{-\langle d, \cdot \rangle}(t)$  for  $d \in \Lambda_\mu^+$ .

**COROLLARY 3.5.** *The series  $S_\mu^{(d)}(t)$  for  $d \in \Lambda_\mu^+$  are given by the functional equations*

$$S_\mu^{(d)}(t) = \prod_{e \in \Lambda_\mu^+} (1 - t^e S_\mu^{(e)}(t))^{\langle d, e \rangle \cdot \chi(M_e^{\Theta\text{-st}}(Q))}.$$

*Proof.* By the definitions and Theorem 3.4, we have

$$S_\mu^{(d)}(t) = \prod_{e \in \Lambda_\mu^+} R^{[e]}(t)^{-\langle d, e \rangle \cdot \chi(M_e^{\Theta\text{-st}}(Q))}.$$

The functional equation in Theorem 3.4 can be restated as

$$R^{[e]}(t) = \left( 1 - t^e \prod_{f \in \Lambda_\mu^+} R^{[f]}(t)^{-\langle e, f \rangle \cdot \chi(M_f^{\Theta\text{-st}}(Q))} \right)^{-1};$$

thus,

$$R^{[e]}(t) = (1 - t^e S_\mu^{(e)}(t))^{-1}.$$

Substituting this in the factorization of  $S_\mu^d(t)$  yields the desired equation. □

### 3.4 Application to wall-crossing formulas

We apply the results of the previous sections to the setup of [Rei10]. We assume that  $Q$  is a quiver without oriented cycles; thus, we can order the vertices as  $I = \{i_1, \dots, i_r\}$  in such a way that  $k > l$  provided there exists an arrow  $i_k \rightarrow i_l$ . We also choose a stability  $\Theta$  for  $Q$ .

Denote by  $\{-, -\}$  the skew symmetrization of  $\langle -, - \rangle$ ; thus,  $\{d, e\} = \langle d, e \rangle - \langle e, d \rangle$ . Define  $b_{ij} = \{i, j\}$  for  $i, j \in I$ .

We consider the formal power series ring  $B = \mathbf{Q}[[\Lambda^+]] = \mathbf{Q}[[x_i : i \in I]]$  with topological basis  $x^d = \prod_{i \in I} x_i^{d_i}$  for  $d \in \Lambda^+$ . The algebra  $B$  becomes a Poisson algebra via the Poisson bracket

$$\{x_i, x_j\} = b_{ij} x_i x_j \quad \text{for } i, j \in I.$$

Define Poisson automorphisms  $T_i$  of  $B$  by

$$T_i(x_j) = x_j \cdot (1 + x_i)^{\{i, j\}}$$

for all  $i, j \in I$ .

We study a factorization property in the group  $\text{Aut}(B)$  of Poisson automorphisms of  $B$  involving a descending product  $\prod_{\mu \in \mathbf{Q}}^{\leftarrow}$  indexed by rational numbers, which is indeed well defined (see [Rei10]). The index  $\mu$  plays the role of slopes of representations; thus, the product depends on the choice of the stability  $\Theta$ . The main result of [Rei10] is the following theorem.

**THEOREM 3.6.** *In the group  $\text{Aut}(B)$ , we have a factorization*

$$T_{i_1} \circ \dots \circ T_{i_r} = \prod_{\mu \in \mathbf{Q}}^{\leftarrow} T_\mu,$$

where

$$T_\mu(x^d) = x^d \cdot Q_\mu^{\{\cdot, d\}}(x).$$

Here  $Q_\mu^{(\eta)}(x) = \prod_{i \in I} (\sum_{d \in \Lambda_\mu^+} \chi(M_{d,i}^{\Theta}(Q)) x^d)^{\eta(i)}$  denotes the specialization of the series  $Q_\mu^{(\eta)}(t)$  from the variables  $t_i$  to the variables  $x_i$ .

Let  $\Phi \in \text{Aut}(\Lambda)$  be the map induced on dimension vectors by the inverse Auslander–Reiten translation;  $\Phi$  is a Coxeter element of the corresponding Weyl group determined by the property

$$\langle \Phi(d), e \rangle = -\langle e, d \rangle.$$

Then we have

$$\{ -, d \} = \langle -(\text{id} + \Phi)d, - \rangle$$

and thus, using Corollary 3.5, we have the following corollary.

COROLLARY 3.7. *The automorphisms  $T_\mu$  of Theorem 3.6 can be written as*

$$T_\mu(x^d) = x^d \cdot S_\mu^{(\text{id} + \Phi)d}(x).$$

#### 4. Duality for Euler products

This section is independent of the others; in particular, none of the notation referring to quivers and their moduli spaces is used.

