SUBNORMALITY AND GENERALIZED COMMUTATION RELATIONS

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1. In the theory of Hilbert space operators an important question is whether an operator is subnormal [**3**], [**4**], [**7**], [**8**]. A densely defined linear operator S in a complex Hilbert space H is subnormal if there exists a normal operator N in a complex Hilbert space $K \supset H$ such that $S \subset N$.

In [**7**] it has been proved that S, with its domain $D(S)$ invariant under S, is subnormal provided S has a total set of quasianalytic vectors and satisfies the Halmos–Bram condition

$$\sum_{i,j=0}^{n} \langle S^j f_i, S^i f_j \rangle \geq 0 \text{ for all natural numbers } n \text{ and all finite sequences } \{f_i\} \text{ from the domain } D(S) \text{ of } S. \tag{1.1}$$

In this paper it is shown that all operators S satisfying the generalized commutation relation (i.e. $(S^*S - SS^*)f = E^2 f$, $EAf = AEf$, for each $f \in D(S)$, with suitable symmetric operator E) satisfy the Halmos–Bram condition. A similar result with $E = I$ has been proved by Jorgensen [**5**], but in a more involved way.

2. In this section it will be shown that each operator satisfying the generalized commutation relation automatically satisfies the Halmos–Bram condition.

First we prove the following lemma.

Lemma. *Let S be a densely defined linear operator in H. Let M be a dense linear subspace of H such that $M \subset D(S) \cap D(S^*)$, $SM \subset M$ and $S^*M \subset M$. If there exists an operator C such that*

$$\begin{aligned} &\text{(i) } M \subset D(C) \cap D(C^*), \\ &\text{(ii) } (S^*S - SS^*)f = Cf, \quad SCf = CSf, \quad \textit{for each } f \in M, \end{aligned} \tag{2.2}$$

then

$$S^*Cf = CS^*f, \quad \textit{for each } f \in M$$

and

$$(S^*)^i S^j f = \sum_{k=0}^{\infty} k! \binom{j}{k} \binom{i}{k} S^{j-k} (S^*)^{i-k} C^k f, \quad \textit{for each } f \in M, \tag{2.3}$$

where, by definition

$$S^{-l} = (S^*)^{-l} = 0 \quad \textit{if} \quad l > 0, \qquad \binom{i}{j} = 0 \quad \textit{if} \quad j > i.$$

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Proof. Since $S^*S - SS^*|_M = C|_M$, $C|_M$ is symmetric and $C(M) \subset M$. This and (i) imply that $C|_M = C^*|_M$, so $C^*(M) \subset M$. Thus $\langle S^*Cf, g\rangle = \langle f, C^*Sg\rangle = \langle f, CSg\rangle = \langle f, SCg\rangle$ for all $f, g \in M$. Since M is dense in H, $S^*Cf = CS^*f$ for all $f \in M$. Now we prove the condition (2.3) by induction on j. It is clear that the equation (2.3) holds for $j = 0$. Now we prove this equation for $j = 1$ using induction on i. It is clear that (2.3) holds for $i = 0, 1$. Let $i \geq 2$. The inductive assumption and the condition (2.1) imply that, for all $f \in M$,

$$\begin{aligned}
(S^*)^iSf = S^*(S^*)^{i-1}Sf &= S^* \sum_{k=0}^{\infty} k!\binom{1}{k}\binom{i-1}{k} S^{1-k}(S^*)^{i-1-k}C^kf \\
&= S^*[S(S^*)^{i-1}f + (i-1)(S^*)^{i-2}Cf] \\
&= S^*S(S^*)^{i-1}f + (i-1)(S^*)^{i-1}Cf \\
&= [SS^* + C](S^*)^{i-1}f + (i-1)(S^*)^{i-1}Cf \\
&= S(S^*)^if + i(S^*)^{i-1}Cf \\
&= \sum_{k=0}^{\infty} k!\binom{1}{k}\binom{i}{k} S^{1-k}(S^*)^{i-k}C^kf.
\end{aligned}$$

Now we show that the inductive step with respect to j holds. The inductive assumption and the condition (2.2) for $j = 1$ imply that

$$\begin{aligned}
(S^*)^iS^jf = (S^*)^iS^{j-1}Sf &= \sum_{k=0}^{\infty} k!\binom{j-1}{k}\binom{i}{k} S^{j-1-k}(S^*)^{i-k}C^kSf \\
&= \sum_{k=0}^{\infty} k!\binom{j-1}{k}\binom{i}{k} S^{j-1-k}[(S^*)^{i-k}S]C^kf \\
&= \sum_{k=0}^{\infty} k!\binom{j-1}{k}\binom{i}{k} S^{j-1-k}[S(S^*)^{i-k} + (i-k)(S^*)^{i-k-1}C]C^kf \\
&= \sum_{k=0}^{\infty} k!\binom{j-1}{k}\binom{i}{k} S^{j-k}(S^*)^{i-k}C^kf \\
&\quad + \sum_{k=0}^{\infty} (i-k)k!\binom{j-1}{k}\binom{i}{k} S^{j-k-1}(S^*)^{i-k-1}C^{k+1}f \\
&= [S^j(S^*)^if + \sum_{k=1}^{\infty} k!\binom{j-1}{k}\binom{i}{k} S^{j-k}(S^*)^{i-k}C^kf] \\
&\quad + \sum_{k=1}^{\infty} (i-s+1)(s-1)!\binom{j-1}{s-1}\binom{i}{s-1} S^{j-s}(S^*)^{i-s}C^sf \\
&= S^j(S^*)^if + \sum_{k=1}^{\infty} \Bigg[k!\binom{j-1}{k}\binom{i}{k} \\
&\quad + (i-k+1)(k-1)!\binom{j-1}{k-1}\binom{i}{k-1}\Bigg] S^{j-k}(S^*)^{i-k}C^kf
\end{aligned}$$

