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A DUAL CHARACTERISATION OF THE RADON-NIKODYM PROPERTY

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We prove that a Banach space X has the Radon-Nikodym property if, and only if, every weak*-lower semicontinuous convex continuous function f of X^* is Gâteaux differentiable at some point of its domain with derivative in the predual space X.

1. INTRODUCTION

Collier [5] has shown that a Banach space X has the Radon-Nikodym property if, and only if, all weak*-lower semicontinuous convex continuous functions on the dual space X^* are generically Fréchet differentiable. (Such a dual space was called in [5] weak*-Asplund.) In this article we give the following characterisation of the Radon-Nikodym property in terms of Gâteaux derivatives.

THEOREM 1. *A Banach space X has the Radon-Nikodym property if, and only if, every weak*-lower semicontinuous convex continuous function on X^* is Gâteaux differentiable at some point of its domain with derivative in the predual space X.*

Since Fréchet derivatives of weak*-lower semicontinuous convex continuous functions of X^* are always elements of X (see [7], for example), the improvement upon the aforementioned result of Collier consists on replacing the Fréchet derivative by Gâteaux and on passing from a dense differentiability assumption to the existence of the derivative at one point.

If X does not have the Radon-Nikodym property, then it is possible to have nowhere Fréchet differentiable weak*-lower semicontinuous convex continuous functions on X^* for which the set of points where the Gâteaux derivative exists and belongs to the predual space is dense (see Proposition 8). Concurrently, it is also possible to have weak*-lower semicontinuous convex continuous functions on X^* that are generically Gâteaux differentiable with all derivatives in $X^{**} \setminus X$. Indeed, consider the Banach space $X = c_0(\mathbb{N})$, its dual space $X^* = \ell^1(\mathbb{N})$ and the function $g(x) = \|x\|_1$, see [10, Example 1.4 (b)] for details.

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Let us finally note that characterisations of the Radon-Nikodym property for dual Banach spaces in terms of the Gâteaux derivative are recently established by Giles in [8, Theorem 2].

The proof of Theorem 1 is given in Section 3, while in the following section we fix our notation and we recall relevant definitions.

2. Preliminaries

In the sequel, $(X, \|.\|)$ will be a Banach space and $(X^*, \|.\|)$ will be its dual. We denote by B_X the closed unit ball of X and by $\mathbb{R}$ (respectively, $\mathbb{N}$) the set of all real (respectively, positive integer) numbers. For any $x \in X$ and any $p \in X^*$ we denote by $\langle p, x\rangle$ the value of the functional p at the point x. Similarly, for any z^{**} in X^{**} we denote by $\langle p, z^{**}\rangle$ the value of z^{**} at p. We also denote by $\overline{\mathrm{co}}\, F$ the closed convex hull of the set F. For any non-empty closed bounded subset F of X we denote by ψ_F the indicator function of F ($\psi_F(x) := 0$, if $x \in F$ and $+\infty$, if $x \notin F$) and by ψ_F^* its Fenchel conjugate, that is, for all $p \in X^*$

$$\psi_F^*(p) = \sup_{x \in F} \langle p, x\rangle. \tag{1}$$

It is known that ψ_F^* is a weak*-lower semicontinuous convex continuous function. (The latter follows from the fact that the boundedness of F yields $\operatorname{dom} \psi_F^* = X^*$.) We also recall that every weak*-lower semicontinuous convex continuous function $g : X^* \to \mathbb{R} \cup \{+\infty\}$ coincides with the first conjugate f^* of a lower semicontinuous convex function f defined on X (take $f := g^*$). We denote by $\operatorname{dom} g := \{p \in X^* : g(p) < +\infty\}$ the domain of the function g. Then, the Fenchel-Moreau subdifferential ∂g of g at any $p_0 \in \operatorname{dom} g$ is defined as follows:

$$\partial g(p_0) = \{z^{**} \in X^{**} : g(p) - g(p_0) \geqslant \langle p - p_0, z^{**}\rangle,\ \forall p \in X^*\} \tag{2}$$

If $p_0 \in X^* \setminus \operatorname{dom} g$, then we set $\partial g(p_0) = \emptyset$.