Let  $F(t) \in \mathbf{Z}[[t]]$  be a formal power series with constant term  $F(0) = 1$ . Then we can write  $F(t)$  as an Euler product

$$F(t) = \prod_{i \geq 1} (1 - (-t)^i)^{-ia_i} \tag{3}$$

for  $a_i \in \mathbf{Q}$  (note the sign convention, which is essential in the following; see the example at the end of this section). We can also characterize  $F(t)$  as the unique solution of a functional equation of the form

$$F(t) = \prod_{i \geq 1} (1 - (tF(t))^i)^{ib_i} \tag{4}$$

for  $b_i \in \mathbf{Q}$ ; see the remark below for the proof.

The main result of this section is the following.

THEOREM 4.1. *In the above notation, we have  $b_i \in \mathbf{Z}$  for all  $i \geq 1$  if and only if  $a_i \in \mathbf{Z}$  for all  $i \geq 1$ .*

*Remark.* Writing  $H(t) = -tF(t)$ , we have, by a straightforward calculation,

$$H(t) = -t \prod_{i \geq 1} (1 - (-t)^i)^{-ia_i}$$

and

$$t = -H(t) \prod_{i \geq 1} (1 - (-H(t))^i)^{-ib_i}.$$

This means that  $H(t)$  is the compositional inverse of the series

$$-t \prod_{i \geq 1} (1 - (-t)^i)^{-ib_i}.$$

This shows that the series  $F(t)$  can be characterized by a functional equation of the form (4) for unique  $b_i$ , and it shows the symmetry of the statement in the theorem. Thus, we only have to prove integrality of the  $a_i$  given integrality of the  $b_i$ .

As the first step towards the proof of the theorem, we will derive an explicit formula for the  $a_i$  in terms of the  $b_i$  by applying Lagrange inversion to the functional equation (4). We use the following version of Lagrange inversion.

LEMMA 4.2. Suppose that power series  $F(t), G(t) \in \mathbf{Z}[[t]]$  with  $G(0) \neq 0$  are related by  $F(t) = G(tF(t))$ . Then, for all  $k, d \in \mathbf{Z}$ , we have

$$(k + d)[t^d]F(t)^k = k[t^d]G(t)^{k+d},$$

where  $[t^d]F(t)$  denotes the  $t^d$ -coefficient of the series  $F(t)$ .

*Proof.* Apply [Sta99, Theorem 5.4.2] using the notation  $f(t) = tF(t)$  and  $d = n - k$ . □

For  $n \geq 1$ , we denote by  $L_n$  the set of sequences  $\mathbf{l} = (l_1, l_2, \dots)$  of non-negative integers such that  $\sum_i il_i = n$ , and by  $L$  the union of all  $L_n$ . For  $\mathbf{l} \in L$ , we write  $|\mathbf{l}| = n$  if  $\mathbf{l} \in L_n$ .

LEMMA 4.3. For all  $d \in \mathbf{N}$  and all  $c_i \in \mathbf{Z}$  for  $i \geq 1$ , we have

$$[t^d] \prod_{i \geq 1} (1 - t^i)^{-c_i} = \sum_{\mathbf{l} \in L_d} \prod_{i \geq 1} \binom{c_i + l_i - 1}{l_i}, \tag{5}$$

the sum ranging over all partitions  $\lambda$  of  $d$ .

*Proof.* For  $c \in \mathbf{Z}$ , we have

$$(1 - t)^{-c} = \sum_{l \geq 0} \binom{c + l - 1}{l} t^l,$$

and therefore

$$\begin{aligned} [t^d] \prod_{i \geq 1} (1 - t^i)^{-c_i} &= [t^d] \prod_{i \geq 1} \sum_{l_i \geq 0} \binom{c_i + l_i - 1}{l_i} t^{l_i} \\ &= [t^d] \sum_{l_1, l_2, \dots \geq 0} \prod_{i \geq 1} \binom{c_i + l_i - 1}{l_i} t^{\sum_i il_i} \\ &= [t^d] \sum_{\mathbf{l} \in L} \prod_{i \geq 1} \binom{c_i + l_i - 1}{l_i} t^{|\mathbf{l}|}, \end{aligned}$$

and the lemma follows. □

*Remark.* Here and in the following, we make frequent use of binomial coefficients  $\binom{a}{b}$  for  $a \in \mathbf{Z}$  using

$$\binom{-a + b - 1}{b} = (-1)^b \binom{a}{b}. \tag{6}$$

Using these preparations, we can state the desired formula relating the coefficients  $a_i$  and  $b_i$ .

PROPOSITION 4.4. With the above notation, we have, for all  $d \geq 1$ ,

$$d^2 a_d = \sum_{e|d} \text{Mo}(d/e) (-1)^e \sum_{\mathbf{l} \in L_e} (-1)^{\sum_i l_i} \prod_{i \geq 1} \binom{ib_i e}{l_i}, \tag{7}$$

where the first sum ranges over all divisors of  $d$ , and Mo denotes the number-theoretic Moebius function.

