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BAND-LIMITED WAVELETS WITH SUBEXPONENTIAL DECAY

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ABSTRACT. It is well known that the compactly supported wavelets cannot belong to the class $C^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$. This is also true for wavelets with exponential decay. We show that one can construct wavelets in the class $C^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$ that are "almost" of exponential decay and, moreover, they are band-limited. We do this by showing that we can adapt the construction of the Lemarié-Meyer wavelets [LM] that is found in [BSW] so that we obtain band-limited, C^∞-wavelets on $\mathbf{R}$ that have subexponential decay, that is, for every $0 < \varepsilon < 1$, there exits $C_\varepsilon > 0$ such that $|\psi(x)| \le C_\varepsilon e^{-|x|^{1-\varepsilon}}$, $x \in \mathbf{R}$. Moreover, all of its derivatives have also subexponential decay. The proof is constructive and uses the Gevrey classes of functions.

1. **Introduction.** An orthonormal wavelet ψ is said to have *exponential decay* if there exist $c > 0$ and $\alpha > 0$ such that $|\psi(x)| \le ce^{-\alpha|x|}$ for all $x \in \mathbf{R}$. The spline wavelets have exponential decay ([Le]) as well as the compactly supported wavelets ([Da]). But, *there is no orthonormal wavelet with exponential decay belonging* to $C^\infty(\mathbf{R})$ such that all its derivatives are bounded. To see this, suppose that such a wavelet ψ exists. The exponential decay of ψ would imply that

$$\hat{\psi}(z) = \int_{\mathbf{R}} e^{-izx}\psi(x)\,dx$$

is a holomorphic function on $|\operatorname{Im} z| < \alpha$. Moreover, the smoothness and decay of ψ would imply that all the moments of ψ are zero. (See Theorem 3.4, Chapter 2, in [HW]). Hence, $\frac{d^n\hat{\psi}}{d\xi^n}(0) = 0$ for all $n = 0, 1, 2, \ldots$. The expansion of $\hat{\psi}(z)$ in powers of z around the origin shows that $\hat{\psi} \equiv 0$ in a neighborhood of $z = 0$. Since $\{z \in \mathbf{C} : |\operatorname{Im} z| < \alpha\}$ contains the real line in its interior, ψ must be the zero function on $\mathbf{R}$.

Orthonormal wavelets ψ that belong to $C^\infty(\mathbf{R})$ have been exhibited in [LM]. They are band-limited (*i.e.* the supports of their Fourier transforms are bounded) and belong to the Schwartz class $\mathcal{S}$. They can be constructed using smooth "bell" functions as explained in [AWW] or [HW]. It is impossible, however, for any one of these wavelets to have exponential decay (since $\hat{\psi} \equiv 0$ in a neighborhood of the origin).

DEFINITION 1.1. A real-valued function f defined on $\mathbf{R}$ is said to have *subexponential*

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decay if whenever $0 < \varepsilon < 1$, there exists $C_\varepsilon > 0$ such that

$$|\psi(x)| \leq C_\varepsilon e^{-|x|^{1-\varepsilon}}$$

for all $x \in \mathbf{R}$.

We shall show how to construct band-limited, orthonormal wavelets with subexponential decay belonging to $C^\infty(\mathbf{R})$. The construction is obtained by finding an appropriate "bell" function b whose Fourier transform has subexponential decay. This is accomplished by means of the Gevrey classes of functions, whose definition and properties are presented in the next section.

2. The Gevrey classes.

DEFINITION 2.1. For $\delta > 0$, the *Gevrey* class Γ^δ is the set of all C^∞ real-valued functions defined on $\mathbf{R}$ such that for every compact set $K \subset \mathbf{R}$ there is a constant C_K satisfying

$$|D^n f(x)| \leq C_K C_K^n n^{n\delta},$$

for all $x \in K$ and for all $n = 1, 2, 3, \ldots$.

DEFINITION 2.2. For $\delta > 0$, the (*small*) *Gevrey* class γ^δ is the set of all C^∞ real-valued functions defined on $\mathbf{R}$ such that for every compact set $K \subset \mathbf{R}$ and every $\varepsilon > 0$, there is a constant $C_{K,\varepsilon}$ satisfying

$$|D^n f(x)| \leq C_{K,\varepsilon}\varepsilon^n (n!)^\delta,$$

for all $x \in K$ and for all $n = 1, 2, 3, \ldots$.

We have taken the above definitions from [Ho1] (pp. 280–281) and [Ho2] (p. 137). Since $n! \leq n^n$ it is clear that for every $\delta > 0$

$$\gamma^\delta \subset \Gamma^\delta. \tag{2.3}$$

LEMMA 2.4. *If* $0 < \delta' < \delta$ *then* $\Gamma^{\delta'} \subset \gamma^\delta$.