Given a closed subset F of X and a point x_0 of F we say that x_0 is a *strongly exposed* point of F, if there exists $p_0 \in X^*$ such that any sequence $\{x_n\}_{n \geqslant 1}$ in F satisfying $\lim_{n \to +\infty} \langle p_0, x_n\rangle = \sup_{x \in F} \langle p_0, x\rangle$ converges to x_0 in the norm topology. In such case we say that the functional p_0 strongly exposes x_0 in F. We denote by $\operatorname{se}(F)$ the set of strongly exposed points of F.

We now introduce the notion of a *weakly exposed* point, which will be useful in the sequel, see Lemma 5.

Definition 2. *Let F be a closed subset of X. A point $x_0 \in F$ is called a weakly exposed point in F, if there exists $p_0 \in X^*$ such that any sequence $\{x_n\}_{n \geqslant 1}$ in F with $\lim_{n \to +\infty} \langle p_0, x_n\rangle = \sup_{x \in F} \langle p_0, x\rangle$ weakly converges to x_0.*

In the case of the above definition we say that the functional p_0 weakly exposes x_0 in F. It follows easily that p_0 attains its unique maximum on F at x_0, hence in particular x_0 is an extreme point of F. We denote by $\mathrm{we}(F)$ the set of weakly exposed points of F. Furthermore, a point x_0 is called a *point of continuity* of F, if the identity mapping $\mathrm{id} : (F, \Im_w) \to (F, \Im_{\|\cdot\|})$ is continuous, where $\Im_w$ (respectively, $\Im_{\|\cdot\|}$) denotes the relative weak (respectively, norm) topology of F. It follows directly that x_0 is a strongly exposed point of F if, and only if, it is both weakly exposed and a point of continuity of F. Finally, a point x_0 is called *weakly denting* (or *strongly extreme*, according to the terminology in [**4**, p.67]), if for any relatively weakly open subset W in F containing x_0 there exist $p \in X^*$ and $\alpha > 0$ such that the set $\{x \in F : \langle p, x\rangle > \langle p, x_0\rangle - \alpha\}$ is included in W.

3. Proof of the main result

The proof of Theorem 1 is based on the following result of Bourgain [**3**, Chapter 1; Theorem 4]. (For a proof in English, see [**4**, Corollary 3.7.6].)

Theorem 3. *A Banach space X has the Radon-Nikodym property if, and only if, every nonempty closed convex bounded subset F of X has at least one weakly denting point.*

We can easily deduce the following corollary. The analogous result for dual Banach spaces is given in [**8**, Theorem 4].

Corollary 4. *For a Banach space X, the following are equivalent:*

(i) *X has the Radon-Nikodym property*

(ii) *Every closed convex bounded subset of X is the closed convex hull of its weakly exposed points.*

(iii) *Every nonempty closed convex bounded subset of X has at least one weakly exposed point.*

Proof: It is known ([**4**, Corollary 3.5.7], [**10**, Theorem 5.21], for example) that a Banach space X has the Radon-Nikodym property if, and only if, every closed convex bounded subset of X is the closed convex hull of its strongly exposed points. This shows that (i)$\Longrightarrow$(ii). Implication (ii)$\Longrightarrow$(iii) is trivial, while (iii)$\Longrightarrow$(i) follows from Theorem 3 and the observation that every weakly exposed point of F is weakly denting. ∎

Remark 1. A weakly denting point is not in general weakly exposed, even in finite dimensions. Indeed, let $X = \mathbb{R}^2$, $F = \{(x_1, x_2) : f(x_1) \leqslant x_2 \leqslant g(x_1)\}$, where $f(x) = \max\{0, x^3\}$ and $g(x) = x + 1$, and $\overline{x} = (0,0)$. Then $\overline{x}$ is a weakly denting point of the compact convex set F, without being weakly exposed.

