SELF-POLAR DOUBLE-N's DEFINED BY CERTAIN PAIRS OF NORMAL RATIONAL CURVES

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Introduction

This paper is a sequel to T. G. Room's "Self-polar double configurations in projective geometry, I and II" ([2]). I would like to thank Professor Room for supervising and inspiring my work, and to acknowledge the financial assistance of the C.S.I.R.O.

In part I of his paper, Room gives a (sufficient) condition for the self-polarity of the double $-N$ determined by a $p \times q$ matrix of linear forms in the (homogeneous) coordinate variables of a projective space Π_{p+q-3}(over the field of complex numbers), namely that, of the 2×2 minors of the matrix, no $\frac{1}{2}(p+q)(p+q-3)+1$ are linearly independent.

In part II, Room shows how Coble's self-polar double $-\binom{q+1}{2}$ of lines and Π_{q-2}'s in Π_q (cf. [3]) may be constructed if we are given a n.r.c. (normal rational curve) r, a quadric S inpolar to r, and a pair of lines conjugate with respect to S and chordal to r. S polarizes the constructed double $-\binom{q+1}{2}$.

In the present paper, we define a class of self-polar double $-N$'s of Π_{p-2}'s and Π_{q-2}'s associated with pairs of very specially related ("$\mathscr{S}$-related") n.r.c.'s in Π_{p+q-3}. The matrices defining these double $-N$'s satisfy Room's criterion for self-polarity.

In §1 we define and discuss "$\mathscr{S}$-related" pairs of n.r.c.'s. In §2 we examine the case $p = q = 4$, and in §3 indicate how the results obtained in §2 may be generalized, and find the freedom (except when $p = q = 3$) of the locus with which our configuration is associated.

Finally, in §4, we prove that our class of self-polar double $-N$'s, like Coble's class of self-polar double $-\binom{q+1}{2}$'s, includes the general double-six of lines in Π_3.

1. Pairs of $\mathscr{S}$-related normal rational curves in Π_n

We say that two n.r.c.'s in Π_n have "contact of the highest order" at P if they both pass through P and they have the same tangent line, osculating $\Pi_2, \cdots$, osculating Π_{n-1} at P.

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A pair of n.r.c.'s in Π_n which have contact of the highest order at some point P and intersect at n further points $P_1, \cdots, P_n$ whose join is a prime π not incident with P, shall be called an "$\mathscr{S}$-related" pair. $P, P_1, \cdots, P_n$ shall be called the $\mathscr{S}$-simplex[1], P the $\mathscr{S}$-point, and π the $\mathscr{S}$-prime, of the pair.

It is easily verified that the two n.r.c.'s

$$(1) \qquad \left\| \begin{matrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \end{matrix} \right\|_1 = 0$$

and

$$(2) \qquad \left\| \begin{matrix} x_0 x_1 & \cdots & x_{n-2} & x_{n-1} \\ x_1 x_2 & \cdots & x_{n-1} & b_\delta x_\delta \end{matrix} \right\|_1 = 0$$

(b_δ constants, $\delta = 0, \cdots, n$; $b_n \neq 0$ or 1) are an $\mathscr{S}$-related pair: their $\mathscr{S}$-point is A_n[2], and their $\mathscr{S}$-prime is given by

$$b_\delta x_\delta = x_n.$$

THEOREM 1. *A coordinate-system and a set of constants b_δ can be found such that a given pair r, ρ of $\mathscr{S}$-related n.r.c.'s is represented by equations* (1) *and* (2).

PROOF. Select as A_n the $\mathscr{S}$-point, and as A_0 and the unit point any other two (distinct) points on r. Then points $A_1, \cdots, A_{n-1}$ are uniquely determined by the condition that equations (1) represent r.