PROOF. Let $K \subset \mathbf{R}$ be compact and $\varepsilon > 0$. For $f \in \Gamma^{\delta'}$ we can find $C_K > 0$ such that

$$|D^n f(x)| \leq C_K C_K^n n^{\delta' n},$$

for all $x \in K$ and all $n = 1, 2, 3, \ldots$. By Stirling's formula ($n! \sim \sqrt{2\pi n}\, n^n e^{-n}$) we can write

$$|D^n f(x)| \leq C'_K (n!)^\delta (C'_K)^n \frac{(n!)^{\delta'-\delta} e^{n\delta'}}{(\sqrt{2\pi n})^{\delta'}}, \quad x \in K.$$

The sequence $A_n = C'_K (n!)^{\frac{\delta'-\delta}{n}} e^{\delta'} / (\sqrt{2\pi n})^{\delta'/n}$ tends to zero as $n \longrightarrow \infty$ since $\delta' < \delta$. Thus, there exists $N(\varepsilon) \in \mathbf{N}$ such that for all $n \geq N(\varepsilon)$, $A_n \leq \varepsilon$. Hence, for all $x \in K$,

$$|D^n f(x)| \leq C'_K \varepsilon^n (n!)^\delta,$$

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for all $n \geq N(\varepsilon)$. This inequality is also true for $n = 1, 2, \ldots, N(\varepsilon) - 1$ by enlarging the constant if necessary. ∎

The Gevrey classes satisfy $\gamma^{\delta_1} \subset \gamma^{\delta_2}$ and $\Gamma^{\delta_1} \subset \Gamma^{\delta_2}$ when $0 < \delta_1 < \delta_2$. When $\delta > 1$ the classes γ^δ and Γ^δ contain "cutoff" functions. This follows from Theorem 1.3.5 in [Ho1]. We feel that it is worthwhile for the reader to present the essential ingredients of this result. With $\chi = \chi_{[0,1]}$ write $\chi_a = \frac{1}{a}\chi(\frac{x}{a})$. For any sequence $a_1 \geq a_2 \geq \cdots > 0$ such that $a = \sum_{j=1}^\infty a_j < \infty$, the function

$$\varphi_k = \chi_{a_1} * \cdots * \chi_{a_k}$$

belongs to $C^{k-1}(\mathbf{R})$, has support in $[0, a]$ and converges as $k \longrightarrow \infty$ to a function $\varphi \in C^\infty(\mathbf{R})$, with support in $[0, a]$, such that $\int_{\mathbf{R}} \varphi(x)\,dx = 1$ and

$$|D^n \varphi(x)| \leq \frac{2^n}{a_1 \cdots a_n}. \tag{2.5}$$

By taking $a_n = n^{-\delta}$ in the above construction it follows that $\varphi \in \Gamma^\delta$ when $\delta > 1$. (Observe that in this case $\sum_{n=1}^\infty n^{-\delta}$ is a convergent series.)

This result shows that there are "cutoff" functions in every class Γ^δ and γ^δ when $\delta > 1$. A modification of the above regularization procedure shows that there exists a "cutoff" function which belongs to every Γ^δ and γ^δ for all $\delta > 1$.

PROPOSITION 2.6. *For every $a > 0$ there exists $\varphi_a \in \Gamma^\delta$ for every $\delta > 1$. Moreover, $\varphi_a \geq 0$, $\sup \varphi_a \subset [-a, a]$ and $\int_{\mathbf{R}} \varphi_a(x)\,dx = \pi/2$.*

PROOF. Since Γ^δ is invariant under dilations and multiplication by constants, it is enough to show the result for $a = 1$ and show that $\int_{\mathbf{R}} \varphi_a(x)\,dx < \infty$. Let h be an even function such that $h \in C^\infty([-1, 1])$, $h \geq 0$, and $\int_{-1}^1 h(x)\,dx = 1$. Choose $\delta_m = 1 + \frac{1}{m}$ and let N_m be an increasing sequence of positive integers such that

$$\sum_{n \geq N_m} \frac{1}{n^{\delta_m}} < \frac{1}{2^m}.$$

Choose $a_n = n^{-\delta_m}$ when $N_m \leq n < N_{m+1}$. Observe that

$$\sum_{n \geq N_1} a_n \leq \sum_{m=1}^\infty \frac{1}{2^m} = 1.$$

Define

$$\varphi_{(n)} = h_{a_{N_1}} * h_{a_{N_1+1}} * \cdots * h_{a_n}$$

where $h_a(x) = \frac{1}{a}h(\frac{x}{a})$, so that $\int_{\mathbf{R}} h_a(x)\,dx = 1$. Obviously $\sup \varphi_{(n)} \subset [-1, 1]$. We shall show that for every $\delta > 1$, there exists $C = C_\delta$ such that for all $x \in \mathbf{R}$ and all $N = 1, 2, 3, \ldots,$