Remark 2. A Banach space X has the Radon-Nikodym property if, and only if, for every nonempty closed convex bounded subset F of X we have $\mathrm{se}(F) \neq \emptyset$. However, if X does not have the Radon-Nikodym property, then the fact that $\mathrm{we}(F) \neq \emptyset$ (or even that $\overline{\text{co}}\,(\text{we}\,(F)) = F)$ for some closed convex bounded subset F of X does not necessarily imply that $\text{se}(F) \neq \emptyset$. (Consider the subset F of $c_0(\mathbb{N})$ defined by (12) in Proposition 8 and Claims 1 and 2 therein.)

We shall finally need the following lemma.

Lemma 5. *Let X be a Banach space and F be any non-empty closed convex bounded subset X. Then the following are equivalent:*

(i) *The function ψ_F^* is Gâteaux differentiable at p_0 with derivative $x_0 \in X$.*

(ii) *$x_0 \in F$ and the functional p_0 is weakly exposing x_0 in F.*

Proof: (i)$\Longrightarrow$(ii): Assume that (i) holds. Since $x_0 = \nabla^G \psi_F^*(p_0)$ (where $\nabla^G \psi_F^*$ denotes the Gâteaux derivative of ψ_F^*), we obviously have $x_0 \in \partial\psi_F^*(p_0)$, that is for all $p \in X^*$

$$\psi_F^*(p) - \psi_F^*(p_0) \geqslant \langle p - p_0, x_0 \rangle.$$

For $p = 0$ we obtain

$$\psi_F^*(p_0) := \sup_{x \in F} \langle p_0, x \rangle = \langle p_0, x_0 \rangle. \tag{3}$$

Let now $\{x_n\}_{n \geqslant 1}$ be a sequence in F such that

$$\lim_{n \to +\infty} \langle p_0, x_n \rangle = \sup_{x \in F} \langle p_0, x \rangle. \tag{4}$$

It suffices to show that $\{x_n\}_{n \geqslant 1}$ weakly converges to x_0. (Then, since the weak and the norm closure of the convex set F coincide, it will also follow that $x_0 \in F$.)

Let us assume, towards a contradiction, that there exists a subsequence $\{x_{n_k}\}_{k \geqslant 1}$ of $\{x_n\}_{n \geqslant 1}$, $h \in X^*$ and $\alpha > 0$ such that for all $k \geqslant 1$

$$\langle h, x_{n_k} \rangle - \langle h, x_0 \rangle > \alpha. \tag{5}$$

Thanks to (3) and (4), we can consider $\varepsilon_n \searrow 0^+$ in a way that

$$\langle p_0, x_n \rangle \geqslant \langle p_0, x_0 \rangle - \varepsilon_n. \tag{6}$$

Since $\psi_F^*(p) \geqslant \langle p, x_n \rangle$, using (3) we get

$$\psi_F^*(p) \geqslant \psi_F^*(p_0) + \langle p, x_n \rangle - \langle p_0, x_0 \rangle,$$

which in view of (6) yields

$$\psi_F^*(p) \geqslant \psi_F^*(p_0) + \langle p - p_0, x_n \rangle - \varepsilon_n. \tag{7}$$

Set $t_n = 2\varepsilon_n/\alpha$. Then for $p = p_0 + t_n h$ relation (7) yields

$$(\psi_F^*)(p_0 + t_n h) - (\psi_F^*)(p_0) \geqslant \langle t_n h, x_n \rangle - \varepsilon_n$$

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for all $n \geqslant 1$. In view of (5) this implies

$$\frac{(\psi_F^*)(p_0 + t_{n_k} h) - (\psi_F^*)(p_0)}{t_{n_k}} - \langle h, x_0 \rangle \geqslant \frac{\alpha}{2} > 0$$

for all $k \geqslant 1$. It follows that x_0 is not the Gâteaux derivative of ψ_F^* at p_0, hence a contradiction.