Constants $a_{\alpha\delta}$ ($\alpha, \delta = 0, \cdots, n$) can be found such that the equations

$$\left\| \begin{matrix} y_0 y_1 & \cdots & y_{n-1} \\ y_1 y_2 & \cdots & y_n \end{matrix} \right\|_1 = 0,$$

where $y_\alpha \equiv a_{\alpha\delta} x_\delta$, represent ρ. The Π_{s-1} ($s = 1, \cdots, n$) which osculates r at the $\mathscr{S}$-point may now be represented by either of the two sets of equations

$$x_0 = \cdots = x_{n-s} = 0 \text{ or } y_0 = \cdots = y_{n-s} = 0.$$

Thus

$$a_{\beta\gamma} = 0 \text{ and } a_{\delta\delta} \neq 0 \begin{cases} \beta = 0, \cdots, n-1 \\ \gamma = \beta+1, \cdots, n \\ \delta = 0, \cdots, n \end{cases},$$

so that the equations of ρ may be reduced to the form

$$\left\| \begin{matrix} b_{00} x_0 & b_{11} x_1 & \cdots & b_{n-1,\, n-1} x_{n-1} \\ z_1 & z_2 & \cdots & z_n \end{matrix} \right\|_1 = 0,$$

[1] For convenience, we shall use the term $\mathscr{S}$-simplex even though the $n+1$ points need not be distinct.

[2] The vertices of the simplex of reference shall be denoted by A_δ ($\delta = 0, \cdots, n$).

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where $z_\alpha \equiv b_{\alpha\gamma} x_\gamma \quad \begin{cases} \alpha = 1, \cdots, n \\ \gamma = 0, \cdots, \alpha \end{cases}$

and $b_{10} = 0.$

Substitute θ^δ for x_δ in these equations. Each of the resulting equations which is of degree less than n in θ must be an identity in θ, since r and ρ intersect at n points, apart from A_n (which is given by $1/\theta = 0$). It follows from these identities that

$$b_{\varepsilon\phi} = 0 \qquad \begin{cases} \varepsilon = 2, \cdots, n-1 \\ \phi = 0, \cdots, \varepsilon-1 \end{cases}$$

and that

$$\frac{b_{00}}{b_{11}} = \frac{b_{11}}{b_{22}} = \cdots = \frac{b_{n-2,\,n-2}}{b_{n-1,\,n-1}},$$

so that equations (2), when $b_\delta = (b_{00}/b_{n-1,\,n-1})\, b_{n\delta}$ $(\delta = 0, \cdots, n)$, represent ρ relative to the chosen coordinate-system.

Such a system of coordinates shall be called an $\mathscr{S}$-system for r, ρ; and the constants b_δ determined by an $\mathscr{S}$-system shall be called a set of $\mathscr{S}$-constants for r, ρ.

We now find the freedom of $\mathscr{S}$-related pairs in Π_n and establish some important relations between $\mathscr{S}$-related pairs and tangential quadrics in Π_n.

THEOREM 2. *The freedom of $\mathscr{S}$-related pairs in Π_n is n^2+3n-1.*

PROOF. The freedom of n.r.c.'s in Π_n is $(n-1)(n+3)$ (cf. [1], p. 220). The freedom of simplexes inscribed in a given n.r.c. is $n+1$.

Given a simplex inscribed in a n.r.c., r, let ρ and σ be any two n.r.c.'s $\mathscr{S}$-related to r such that the given simplex is the $\mathscr{S}$-simplex of the pairs r, ρ and r, σ and these two pairs have the same $\mathscr{S}$-point. Then, if b_δ are the $\mathscr{S}$-constants of r, ρ and c_δ the $\mathscr{S}$-constants of r, σ in an $\mathscr{S}$-system for r, ρ (and r, σ), the two equations

$$b_\delta x_\delta = x_n \text{ and } c_\delta x_\delta = x_n$$

represent the same prime.