*Proof.* We apply Lemma 4.2 to the functional equation (4) using

$$G(t) = \prod_{i \geq 1} (1 - t^i)^{ib_i}$$

and get

$$(k + d)[t^d] \prod_{i \geq 1} (1 - (-t)^i)^{-ia_i k} = k[t^d] \prod_{i \geq 1} (1 - t^i)^{ib_i(k+d)}. \tag{8}$$

Lemma 4.3 allows us to write the left-hand side of (8) as

$$(k + d)(-1)^d \sum_{\mathbf{l} \in L_d} \prod_{i \geq 1} \binom{ia_i k + l_i - 1}{l_i},$$

and the right-hand side of (8) as

$$k \sum_{\mathbf{l} \in L_d} \prod_{i \geq 1} \binom{-ib_i(k + d) + l_i - 1}{l_i}.$$

We use (6) and substitute  $k$  by  $X$  to rewrite (8) as

$$X \sum_{\mathbf{l} \in L_d} (-1)^{\sum_i l_i} \prod_{i \geq 1} \binom{ib_i(X + d)}{l_i} = (-1)^d (X + d) \sum_{\mathbf{l} \in L_d} \prod_{i \geq 1} \binom{ia_i X + l_i - 1}{l_i}. \tag{9}$$

Both sides behaving polynomially in  $X$ , equality for all  $X \in \mathbf{Z}$  thus implies equality of the polynomials. We want to compare the linear  $X$ -terms (the constant terms being 0) of both sides. Note the following property.

The polynomial  $\binom{aX+b+c-1}{c}$  has constant  $X$ -coefficient  $\binom{b+c-1}{c}$ , and the polynomial  $\binom{aX+c-1}{c}$  has linear  $X$ -coefficient  $a/c$ .

Applying this, we see that the left-hand side of (9) has linear  $X$ -coefficient

$$\sum_{\mathbf{l} \in L_d} (-1)^{\sum_i l_i} \prod_{i \geq 1} \binom{ib_i d}{l_i}.$$

To analyze the linear  $X$ -coefficient of the right-hand side of (9), note first that the constant  $X$ -coefficient of each product

$$\prod_{i \geq 1} \binom{ia_i X + l_i - 1}{l_i} \tag{10}$$

equals zero. Its linear  $X$ -term is non-zero only if exactly one factor appears, that is, if there is only one non-zero entry  $l_i$  in  $\mathbf{l}$ . In this case, we have  $l_i = d/i$  and  $i$  is a divisor of  $d$ . Thus, the product (10) reduces to

$$\binom{ia_i X + d/i - 1}{d/i},$$

having linear  $X$ -coefficient  $(ia_i)/(d/i) = i^2 a_i/d$  by the above. We conclude that the linear  $X$ -coefficient of the right-hand side of (9) equals

$$(-1)^d \sum_{i|d} i^2 a_i.$$

Comparison of both linear  $X$ -coefficients thus yields

$$\sum_{i|d} i^2 a_i = (-1)^d \sum_{\mathbf{l} \in L_d} (-1)^{\sum_i l_i} \prod_{i \geq 1} \binom{ib_i d}{l_i}.$$

After Moebius inversion, we arrive at the claimed formula (7). □

To prove integrality of the  $a_d$  given integrality of all  $b_i$ , we thus have to prove that the right-hand side of (7) is divisible by  $d^2$ . This can be tested on the prime divisors of  $d$ . Denoting by

$$m(d) = m_p(d) = \max\{m : p^m | d\}$$

the multiplicity of a prime  $p$  as a divisor of  $d$ , we thus have to prove divisibility by  $p^{2m_p(d)}$  of the right-hand side of (7) for all primes  $p$ . We prepare this proof by stating certain divisibility/congruence properties of binomial coefficients.

LEMMA 4.5. *Let  $p$  be a prime. For  $a, b \in \mathbf{Z}$  and  $b \geq 0$ , we have*

$$p^{\max(m_p(a)-m_p(b),0)} \mid \binom{a}{b}.$$

*Proof.* By a result of Kummer (see, for example, [Gra97]), the exact power of  $p$  dividing  $\binom{a}{b}$  equals the number of ‘carries’ when subtracting  $b$  from  $a$  in base  $p$ , at least when  $a \geq 0$ . This can be generalized to  $a \in \mathbf{Z}$  using

$$\binom{-a}{b} = (-1)^b \frac{a}{a+b} \binom{a+b}{b}. \tag{11}$$

The lemma follows. □

LEMMA 4.6. *Let  $p$  be a prime, and define  $\mu_p = 0, 1, 2$  provided that  $p = 2, p = 3, p \geq 5$ , respectively. Assume that  $p|a, b$  for integers  $a, b$  with  $b \geq 0$ . Define  $\eta$  as  $-1$  if  $p = 2$  and  $b \equiv 2 \equiv a - b \pmod 4$ , and as  $1$  otherwise. Then*

$$\binom{a}{b} \equiv \eta \binom{a/p}{b/p} \pmod{p^r}$$

for

$$r \leq m_p(a) + m_p(b) + m_p(a - b) + m_p\left(\binom{a/p}{b/p}\right) - \mu_p.$$