(ii)$\to$(i): Suppose that p_0 is weakly exposing x_0 in F, hence in particular $\langle p_0, x_0 \rangle = \sup_{x \in F} \langle p_0, x \rangle$. It follows easily from (2) that $x_0 \in \partial \psi_F^*(p_0)$. Let us now suppose that (i) does not hold. Then there exist $\varepsilon > 0$, $h \in X^*$ with $\|h\| \leqslant 1$ and $t_n \searrow 0^+$ such that

$$(\psi_F^*)(p_0 + t_n h) - (\psi_F^*)(p_0) > \langle t_n h, x_0 \rangle + \varepsilon t_n. \tag{8}$$

For every $n \geqslant 1$, choose x_n in F such that

$$\langle p_0 + t_n h, x_n \rangle > (\psi_F^*)(p_0 + t_n h) - \frac{t_n}{n}. \tag{9}$$

Since $(\psi_F^*)(p_0) \geqslant \langle p_0, x_n \rangle$, the above inequality yields

$$\langle p_0 + t_n h, x_n \rangle - \langle p_0, x_n \rangle > (\psi_F^*)(p_0 + t_n h) - (\psi_F^*)(p_0) - \frac{t_n}{n}.$$

Hence

$$\langle t_n h, x_n \rangle > (\psi_F^*)(p_0 + t_n h) - (\psi_F^*)(p_0) - \frac{t_n}{n}. \tag{10}$$

Combining (8) and (10) we conclude

$$\langle h, x_n - x_0 \rangle > \varepsilon - \frac{1}{n},$$

which shows that $\{x_n\}_{n \geqslant 1}$ does not weakly converge to x_0. However, since the sequence $\{x_n\}_{n \geqslant 1}$ is bounded and the function ψ_F^* is continuous, relation (9) yields $\lim_{n \to +\infty} \langle p_0, x_n \rangle = (\psi_F^*)(p_0)$, obtaining thus a contradiction to Definition 2. ∎

Remark. The above proof was inspired from techniques developed in [2] where a connection between well-posed problems and differentiability was established. Results in the same spirit are also established in [6, Section 1], via a different approach. We are grateful to C. Zălinescu for bringing the aforementioned reference to our attention.

Proof of Theorem 1: The "only if" part follows from the result of Collier [5] and the fact that the Fréchet derivatives of weak*-lower semicontinuous convex continuous functions on X^* always belong to the predual space X (see [7], for example).

For the "if" part, let F be any closed convex bounded subset of X. Then the function ψ_F^* of X^* (given in (1)) is weak*-lower semicontinuous convex and continuous. From our hypothesis and Lemma 5 we conclude that $\mathrm{we}(F) \neq \emptyset$. Since F is arbitrary, Corollary 4 asserts that X has the Radon-Nikodym property. ∎

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Let us recall that a Banach space X is called *weakly sequentially complete*, if every weakly Cauchy sequence of X is weakly converging in X. A typical example of a non-reflexive weakly sequentially complete Banach space is the space $L^1(\mu)$, where μ is a σ-finite positive measure. The following remark is due to Godefroy.

COROLLARY 6. *Let X be a weakly sequentially complete Banach space. Then X has the Radon-Nikodym property if, and only if, every weak*-lower semicontinuous convex continuous function on X^* is Gâteaux differentiable at some point of its domain.*

PROOF: The "only if" part is a direct consequence of Theorem 1. The "if" part follows from the following observation: if F is a nonempty closed convex bounded subset of X, and if $\nabla^G\psi_F^*(p)$ is the Gâteaux derivative of the function ψ_F^* at $p \in X^*$, then there exists $\{x_n\}_{n\geqslant 1}$ in F that weakly*-converges to $\nabla^G\psi_F^*(p)$ (see the proof of Lemma 5 (i)$\Longrightarrow$(ii)). It follows that $\{x_n\}_{n\geqslant 1}$ is a weakly Cauchy sequence, hence in view of our hypothesis $\nabla^G\psi_F^*(p) \in X$. (For similar considerations, see also [**9**].) We conclude by Lemma 5 (i)$\Longrightarrow$(ii) and Corollary 4 (iii)$\Longrightarrow$(i). ∎

Lemma 5 has also the following consequence. (The proof below is similar to [**10**, Theorem 5.20].)