The freedom of $\mathscr{S}$-related pairs in Π_n is therefore

$$(n-1)(n+3)+(n+1)+1 = n^2+3n-1.$$

Until §4, r and ρ shall always be understood to be the members of a given $\mathscr{S}$-related pair, P its $\mathscr{S}$-point, π its $\mathscr{S}$-prime; and the coordinates shall be an $\mathscr{S}$-system, the constants b_δ a set of $\mathscr{S}$-constants for r, ρ, and b'_δ the same constants except that $b'_n = b_n - 1$.

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THEOREM 3. *The $\mathscr{S}$-simplex of r, ρ is self-polar*[3] *with respect to a (tangential) quadric S if and only if S is inpolar to r, ρ and the (point) quadric $b'_\delta x_\delta x_{n-1} = 0$.*

PROOF. If the $\mathscr{S}$-simplex is self-polar with respect to a tangential quadric then the latter is inpolar to r and ρ since the $\mathscr{S}$-simplex is inscribed in both.

So let S be any quadric inpolar to both r and ρ. Then S is given by a matrix $K = (k_{\rho+\sigma-2})$ where

$$b'_\delta k_{\delta+\alpha} = 0 \qquad \begin{cases} \alpha = 0, \cdots, n-2. \\ \delta = 0, \cdots, n. \end{cases}$$

S will be inpolar to $b'_\delta x_\delta x_{n-1} = 0$ if and only if $b'_\delta k_{\delta+n-1} = 0$.

But the $\mathscr{S}$-simplex will be self-polar with respect to S if and only if P is the pole of π (cf. [1], pp. 225-228), i.e. each of the primes $x_\gamma = 0$ ($\gamma = 0, \cdots, n-1$) is conjugate to π, i.e.

$$b'_\delta k_{\delta+\gamma} = 0 \qquad \gamma = 0, \cdots, n-1.$$

A Π_m $(0 < m < n-1)$, chordal to r, whose polar space with respect to a quadric S is a chordal Π_{n-m-1} of ρ, shall be called a space $\mathfrak{H}_m$ (relative to S).

THEOREM 4. *If S is inpolar to both r and ρ then there exists a space $\mathfrak{H}_m$, such that neither $\mathfrak{H}_m$ nor its polar space is incident with P, if and only if the $\mathscr{S}$-simplex is self-polar. If the $\mathscr{S}$-simplex is self-polar then there are at least ∞^1 spaces $\mathfrak{H}_m$.*

PROOF. S is given by a matrix $K = (k_{\rho+\sigma-2})$ where

$$b'_\delta k_{\delta+\gamma} = 0 \qquad \begin{cases} \gamma = 0, \cdots, n-2 \\ \delta = 0, \cdots, n. \end{cases} \tag{3}$$

A Π_m, chordal to r, is given by equations

$$\lambda_\alpha x_{\alpha+\varepsilon} = 0 \qquad \begin{cases} \alpha = 0, \cdots, m+1 \\ \varepsilon = 0, \cdots, n-m-1, \end{cases}$$

while a Π_{n-m-1}, chordal to ρ, is given by equations

$$\mu_\beta x'_{\beta+\phi} = 0 \qquad \begin{cases} \beta = 0, \cdots, n-m \\ \phi = 0, \cdots, m \end{cases}$$

(where $x'_\nu \equiv x_\nu$, $\nu \neq n$; $x'_n \equiv b_\delta x_\delta$).[4]

[3] Here, and in similar situations, we assume that the vertices of the $\mathscr{S}$-simplex are distinct.

[4] x'_δ is used with this meaning throughout this paper.