$$|D^N \varphi_{(n)}(x)| \leq C_\delta (C_\delta)^N N^{\delta N}, \tag{2.7}$$

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for all $n \geq n(C_\delta, N)$. Take m and n so large that $\delta_m < \delta$, and $N_m + N < n$. Then,

$$D^N \varphi_{(n)} = h_{a_{N_1}} * h_{a_{N_1+1}} * \cdots * h_{a_{N_m}} * Dh_{a_{N_m+1}} * \cdots * Dh_{N_m+N} * \cdots * h_{a_n}.$$

We have

$$\|Dh_{a_n}\|_1 = \frac{1}{a_n}\int_{\mathbf{R}} \frac{1}{a_n}\left|Dh\left(\frac{x}{a_n}\right)\right| dx \leq \frac{C}{a_n} \leq Cn^{\delta_m}$$

if $n \geq N_m$. Thus, using $\int u * v = (\int u)(\int v)$, and $\int_{\mathbf{R}} h_a = 1$, we deduce,

$$\begin{aligned} |D^N \varphi_{(n)}(x)| &\leq C^N (N_m+1)^{\delta_m} \cdots (N_m+N)^{\delta_m} \\ &\leq C^N (N_m+N)^{\delta_m N} \leq C^N N_m^{\delta_m N} N^{\delta N} \\ &\leq C^N N_m^{2N} N^{\delta N} \leq C_\delta (C_\delta)^N N^{\delta N}, \end{aligned}$$

where $C_\delta = CN_m^2$ (observe that N_m depends on δ). One can show that $\{D^N \varphi_{(n)} : n = N_{1,\ldots}\}$ is a Cauchy sequence for every $N = 0, 1, 2, \ldots$. Thus, $\varphi_{(n)}$ converges to a function φ which satisfies 2.7 with $\varphi_{(n)}$ replaced by φ. Hence, $\varphi \in \Gamma^\delta$ for all $\delta > 1$ and $\sup \varphi \subset [-1, 1]$. ∎

The behaviour of the Fourier transforms of functions with compact support that are contained in γ^δ is given in the following result.

PROPOSITION 2.8. *Let $\delta > 0$. Suppose f is a function such that* $\sup f \subset [-A, A]$ *and $f \in \gamma^\delta$. Then, for every $B > 0$ there exists a constant C_B such that*

$$|\hat{f}(z)| \leq C_B e^{A|\operatorname{Im}(z)|} e^{-B|\operatorname{Re} z|^{1/\delta}}, \quad z \in \mathbf{C}.$$

Proposition 2.8 is a generalization of one of the implications in the Paley-Wiener theorem and its proof can be found in Lemma 12.7.4. of [Ho2].

3**. The construction.** For fixed $a > 0$ choose a "cutoff" function φ_a as in Proposition 2.6. In particular, $\varphi_a \in \Gamma^\delta$ for every $\delta > 1$. Set

$$\theta_a(x) = \int_{-\infty}^{x} \varphi_a(t)\, dt.$$

Observe that $\theta_a \in \Gamma^\delta$ for every $\delta > 1$. As in [AWW] we consider $S_a(x) = \sin\big(\theta_a(x)\big)$ and $C_a(x) = \cos\big(\theta_a(x)\big)$, so that

$$b_a(x) = S_a(x-\pi) C_{2a}(x-2\pi), \quad a \leq \frac{\pi}{3}, \tag{3.1}$$

is a bell function associated with the interval $[\pi, 2\pi]$ as considered in [AWW] or [BSW]. Let us assume for the moment (see Theorem 3.3 below) that S_a and C_a belong to Γ^δ for every $\delta > 1$. Since Γ^δ is an algebra (Proposition 8.4.1 in [Ho1]) and it is invariant under translations, it follows that $b_a \in \Gamma^\delta$ for every $\delta > 1$. Extending b_a evenly to $[-\infty, 0]$ it is proved in [AWW] (see also Corollary 4.7 of Chapter 1 in [HW]) that the function ψ defined by

$$\hat{\psi}_a(\xi) = e^{i\xi/2} b_a(\xi) \tag{3.2}$$

is an orthonormal wavelet in $L^2(\mathbf{R})$.

The following result shows, as a particular case, that the functions S_a and C_a, constructed as the composition of the sine and cosine functions with θ_a, belong to Γ^δ for every $\delta > 1$.