In the case  $p = 2$ , we also have

$$\binom{a}{b} \equiv \binom{a/2}{b/2} \pmod 4.$$

*Proof.* The general statement (usually [Ges83, Gra97] attributed to Jacobsthal and co-workers [BSFTALJ52]) is proved in [Ges83, Theorem 2.2], with the assumption  $a \geq 0$  there removed by (11). For the congruence modulo 4, we calculate as in the proof of [Ges83, Theorem 2.2]

$$\begin{aligned} \binom{a}{b} &= \binom{a/2}{b/2} \prod_{i=1}^b (1 + 2(a - b)/i) \equiv \binom{a/2}{b/2} \left(1 + 2(a - b) \sum_{i=1}^b \frac{1}{2^i}\right) \\ &\equiv \binom{a/2}{b/2} (1 + (a - b)(b/2)^2) \pmod 4. \end{aligned}$$

The term  $(a - b)(b/2)^2$  is congruent to  $1 \pmod 4$  except when  $b/2$  is odd and  $a/2$  is even, in which case it is congruent to  $-1 \pmod 4$ . But, in this case,  $\binom{a/2}{b/2}$  is even by Lemma 4.5. □

From the previous two lemmas, we derive divisibility/congruence properties of the product of binomial coefficients appearing in (7).

LEMMA 4.7. *Let  $p$  be a prime dividing  $e \geq 0$ . If  $\mathbf{l} \in Q_e$  is not divisible by  $p$  (that is, some entry  $l_i$  is not divisible by  $p$ ), we have*

$$p^{2m(e)} \left| \prod_{i \geq 1} \binom{ib_i e}{l_i} \right|$$

*Proof.* To shorten notation, we write  $m = m_p(e)$ . Lemma 4.5 yields

$$p^{\max(m+m(i)-m(l_i), 0)} \left| \binom{ib_i e}{l_i} \right|;$$

we thus have to prove that

$$\sum_{i:l_i \neq 0} \max(m + m(i) - m(l_i), 0) \geq 2m \tag{12}$$

provided that some  $l_i \neq 0$  is not divisible by  $p$ . Let  $i_0$  be an index such that  $m(l_{i_0}) = 0$ .

Let  $m_0$  be the minimum over all  $m(i) + m(l_i)$ . Since  $e = \sum_i il_i$ , we can distinguish two cases: either  $m_0 = m$  (case 1), or  $m_0 < m$  and the minimum is obtained at least twice (case 2). For case 1 we have, in particular,  $m(i_0) \geq m$ ; thus,

$$\max(m + m(i_0) - m(l_{i_0}), 0) \geq 2m,$$

and (12) follows.

For case 2, let  $i_1, i_2$  be two different indices where the minimum  $m_0$  is obtained. For  $s = 1, 2$ , we have  $m + m(i_s) - m(l_{i_s}) \geq 0$ , since, otherwise,

$$m > m_0 = m(l_{i_s}) + m(i_s) \geq m(l_{i_s}) > m + m(i_s),$$

a contradiction. Again we distinguish two cases: first, assume that  $i_0$  coincides with, say,  $i_1$ . Then we can estimate

$$\begin{aligned} \sum_{i:l_i \neq 0} \max(m + m(i) - m(l_i), 0) &\geq \max(m + m(i_0) - m(l_{i_0}), 0) + \max(m + m(i_2) - m(l_{i_2}), 0) \\ &= 2m + m_0 + m(i_2) - m(l_{i_2}) = 2m + 2m(i_2) \geq 2m, \end{aligned}$$

and (12) follows. Second, assume that  $i_0$  differs from  $i_1, i_2$ . As in the previous case, we can estimate

$$\begin{aligned} \sum_{i:l_i \neq 0} \max(m + m(i) - m(l_i), 0) &\geq 3m + m(i_0) + m(i_1) + m(i_2) - m(l_{i_1}) - m(l_{i_2}) \\ &\geq 2m + m - m_0 + 2m(i_1) + 2m(i_2) \geq 2m, \end{aligned}$$

and (12) follows again. □

LEMMA 4.8. *Let  $p$  be a prime dividing  $e \geq 0$ . If  $\mathbf{l} = p\mathbf{l}'$  is an element of  $L_e$  divisible by  $p$  (that is, each  $l_i = pl'_i$  is divisible by  $p$ ), then*

$$\prod_{i \geq 1} \binom{ib_i e}{l_i} \equiv (-1)^{(p-1)(e/p + \sum_i l'_i)} \prod_{i \geq 1} \binom{ib_i e/p}{l'_i} \pmod{p^{2m_p(e)}}. \tag{13}$$