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These spaces are polars if and only if each prime $\lambda_\alpha x_{\alpha+\varepsilon} = 0$ is conjugate to each prime $\mu_\beta x'_{\beta+\phi} = 0$, i.e.

$$(4) \qquad \lambda_\alpha k_{\alpha+\beta+\tau}\mu_\beta = 0 \qquad \tau = 0, \cdots, n-2$$

and

$$(5) \qquad \lambda_\alpha k_{\alpha+\beta+\sigma}\mu_\beta + \lambda_\alpha b'_\delta k_{\delta+\alpha+\sigma-m}\mu_{n-m} = 0 \qquad \sigma = m, \cdots, n-1.$$

It is easily shown that (3), (4) and (5) are together equivalent to (3),

$$(6) \qquad \lambda_\alpha k_{\alpha+\beta+\chi}\mu_\beta + \lambda_\alpha b'_\delta k_{\delta+\alpha+\chi-m}\mu_{n-m} = 0 \qquad \chi = 0, \cdots, n-1$$

and

$$(7) \qquad \lambda_{m+1} b'_\delta k_{\delta+n-1}\mu_{n-m} = 0.$$

(7) reduces to $b'_\delta k_{\delta+n-1} = 0$ if $\lambda_{m+1}\mu_{n-m} \neq 0$, i.e. if neither space is incident with P. So P is the pole of π if there exists a space $\mathfrak{H}_m$ satisfying the given conditions.

Suppose now that P is the pole of π, so that $b'_\delta k_{\delta+n-1} = 0$. The sets of λ_α's for which there is a set of μ_β's which satisfy the n equations (6) generate a determinantal locus (cf. [1], p. 33) of type $(|n-m+1, n|, [m+1])$ in the Π_{m+1} whose points represent the chordal Π_m's of r in the natural way.

The dimension of the general locus

$$(|n-m+1, n|, [m+1]) \text{ is } (m+1)-n+(n-m+1)-1 = 1$$

(cf. [1], p. 34), so that the dimension of our special locus is not less than one. If S is non-singular, the dimension is certainly only one, since there are then only n points of ρ conjugate to any given point of r.

THEOREM 5. *If S is any non-singular quadric which polarizes the $\mathscr{S}$-simplex of r, ρ then there is a linear series g_1^{n+1} on r such that the m-edges $(0 < m < n-1)$ of the simplexes defined by the series are spaces $\mathfrak{H}_m$ (relative to S).*

PROOF. Since the order of an $(|n, n|, [2])$ is n, there are n spaces $\mathfrak{H}_1$ through any point L_{n+1} on r. These define n more points $L_1, \cdots, L_n$ on r. We show that every 1-edge of the simplex $L_1, \cdots, L_{n+1}$ is a space $\mathfrak{H}_1$.

The polar prime of L_{n+1} meets ρ in n points $M_1, \cdots, M_n$. The order of a $(|3, n|, [n-1])$ is $\binom{n}{2}$ (cf. [1], p. 42), so that, through any M_i, there are $\binom{n}{2}$ chordal Π_{n-2}'s of ρ whose polar lines are chords of r. Each of these $\binom{n}{2}$ spaces $\mathfrak{H}_1$ lies in the polar prime of M_i, namely the prime through $L_1, \cdots, L_{n+1}$ (L_i omitted). $L_\alpha L_\beta$ $(\alpha, \beta \neq i)$ are the only chords of r in this prime, so they are all spaces $\mathfrak{H}_1$.

Now the polar spaces of the 1-edges of any simplex $L_1, \cdots, L_{n+1}$

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(determined by L_{n+1} as above) are the $(n-2)$-edges of a simplex inscribed in ρ, so that the m-edges of the simplex $L_1, \cdots, L_{n+1}$ must be spaces $\mathfrak{H}_m$. Since the order of an $(|n-m+1, n|, [m+1])$ is $\binom{n}{n-m} = \binom{n}{m}$, there are only $\binom{n}{m}$ spaces $\mathfrak{H}_m$ through any point of r, i.e. the linear series gives all the spaces $\mathfrak{H}_m$.