THEOREM 3.3. *Let $\delta \geq 1$. Suppose that F is an entire function and $f \in \Gamma^\delta$. Then, $g(x) = F\big(f(x)\big) \in \Gamma^\delta$.*

PROOF. We have to show that for every compact set $K \subset \mathbf{R}$ there is a constant C_0 such that

$$|D^N g(x_0)| \leq C_0 C_0^N N^{\delta N}$$

for all $x_0 \in K$ and all $N = 1, 2, \ldots$. Using the Taylor expansion we can write

$$f(x) = \sum_{n=0}^{N} \frac{1}{n!} D^n f(x_0)(x - x_0)^n + R_N(x; x_0) \equiv f_N(x) + R_N(x).$$

Obviously, $D^N g(x_0) = D^N[F(f_N)](x_0)$. By the assumption, $F\big(f_N(z)\big)$, $z \in \mathbf{C}$, is analytic, and by the Cauchy formula we can write

$$D^N g(x_0) = \frac{N!}{2\pi i} \int_{\omega_N} \frac{F\big(f_N(z)\big)}{(z - x_0)^{N+1}}\, dz,$$

where $\omega_N = \{z \in \mathbf{C} : |z - x_0| = \frac{1}{2eC} N^{1-\delta}\}$ and C is the constant such that $|D^n f(x)| \leq CC^n n^{\delta n}$ for all $x \in K$ and all $n = 1, 2, \ldots$. If $z \in \omega_N$, we use Stirling's formula to obtain

$$\begin{aligned}
|f_N(z)| &\leq \sum_{n=0}^{N} \frac{1}{n!} CC^n n^{\delta n} \Big(\frac{1}{2eC} N^{1-\delta}\Big)^n \\
&\leq \sum_{n=0}^{N} \frac{1}{n!} C n^{\delta n} \frac{1}{(2e)^n} N^{n - n\delta} \\
&\leq C' \sum_{n=0}^{N} n^{\delta n - n} e^n \frac{1}{(2e)^n} N^{n - n\delta} \\
&= C' \sum_{n=0}^{N} \frac{1}{2^n} \Big(\frac{n}{N}\Big)^{\delta n - n}.
\end{aligned}$$

Since $\delta \geq 1$, we have $|f_N(z)| \leq C' \sum_{n=0}^{\infty} \frac{1}{2^n} = 2C'$. Since F is analytic we obtain $\big|F\big(f_N(z)\big)\big| \leq C''$ on ω_N. Thus,

$$\begin{aligned}
|D^N g(x_0)| &\leq \frac{N!}{2\pi} C'' \frac{2\pi}{2eC} N^{1-\delta} \Big(\frac{1}{2eC} N^{1-\delta}\Big)^{-(N+1)} \\
&\leq C_1\, N! (2eC)^N N^{(\delta-1)N}.
\end{aligned}$$

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Using $N! \leq N^N$ we obtain

$$|D^N g(x_0)| \leq C_1(2eC)^N N^{\delta N} \leq C_0 C_0^N N^{\delta N}$$

where $C_0 = \max\{C_1, 2eC\}$. ■

REMARK. One can find in the literature that if F is an entire function and $f \in \Gamma^\delta$, then $h(x) = f\big(F(x)\big) \in \Gamma^\delta$ (see Proposition 8.4.1 in [Ho1]).

COROLLARY 3.4. *There exist band-limited, C^∞, orthonormal wavelets in $L^2(\mathbf{R})$ with subexponential decay. Moreover all of their derivatives have also exponential decay.*

PROOF. Let $0 < \varepsilon < 1$ and choose $\delta = \frac{1}{1-\varepsilon} (\delta > 1)$. The function b_a defined by 3.1, as well as its even extension to $(-\infty, 0]$ belong to Γ^δ for every $\delta > 1$ by Theorem 3.3. By Lemma 2.4, $b_a \in \gamma^\delta$ for every $\delta > 1$. By Proposition 2.8 (with $B = 1$) the orthonormal wavelet ψ_a given by 3.2 satisfies

$$|\psi_a(x)| = C\left|\hat{b}_a\left(x + \frac{1}{2}\right)\right| \leq C_\varepsilon e^{-|x+\frac{1}{2}|^{1/\delta}} \leq C_\varepsilon e^{-|x|^{1-\varepsilon}}, \quad x \in \mathbf{R}.$$

That ψ_a is band-limited is obvious from the definition of b_a. The fact that all of its derivatives have also exponential decay follows from

$$|D^n \psi_a(x)| = C\left|\left(\xi^n e^{i\xi/2} b_a(\xi)\right)\hat{}\,(x)\right|$$

and

$$\xi^n e^{i\xi/2} b_a(\xi) \in \gamma^\delta$$

for every $\delta > 1$. ■

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