*Proof.* So, assume that  $\mathbf{l} = p\mathbf{l}'$ , and denote again  $m = m(e)$ . Applying the general congruence of Lemma 4.6 to a non-trivial (that is,  $l_i \neq 0$ ) factor of the left-hand side of (13), we get

$$\binom{ib_i e}{l_i} \equiv \eta_i \binom{ib_i e/p}{l_i/p} \pmod{p^{r_i}},$$



where the sign  $\eta_i$  is  $-1$  only in the case  $p = 2$ ,  $l_i/2$  odd,  $ib_i e/2 - l_i/2$  odd, and

$$\begin{aligned} r_i &= m(ib_i e) + m(l_i) + m(ib_i e - l_i) + m\binom{ib_i e/p}{l_i/p} - \mu_p \\ &\geq m(e) + m(i) + m(l_i) + \min(m(e) + m(i), m(l_i)) \\ &\quad + \max(m(e) + m(i) - m(l_i), 0) - \mu_p \\ &= 2m + 2m(i) + m(l_i) - \mu_p. \end{aligned} \tag{14}$$

Suppose first that  $p \geq 3$ . Then  $r_i \geq 2m$  using  $m(l_i) \geq 1$  and  $\mu_p \leq 1$ . The sign in (13) vanishes due to the even factor  $p - 1$ , and  $\eta_i = 1$ . The congruence (13) follows.

Next, assume that  $p = 2$  and  $m \geq 2$ . Then the estimate (14) only assures congruence of the binomial coefficients mod  $2^{2m-1}$  in the case  $i$  and  $l_i/2$  are odd; thus,

$$\binom{ib_i e}{l_i} \equiv \eta_i \binom{ib_i e/2}{l_i/2} + \varepsilon_i \pmod{2^{2m}},$$

where  $\varepsilon_i \in \{0, 2^{2m-1}\}$ , non-triviality only being possible if  $i$  and  $l_i/2$  are odd. Then

$$\begin{aligned} \prod_{i \geq 1} \binom{ib_i e}{l_i} &\equiv \prod_{i \geq 1} \left( \eta_i \binom{ib_i e}{l_i/2} + \varepsilon_i \right) \\ &\equiv \prod_{i \geq 1} \eta_i \binom{ib_i e/2}{l_i/2} + \sum_{i \geq 1} \varepsilon_i \prod_{j \neq i} \eta_j \binom{jb_j e}{l_j/2} \pmod{2^{2m}}, \end{aligned} \tag{15}$$

since all multiple products of the  $\varepsilon_i$  vanish mod  $2^{2m}$ . For the same reason, we only have to consider summands in (15) for which  $\varepsilon_i \neq 0$  and each factor

$$\binom{jb_j e/2}{l_j/2}$$

is odd. Since  $m \geq 2$ , this can only happen (using Lemma 4.5) in the case  $m(l_j) \geq m(j) + m$  for all  $j \neq i$  such that  $l_j \neq 0$ . But then

$$2^m | e - \sum_{j \neq i: l_j \neq 0} jl_j = il_i,$$

a contradiction to the assumptions  $m(il_i) = 1$  (by  $\varepsilon_i \neq 0$ ) and  $m \geq 2$ . Thus, we have proved that

$$\prod_{i \geq 1} \binom{ib_i e}{l_i} \equiv \prod_{i \geq 1} \eta_i \cdot \prod_{i \geq 1} \binom{ib_i e/2}{l_i/2} \pmod{2^{2m}},$$

and we have to compare the sign  $\prod_i \eta_i = (-1)^u$  to the sign of (13). Using  $m \geq 2$ , we have

$$\begin{aligned} u &= |\{i \geq 1 : l_i/2 \text{ odd, } ib_i e/2 - l_i/2 \text{ odd}\}| \\ &= |\{i \geq 1 : l_i/2 \text{ odd}\}|. \end{aligned}$$

The sign in (13) equals

$$(-1)^{e/2 + \sum_i l_i/2},$$

and we are done.