2. A class of self-polar double-twenties of planes in Π_5

If r, ρ is any $\mathscr{S}$-related pair in Π_5, then in an $\mathscr{S}$-system for r, ρ the equation

$$|X^4_{b,c}| = 0, \text{ where } X^4_{b,c} = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & b_\delta x_\delta \\ x_3 & x_4 & x_5 & c_\delta x_\delta \end{bmatrix},$$

represents, for each choice of a set of constants c_δ, a determinantal quartic primal $D^4_{b,c}$ on which both r and ρ are components of the double curve. [5]

It is easily verified that every 2×2 minor of $X^4_{b,c}$ is linearly dependent upon the set of 20 forms consisting of:

(A) the 10 2×2 minors of $\begin{bmatrix} x_0 x_1 \cdots x_4 \\ x_1 x_2 \cdots x_5 \end{bmatrix}$;

(B) the 4 forms $\begin{vmatrix} x_\alpha & x_4 \\ x_{\alpha+1} & b_\delta x_\delta \end{vmatrix}$ $\alpha = 0, \cdots, 3$;

and

(C)
$$\begin{array}{ll} x_0 c_\delta x_\delta - x_3^2 & x_1 c_\delta x_\delta - x_3 x_4 \\ x_2 c_\delta x_\delta - x_3 x_5 & x_3 c_\delta x_\delta - x_4 x_5 \\ x_3 c_\delta x_\delta - x_4 b_\delta x_\delta & x_4 c_\delta x_\delta - x_5 b_\delta x_\delta . \end{array}$$

Thus, by Room's criterion,

THEOREM 1. *The double-twenty of planes on $D^4_{b,c}$ is self-polar.*

We denote this self-polar double-twenty by $CD^4_{b,c}$. It is polarized by a quadric S (generally unique) inpolar to each of the 20 quadrics represented by the vanishing of the above forms (cf. [2], p. 68). Since it is inpolar to the quadrics determined by (A) and (B), S is inpolar to both r and ρ. S is also inpolar to the quadric $b'_\delta x_\delta x_4 = 0$, so that (by Theorem 3 of §1) the $\mathscr{S}$-simplex is self-polar.

If S is non-singular, there are (by Theorem 4 of §1) ∞^1 chordal planes of r whose polar planes are chordal to ρ. By Theorem 5 of §1, they are the

[5] cf. [1], pp. 429—433 for a treatment of the determinantal quartic primal in Π_5.

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2-edges of the simplexes defined by a g_1^6 on r. We shall prove that they may also be characterized in terms of the double-twenties associated with the primals $D^4_{b,\bar{c}}$ of a pencil containing $D^4_{b,c}$.

Each plane H_i of one row of $CD^4_{b,c}$ is a space on $D^4_{b,c}$ of exceptional dimension in the ∞^3 family of generating spaces h given by the four equations.

$$\lambda_\alpha x_{\alpha+\varepsilon} = 0 \qquad \begin{cases} \alpha = 0, \cdots, 3 \\ \varepsilon = 0, 1, 2 \end{cases}$$

$$\lambda_0 x_3 + \lambda_1 x_4 + \lambda_2 b_\delta x_\delta + \lambda_3 c_\delta x_\delta = 0$$

as the λ_α's vary. The general space h is a line. The first three equations always represent a plane chordal to r. Thus each plane H_i is chordal to r.

Similarly, operating on the columns rather than the rows of $X^4_{b,c}$, it can be seen that each plane K_j of the other row of $CD^4_{b,c}$ is chordal to ρ.

A tangential quadric in Π_5 with matrix $K = (k_{\rho+\sigma-2})$ polarizes $CD^4_{b,a}$ if and only if

$$(1) \qquad \begin{cases} b'_\delta k_{\delta+\tau} = 0 & \begin{cases} \delta = 0, \cdots, 5 \\ \tau = 0, \cdots, 4 \end{cases} \\ a_\delta k_{\delta+\alpha} = k_{6+\alpha} & \alpha = 0, \cdots, 3 \\ a_\delta k_{\delta+4} = b_\delta k_{\delta+5}\,. \end{cases}$$