$$= S^j(S^*)^i f + \sum_{k=1}^{\infty} k!\binom{j}{k}\binom{i}{k} S^{j-k}(S^*)^{i-k}C^k f$$
$$= \sum_{k=0}^{\infty} k!\binom{j}{k}\binom{i}{k} S^{j-k}(S^*)^{i-k}C^k f.$$

Now we can state and prove the main result of the paper.

THEOREM 1. *Let S be a densely defined linear operator in H. Let M be a dense linear subspace of H such that $M \subset D(S) \cap D(S^*)$, $SM \subset M$ and $S^*(M) \subset M$. If there exists an operator E such that*

(I) $M \subset D(E) \cap D(E^*)$, $\quad EM \subset M$,

(II) $(S^*S - SS^*)f = E^2 f$, $\quad SEf = ESf$, *for each* $f \in M$ (II)

(III) $\langle f, Eg\rangle = \langle Ef, g\rangle$, *for each* $f, \; g \in M$,

then the Halmos–Bram condition holds on M.

Proof. The conditions (I) and (III) imply that $E^*(M) \subset M$. Now, using the Lemma with $C = E^2$ and the above assumptions we obtain:

$$\sum_{i,j=0}^{n} \langle S^i f_j, S^j f_i\rangle = \sum_{i,j=0}^{n} \langle (S^*)^i S^j f_i, f_j\rangle$$
$$= \sum_{k=0}^{\infty} k! \sum_{i,j=0}^{n} \binom{j}{k}\binom{i}{k} \langle S^{j-k}(S^*)^{i-k}(E^2)^k f_i, f_j\rangle$$
$$= \sum_{k=0}^{\infty} k! \sum_{i,j=0}^{n} \binom{j}{k}\binom{i}{k} \langle (S^*)^{i-k}E^k f_i, (S^*)^{j-k}E^k f_j\rangle$$
$$= \sum_{k=0}^{\infty} k! \left\langle \sum_{i=0}^{n} \binom{i}{k}(S^*)^{i-k}E^k f_i, \sum_{j=0}^{n} \binom{j}{k}(S^*)^{j-k}E^k f_j \right\rangle$$
$$= \sum_{k=0}^{\infty} k! \left\| \sum_{i=0}^{n} \binom{i}{k}(S^*)^{i-k}E^k f_i \right\|^2 \geq 0.$$

As a simple consequence Theorem 1 we obtain the following result.

THEOREM 2. *Let S, E be as in Theorem 1 and let S have a total set of quasianalytic vectors. Then the operator S is subnormal.*

3. Now we make some comments on the assumptions of Theorem 1. Throughout the whole of Section 3, S, E and M are assumed to satisfy the assumptions of Theorem 1.

If $E = 0$, then $S^*S = SS^*$ on M, so $S|_M$ is formally-normal [**2**].

If $E = I$, then (2.4)(II) takes the form $S^*S - SS^* = I$ on M. This equality, when rewritten via cartesian decomposition of S, is equivalent to a commutation relation [**6**]. This case has been considered by Jorgensen [**5**].

Now let $S \in L(H)$ and $M = H$. The condition (2.4)(II) in the form $E^2S = SE^2$ implies that E^2 is quasinilpotent [**4**], [**6**]. But E^2 is selfadjoint. So $E^2 = 0$ and S is normal in consequence. Thus if one looks for subnormal operators which are not normal, then one must consider unbounded operators in Theorem 1.

Let H be a separable Hilbert space with the orthogonal basis $\{e_i\} \subset M$. Below we show that there is no diagonal operator E with distinct diagonal elements of multiplicity one such that $E^2 \neq 0$ and which satisfies (2.4). If there is such an E, then the condition $ES = SE$ on M implies that $E^2Se_i = SE^2e_i = d_iSe_i$ and thus there exists a complex sequence $\{b_i\}$ such that $Se_i = b_ie_i$. So we can calculate $E^2e_i = (S^*S - SS^*)e_i = 0$, $i \in \mathbb{N}$, contrary to $E^2 \neq 0$.

At the end of this paper we give an example of an operator which satisfies the condition (2.4) with $E \notin \mathbb{C}I$. Let H_1, H_2 be separable Hilbert spaces with orthonormal bases $\{e_i^k : i \in \mathbb{N}\}$, $k = 1, 2$ and A_1, A_2 be the weighted shift operators on H_1, H_2 respectively such that $A_k e_i^k = i e_{i+1}^k$, $k = 1, 2$, $i \in \mathbb{N}$; see also Bargmann's model [**1**]. We define the operator $S = a_1A_1 + a_2A_2$ on $H_1 \oplus H_2$, where $a_1 > a_2 > 0$. Since the operators A_1, A_2 are subnormal, S is subnormal too. A simple calculation shows that the operator S satisfies the condition (2.4) with $M = \text{lin}\{e_i^k : i \in \mathbb{N},\ k = 1, 2\}$ and $E \notin \mathbb{C}I$.

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