Finally, consider the case  $p = 2$  and  $m = 1$ . Then the statement on congruences mod 4 of Lemma 4.6 yields

$$\prod_{i \geq 1} \binom{ib_i e}{l_i} \equiv \prod_{i \geq 1} \binom{ib_i e/2}{l_i/2} \pmod{4},$$

and again we only have to consider the sign. The sign in (13) equals

$$(-1)^{1+\sum_i l_i/2}.$$

We have  $e/2 = \sum_i il_i/2$ ; thus, the sum  $\sum_{2|i} l_i/2$  is odd. Suppose that  $\sum_i l_i/2$  is even (the only case in which the sign of (13) potentially differs from 1). Then  $\sum_{2\nmid i} l_i/2$  is odd. Thus, there exists an even index  $i$  with  $l_i/2$  odd. In this case, the binomial coefficient

$$\binom{ib_i e/2}{l_i/2}$$

is even, and the sign is irrelevant mod 4. □

With these preparations, we can finish the following.

*Proof of Theorem 4.1.* Assume that  $p$  is a prime such that  $m = m(d) = m_p(d) \geq 1$ . The divisors  $e$  of  $d$  for which  $\text{Mo}(d/e)$  is non-zero fulfill  $m(e) = m(d)$  or  $m(e) = m(d) - 1$ , that is, they are of the form  $e$  or  $e/p$  for a divisor  $e$  of  $d$  such that  $m(e) = m(d)$ . We can thus split the right-hand side of (7) into the following difference:

$$\begin{aligned} & \sum_{e|d:m(e)=m(d)} \text{Mo}(d/e)(-1)^e \sum_{\mathbf{l} \in L_e} (-1)^{\sum_i l_i} \prod_{i \geq 1} \binom{ib_i e}{l_i} \\ & - \sum_{e|d:m(e)=m(d)} \text{Mo}(d/e)(-1)^{e/p} \sum_{\mathbf{l}' \in L_{e/p}} (-1)^{\sum_i l'_i} \prod_{i \geq 1} \binom{ib_i e/p}{l'_i}. \end{aligned} \tag{16}$$

Now consider a summand of the first sum of (16) corresponding to a sequence  $\mathbf{l} \in L_e$ . If  $\mathbf{l}$  is not divisible by  $p$ , then Lemma 4.7 shows that the summand is divisible by  $p^{2m(e)} = p^{2m(d)}$ . If  $\mathbf{l} = p\mathbf{l}'$  is divisible by  $p$ , then Lemma 4.8 shows that the summand is congruent mod  $p^{2m(d)}$  to the summand of the second sum of (16) corresponding to the sequence  $\mathbf{l}'$ . In other words, the difference of the two sums in (16) vanishes mod  $p^{2m(d)}$ , proving the theorem. □

For the application to the integrality of certain Donaldson–Thomas type invariants in the following section, we need a slight generalization of Theorem 4.1. We treat this case separately, although a second inspection of the proofs leading to Theorem 4.1 is necessary to avoid additional complications in the notation used so far.

**THEOREM 4.9.** *Let  $F(t) \in \mathbf{Z}[[t]]$  be a power series with  $F(0) = 1$ . For  $N \in \mathbf{Z}$ , write*

$$F(t) = \prod_{i \geq 1} (1 - ((-1)^N t)^i)^{-ia_i}$$

for  $a_i \in \mathbf{Q}$ . We can characterize  $F(t)$  as the solution to a functional equation of the form

$$F(t) = \prod_{i \geq 1} (1 - (tF(t)^N)^i)^{ib_i}$$

for unique  $b_i \in \mathbf{Q}$ . Under these assumptions, we have  $b_i \in \mathbf{Z}$  for all  $i \geq 1$  if and only if  $a_i \in \mathbf{Z}$  for all  $i \geq 1$ .

*Proof.* The argument used in the remark following Theorem 4.1, using the power series  $H(t) = t(-F(t))^N$ , shows existence and uniqueness of the  $b_i$ , as well as the symmetry of the statement of Theorem 4.9. Application of Proposition 4.4 to  $G(t) = F(t)^N$  yields the following explicit formula

for all  $d \geq 1$ :

$$d^2 a_d = \frac{1}{N} \sum_{e|d} \text{Mo}(d/e) (-1)^{Ne} \sum_{\mathbf{l} \in L_e} (-1)^{\sum_i l_i} \prod_{i \geq 1} \binom{N i b_i e}{l_i}. \tag{17}$$

Now any summand of (17) is divisible by  $N$ ; thus, the denominator  $N$  in (17) cancels. Next, note that none of our arguments (Lemmas 4.7 and 4.8) for the proof of Theorem 4.1 uses any divisibility properties of the  $b_i$ ; thus, these arguments are valid when replacing  $b_i$  by  $N b_i$ , yielding an additional divisibility by  $N$ .