So, if S polarizes $CD^4_{b,c}$, it also polarizes all the double-twenties $CD^4_{b,\bar{c}}$, where

$$\bar{c}_\delta = c_\delta + k b'_\delta \text{ for some } k.$$

The primals $D^4_{b,\bar{c}}$ are all the primals, except that given by $\mu = 0$, of the pencil

$$\lambda b'_\delta x_\delta \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{vmatrix} + \mu |X^4_{b,c}| = 0.$$

We have proved:

THEOREM 2. *The planes of one row of each $CD^4_{b,\bar{c}}$ are spaces $\mathfrak{H}_2$ relative to the quadric S which polarizes $CD^4_{b,c}$.*

In fact, if S is non-singular, then it polarizes no configurations $CD^4_{b,a}$ except the ∞^1 configurations $CD^4_{b,\bar{c}}$. For the complete solution of equations (1) is then

$$a_\delta = c_\delta + k b'_\delta \quad k \text{ an arbitrary constant.}$$

Finally, we prove:

THEOREM 3. *If the quadric S which polarizes $CD^4_{b,c}$ is non-singular then each space $\mathfrak{H}_2$ (relative to S) is either paired with its polar space in some $CD^4_{b,\bar{c}}$ or is a 2-edge of the $\mathscr{S}$-simplex.*

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PROOF. Let $\lambda_\alpha x_{\alpha+\varepsilon} = 0$ be a given $\mathfrak{H}_2$, whose polar plane is, say, $\mu_\alpha x'_{\alpha+\varepsilon} = 0$. Then, by the proof of Theorem 4 of §1,

$$\lambda_\alpha k_{\alpha+\beta+\tau}\mu_\beta + \lambda_3 b'_\delta k_{\delta+\tau+1}\mu_3 = 0 \qquad \tau = 0, \cdots, 4. \tag{2}$$

If $\lambda_3\mu_3 \neq 0$, the given $\mathfrak{H}_2$ and its polar plane are paired in the $CD^4_{b,a}$ given by the a_δ's such that the following is an identity (in the x_δ's):

$$\lambda_0\mu_\alpha x_\alpha + \lambda_1\mu_\alpha x_{\alpha+1} + \lambda_2\mu_\alpha x'_{\alpha+2} + \lambda_3(\mu_0 x_3 + \mu_1 x_4 + \mu_2 x_5 + \mu_3 a_\delta x_\delta) \equiv 0.$$

We derive:

$$\sum_{\alpha+\beta=\delta} \lambda_\alpha\mu_\beta + (\lambda_2 b'_\delta + \lambda_3 a_\delta)\mu_3 = 0 \qquad (\alpha, \beta \leqq 3).$$

Thus

$$a_\delta k_{\delta+\tau} = -\frac{1}{\lambda_3\mu_3}[\lambda_\alpha k_{\alpha+\beta+\tau}\mu_\beta - \lambda_3\mu_3 k_{6+\tau}] - \frac{\lambda_2}{\lambda_3} b'_\delta k_{\delta+\tau} \qquad \tau = 0, \cdots, 4$$

and so, by (2),

$$\begin{aligned} a_\delta k_{\delta+\tau} &= b'_\delta k_{\delta+\tau+1} + k_{6+\tau} - \frac{\lambda_2}{\lambda_3} b'_\delta k_{\delta+\tau} \\ &= b'_\delta k_{\delta+\tau+1} + k_{6+\tau} \end{aligned}$$

Hence

$$a_\delta k_{\delta+\alpha} = k_{6+\alpha} \qquad \alpha = 0, \cdots, 3$$

and

$$a_\delta k_{\delta+4} = b_\delta k_{\delta+5}.$$

If $\lambda_3\mu_3 = 0$ elementary considerations show that, by virtue of the self-polarity of the $\mathscr{S}$-simplex and the non-singularity of S, the given $\mathfrak{H}_2$ must be a 2-edge of the $\mathscr{S}$-simplex.

3. A class of self-polar double-N's in Π_{p+q-3}

The results obtained in §2 may be generalized to apply to double-N's of Π_{p-2}'s and Π_{q-2}'s in Π_{p+q-3}, where $3 \leqq p \leqq q$.

Let r, ρ be any $\mathscr{S}$-related pair in Π_{p+q-3}. Then each set of constants c_δ determines a locus $D^{p,q}_{b,c}$ represented by the equations

$$||X^{p,q}_{b,c}||_{p-1} = 0, \text{ where } X^{p,q}_{b,c} = [x_{\alpha\beta}] \qquad \begin{cases} \alpha = 1, \cdots, p \\ \beta = 1, \cdots, q \end{cases}$$

and $x_{\alpha\beta} \equiv x_{\alpha+\beta-2}$, except that $x_{p-1,q} \equiv b_\delta x_\delta$ and $x_{pq} \equiv c_\delta x_\delta$.

Simple calculations show that the family of loci thus defined depends only on the ordered pair r, ρ and not on the particular $\mathscr{S}$-system chosen.

Examination of the 2×2 minors of $X^{p,q}_{b,c}$ shows that Room's criterion is

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satisfied, so that the double-N, say $CD^{p,q}_{b,c}$,[6] associated with $D^{p,q}_{b,c}$ is polarized by a tangential quadric S. S is given by a matrix $K = (k_{\rho+\sigma-2})$, where the k_γ's satisfy:

$$\begin{cases} b'_\delta k_{\delta+\tau} = 0 & \begin{cases} \delta = 0, \cdots, p+q-3 \\ \tau = 0, \cdots, p+q-4 \end{cases} \\ c_\delta k_{\delta+\alpha} = k_{p+q-2+\alpha} & \quad \alpha = 0, \cdots, p+q-5 \\ c_\delta k_{\delta+p+q-4} = b_\delta k_{\delta+p+q-3}\,. \end{cases}$$

S is inpolar to both r and ρ. The $\mathscr{S}$-simplex is self-polar, so that there exist ∞^1 chordal Π_{p-2}'s of r whose polar Π_{q-2}'s are chordal to ρ. If S is non-singular there are only ∞^1 such Π_{p-2}'s and they are the $(p-2)$-edges of the simplexes defined by a g_1^{p+q-2} on r; each is paired with its polar Π_{q-2} in one of the ∞^1 configurations $CD^{p,q}_{b,\bar{c}}$ (where $\bar{c}_\delta = c_\delta + kb'_\delta$ for some k) or else is a $(p-2)$-edge of the $\mathscr{S}$-simplex.

The freedom of the double-N (of Π_{p-2}'s and Π_{q-2}'s in Π_{p+q-3}) associated with a general $p\times q$ matrix of linear forms is $(p-1)(q-1)\ (p+q+1)-pq$ (cf. [1], p. 74).

Theorem. *The freedom of loci of type $D^{p,q}_{b,c}$ in Π_{p+q-3} is*

$$(p+q)^2-2(p+q)-3$$

(except when $p = q = 3$).

Proof. The freedom of $\mathscr{S}$-related pairs in Π_{p+q-3} is

$$(p+q-3)^2+3(p+q-3)-1,$$

by Theorem 2 of §1.

Each $\mathscr{S}$-related pair r, ρ defines at most ∞^{p+q-2} loci of type $D^{p,q}_{b,c}$, since there are ∞^{p+q-2} sets of constants c_δ. But it can be easily shown that, given a general chordal Π_{p-2} of r and a general chordal Π_{q-2} of ρ (not incident with the Π_{p-2}), there is a $CD^{p,q}_{b,c}$ defined by r, ρ in which these spaces are paired. It follows that each $\mathscr{S}$-related pair defines ∞^{p+q-2} loci of type $D^{p,q}_{b,c}$.