The only additional difficulty is the sign in the statement of Lemma 4.8, which now reads

$$(-1)^{(p-1)(Ne/p + \sum_i l'_i)}.$$

Repeating the sign considerations in the proof of Lemma 4.8, we see that we can concentrate on the case  $p = 2$  and  $m(e) = 1$ , where the sign now reads

$$(-1)^{N + \sum_i l_i/2}.$$

The argument of the proof of Lemma 4.8 is still valid in the case  $N$  is odd. On the other hand, if  $N$  is even, we can choose an index  $i$  such that  $l_i/2$  is odd, and Lemma 4.5 shows that the binomial coefficient

$$\binom{N i b_i e/2}{l_i/2}$$

is even, the sign thus being again irrelevant mod 4. □

*Example.* We consider the example  $b_i = 0$  for all  $i \geq 2$  and denote  $b = b_1$ . Then  $F(t)$  is the solution to the functional equation

$$F(t) = (1 - tF(t)^N)^b,$$

and we want to factor  $F(t)$  as

$$F(t) = \prod_{i \geq 1} (1 - ((-1)^N t)^i)^{-i a_i}.$$

The formula (17) gives

$$a_d = \frac{1}{N d^2} \sum_{e|d} \text{Mo}(d/e) (-1)^{(N+1)e} \binom{N b e}{e}.$$

In particular, we have  $a_1 = (-1)^{N+1} b$  and

$$a_2 = \frac{1}{4} b (2 N b - 1 + (-1)^N),$$

and we see that the choice of signs is essential for the integrality of the  $a_d$  given by Theorem 4.9.

The particular case  $N = 1, b = -1$  gives a factorization (3) for the generating function

$$F(t) = \frac{1 - \sqrt{1 - 4t}}{2t}$$

of Catalan numbers with

$$a_d = \frac{1}{d^2} \sum_{e|d} (-1)^e \text{Mo}(d/e) \binom{2e - 1}{e},$$

which is (up to signs) sequence A131868 in [Slo].

More generally, in the case  $N = m - 1 \geq 1$ ,  $b = -1$ , the series  $F(t)$  is the generating series of Euler characteristics of Hilbert schemes for the quiver with a single vertex and  $m$  loops by Proposition 2.3.

5. Integrality of Donaldson–Thomas invariants

We specialize the results of §3 to the generalized Kronecker quiver  $K_m$  with set of vertices  $I = \{i, j\}$  and  $m$  arrows from  $j$  to  $i$ . Choose the generators  $x = -x_i$  and  $y = -x_j$  of  $B$ ; then  $B_m = \mathbf{Q}[[x, y]]$  with Poisson bracket  $\{x, y\} = mxy$ . For  $a, b \in \mathbf{Z}$  with  $a, b \geq 0$  and  $a + b \geq 1$ , we define a Poisson automorphism  $T_{a,b}^{(m)}$  of  $B$  by

$$T_{a,b}^{(m)} : \begin{cases} x \mapsto x(1 - (-1)^{mab}x^ay^b)^{-mb}, \\ y \mapsto y(1 - (-1)^{mab}x^ay^b)^{ma}, \end{cases}$$

as in [KS08, 1.4]. More generally, for an arbitrary series  $F(t) \in \mathbf{Z}[[t]]$  with  $F(0) = 1$ , we define as in [GPS10, 0.1]

$$T_{a,b,F(t)}^{(m)} : \begin{cases} x \mapsto xF(x^ay^b)^{-mb}, \\ y \mapsto yF(x^ay^b)^{ma}. \end{cases}$$

Note that the automorphisms  $T_{a,b}^{(m)}$  for fixed slope  $a/b$  commute; thus,

$$\prod_{i \geq 1} (T_{ia,ib}^{(m)})^{d_i} = T_{a,b,F(t)}^{(m)} \tag{18}$$

for

$$F(t) = \prod_{i \geq 1} (1 - ((-1)^{mab}t)^i)^{id_i}.$$

We can now use our main results (Theorems 4.9 and 3.6) to confirm [KS08, Conjecture 1].

THEOREM 5.1. *Writing*

$$T_{1,0}^{(m)}T_{0,1}^{(m)} = \prod_{b/a \text{ decreasing}}^{\leftarrow} (T_{a,b}^{(m)})^{d(a,b,m)},$$

we have  $d(a, b, m) \in \mathbf{Z}$  for all  $a, b, m$ .

*Remark.* In the setup of [KS08], the integers  $d(a, b, m)$  play the role of *wall-crossing Donaldson–Thomas type invariants*, that is, they describe the behavior of the Donaldson–Thomas type invariants of [KS08] under a change of stability. The above theorem therefore proves the *relative integrality* of these invariants, that is, preservation of integrality under change of stability.

*Proof.* We choose the stability  $\Theta = j^*$  (in fact, the only non-trivial stability; see [Rei08, 5.1]). By Theorem 3.6, we have a factorization

$$T_{1,0}^{(m)}T_{0,1}^{(m)} = T_iT_j = \prod_{\mu \in \mathbf{Q}}^{\leftarrow} T_\mu, \tag{19}$$

where

$$T_\mu(x^d) = x^d \cdot Q_\mu^{\{\cdot, d\}}(x).$$

Given  $\mu \in \mathbf{Q}$ , we write  $\mu = b/(a + b)$  for coprime non-negative  $a, b \in \mathbf{Z}$  and choose integers  $c$  and  $d$  such that  $ac + bd = 1$ . We have thus  $\Lambda_\mu^+ = \mathbf{N}t^f$  for the dimension vector  $f$  with components  $f_i = a$

and  $f_j = b$ , and  $n \cdot f = 1$  for  $n$  with components  $n_i = c$  and  $n_j = d$ . Defining

$$G_\mu(t) = Q_\mu^{(i)}(t)^c Q_\mu^{(j)}(t)^d \in \mathbf{Z}[[\Lambda_\mu^+]],$$

the proof of [Rei10, Theorem 6.1] shows that

$$G_\mu(t)^a = Q_\mu^{(i)}(t) \quad \text{and} \quad G_\mu(t)^b = Q_\mu^{(j)}(t).$$

Similarly to Corollary 3.5, we can find a functional equation for  $G_\mu(t)$ . We denote  $\chi_\mu(k) = \chi(M_{(ka, kb)}^{\Theta\text{-st}}(K_m))$  for  $k \geq 1$  and  $N = -\langle(a, b), (a, b)\rangle = mab - a^2 - b^2$  and apply the first formula of Theorem 3.4:

$$\begin{aligned} G_\mu(t) &= Q_\mu^{(i)}(t)^c Q_\mu^{(j)}(t)^d = Q_\mu^{(n)}(t) \\ &= \prod_{k \geq 1} R^{[kf]}(t)^{\chi_\mu(k) \cdot kn \cdot d} \\ &= \prod_{k \geq 1} R^{[kf]}(t)^{k\chi_\mu(k)}. \end{aligned}$$

Applying the second formula of Theorem 3.4, this yields

$$\begin{aligned} G_\mu(t) &= \prod_{k \geq 1} \left( 1 - t^{kf} \prod_{l \geq 1} R^{[lf]}(t)^{klN\chi_\mu(l)} \right)^{-l\chi_\mu(k)} \\ &= \prod_{l \geq 1} (1 - t^{kf} G_\mu(t)^{kN})^{-k\chi_\mu(k)}. \end{aligned}$$

Thus, the series  $G_\mu(t)$  fulfills the functional equation

$$G_\mu(t) = \prod_{k \geq 1} (1 - (t^f G_\mu(t)^N)^k)^{-k\chi_\mu(k)}. \tag{20}$$

By Theorem 4.9,  $G_\mu(t)$  admits a factorization

$$G_\mu(t) = \prod_{k \geq 1} (1 - ((-1)^N t^f)^k)^{kd_\mu(k)} \tag{21}$$

for  $d_\mu(k) \in \mathbf{Z}$  for all  $k \geq 1$ .

Defining  $F_\mu(t) \in \mathbf{Z}[[t]]$  by  $F_\mu((-1)^{a+b} t^f) = G_\mu(t)$ , we have

$$T_\mu = T_{a,b,F_\mu(t)}^{(m)} \tag{22}$$

(the sign appearing due to the convention  $x = -x_i, y = -x_j$ ) and

$$\begin{aligned} F_\mu(t) &= \prod_{k \geq 1} (1 - ((-1)^{N+a+b} t)^k)^{kd_\mu(k)} \\ &= \prod_{k \geq 1} (1 - ((-1)^{mab} t)^k)^{kd_\mu(k)}. \end{aligned}$$

By (18) and (22), this yields

$$T_\mu = \prod_{k \geq 1} (T_{ka, kb}^{(m)})^{d_\mu(k)}.$$

Together with the factorization (19), this yields the factorization claimed in the theorem, with  $d(ka, kb, m) = d_\mu(k)$ . □

Using a result of Weist, we can also confirm a conjecture in [KS08, 1.4] concerning the diagonal term of the factorization in Theorem 5.1.

THEOREM 5.2. *For all  $k \geq 1$ , we have*

$$d(k, k, m) = \frac{1}{(m-2)k^2} \sum_{i|k} \text{Mo}(k/i)(-1)^{mi+1} \binom{(m-1)^2i-1}{i}.$$

*Proof.* By [Wei09, 6.2], we have  $\chi(M_{d,d}^{\Theta\text{-st}}(K_m)) = 0$  for  $d \geq 2$ , whereas the moduli space  $M_{1,1}^{\Theta\text{-st}}(K_m)$  is isomorphic to  $\mathbf{P}^{m-1}$ . In the notation of (20) and (21) above, we can apply the example at the end of the previous section with  $b = -m$  and  $N = m - 2$  and arrive at the claimed formula.  $\square$

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