The result is established if we prove that a general $D^{p,q}_{b,c}$ can be defined by only finitely many $\mathscr{S}$-related pairs (except when $p = q = 3$). When $p > 3$, this follows from the (easily proved) fact that the dimension of the locus $||X^{p,q}_{b,c}||_2 = 0$ is one. Now $D^{3,q}_{b,c}$ is a special surface of type F_S (cf. [2], p. 70). Suppose $q > 3$. Then the plane representation of F_S (cf. [1], p. 392) shows that the curve r is uniquely defined by $D^{3,q}_{b,c}$ (two irreducible plane $(q+2)$-ics cannot share $\binom{q+1}{2}$ double points), and that each point of r is the

[6] If $p = q$ we abbreviate $X^{p,q}_{b,c}$, $D^{p,q}_{b,c}$, $CD^{p,q}_{b,c}$ to $X^q_{b,c}$, $D^q_{b,c}$, $CD^q_{b,c}$.

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$\mathscr{S}$-point for at most one suitable $\mathscr{S}$-related pair r, ρ (a suitable ρ is represented by a line tangent to the $(q+2)$-ic, which represents r, at the point representing the $\mathscr{S}$-point). The $\mathscr{S}$-point of any suitable $\mathscr{S}$-related pair is represented by one of the $q+2$ points at which the contact conic touches the $(q+2)$-ic. So there are at most $q+2$ suitable $\mathscr{S}$-related pairs.

The above formula is not valid when $p = q = 3$. The work in §4 will show that the freedom of $D^3_{b,c}$ is 19.

4. The double-six of lines

It is well known that the general double-six of lines in Π_3 is self-polar; we can in fact prove that it is a $CD^3_{b,c}$.

The general double-six of lines $a_1, \cdots, a_6$, $b_1 \cdots, b_6$ lies on a non-singular cubic surface F^3, which may be represented by cubic curves γ' through six points A'_i in a plane (cf. [4], pp. 189-192).

Let P be a point on F^3 (but not on any of the lines of F^3), represented by P'. There is a curve γ', say Γ', which has a double point at P'. Let ρ' be the tangent line to one of the two branches of Γ' at P'. The intersection multiplicity of ρ' and Γ' at P' is three.

There is a rational quintic r' which has a double point at each A'_i and passes through P', having ρ' as tangent line at P'. The intersection multiplicity of Γ' and r' at P' is easily computed to be three.

Γ' represents the curve Γ in which the tangent plane μ to F^3 at P meets F^3. ρ' represents a twisted cubic ρ on F^3. ρ passes through P and meets each b_i twice. r' represents another twisted cubic r on F^3. r passes through P and meets each a_i twice. r meets ρ at three points apart from P.

Since Γ' meets ρ' nowhere except at P', and r' nowhere except at P' and at the base points A'_i, μ osculates both ρ and r at P. Furthermore, since ρ' is tangent to r' at P', ρ touches r at P. Thus r, ρ is an $\mathscr{S}$-related pair on F^3.

By Theorem 1 of §1, a coordinate-system and a set of constants b_δ can be found such that r is represented by

$$\left\|\begin{matrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{matrix}\right\|_1 = 0 \text{ and } \rho \text{ by } \left\|\begin{matrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & b_\delta x_\delta \end{matrix}\right\|_1 = 0.$$

Now the cubic surfaces which pass through both r and ρ form a linear family of freedom four. But the surfaces

$$\alpha_0 \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & b_\delta x_\delta \\ 0 & 0 & x_0 \end{vmatrix} + \cdots + \alpha_3 \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & b_\delta x_\delta \\ 0 & 0 & x_3 \end{vmatrix} + \alpha_4 \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & b_\delta x_\delta \\ x_2 & x_3 & 0 \end{vmatrix} = 0$$

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all pass through both r and ρ, so that every cubic surface through both r and ρ is one of these. Those given by $\alpha_4 = 0$ are composite, so that F^3 must be a $D^3_{b,c}$.

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