COROLLARY 7. *Let F be a closed convex bounded subset of X. If ψ_F^* is Gâteaux differentiable in a dense subset of X^* with derivatives in X, then $F = \overline{\mathrm{co}}\big(\mathrm{we}\,(F)\big)$.*

PROOF: Since F is bounded, we have $\mathrm{dom}\,(\psi_F^*) = X^*$. (In particular the function ψ_F^* is convex and Lipschitz.) Since F is closed and convex, we have $\overline{\mathrm{co}}\big(\mathrm{we}\,(F)\big) \subseteq F$. Let us suppose, towards a contradiction, that there exists some x_0 in $F \setminus \overline{\mathrm{co}}\big(\mathrm{we}\,(F)\big)$. Then by applying the Hahn-Banach theorem, we can find $p \in X^*$ $(p \neq 0)$ and $\alpha \in \mathbb{R}$ such that

$$\sup\Big\{\langle p, x\rangle : x \in \overline{\mathrm{co}}\big(\mathrm{we}\,(F)\big)\Big\} < \alpha < \langle p, x_0\rangle.$$

Set $D = \{q \in X^* : \exists\nabla^G(\psi_F^*)(q) \in X\}$. Since D is dense in X^*, we can find $q \in D$ close to p such that $z := \nabla^G(\psi_F^*)(q) \in X$ and

$$\sup\Big\{\langle q, x\rangle : x \in \overline{\mathrm{co}}\big(\mathrm{we}\,(F)\big)\Big\} < \alpha < \langle q, x_0\rangle. \tag{11}$$

By Lemma 5 we conclude that $z \in \mathrm{we}\,(F)$ and that the functional q weakly exposes z. This clearly contradicts (11). ∎

The space $c_0(\mathbb{N})$ is a typical example of a Banach space without the Radon-Nikodym property. In this case, as already mentioned in Section 1, the norm $\|\cdot\|_1$ provides an example of a weak*-lower semicontinuous convex continuous function of $\ell^1(\mathbb{N})$, which is generically Gâteaux differentiable with all derivatives in $X^{**}\setminus X$. In the following proposition we give an example of a (nowhere Fréchet differentiable) weak*-lower semicontinuous convex continuous function of $\ell^1(\mathbb{N})$, which is Gâteaux differentiable with derivatives in the predual space in a dense set.

PROPOSITION 8. *Let $X = c_0(\mathbb{N})$. Then there exists a weak*-lower semicontinuous convex continuous function $f : X^* \to \mathbb{R}$ such that:*

(i) *there exists a dense subset D of X^* such that f is Gâteaux differentiable at every point of D with derivative in the predual space ;*

(ii) *f is nowhere Fréchet differentiable.*

PROOF: Set $X = c_0(\mathbb{N})$ and consider the set

$$F = B_X \cap c_0^+(\mathbb{N}) := \{x = (x^i)_i : \|x\|_\infty \leqslant 1 \text{ and } x^i \geqslant 0\ (\forall i \in \mathbb{N})\}. \tag{12}$$

It is easily seen that F is closed convex bounded and that

$$\operatorname{ext}(F) = \{x \in F : x^i \in \{0,1\} \text{ for all } i\}$$

where $\operatorname{ext}(F)$ denotes the set of the extreme points of F.

CLAIM 1. Let $\overline{x} \in \operatorname{ext}(F)$ and consider the finite set

$$I_{\overline{x}} = \{i \in \mathbb{N} : \overline{x}^i = 1\}. \tag{13}$$

Then any functional $p = (p^i)_i$ of $X^* := \ell^1(\mathbb{N})$ satisfying

$$\begin{aligned} p^i &> 0, \quad \text{if } i \in I_{\overline{x}} \\ p^i &< 0, \quad \text{if } i \in \mathbb{N}\backslash I_{\overline{x}} \end{aligned} \tag{14}$$

weakly exposes the point $\overline{x}$. In particular $\operatorname{ext}(F) = \operatorname{we}(F)$ (and hence $\operatorname{we}(F) \neq \emptyset$).

PROOF OF CLAIM 1: Let $\overline{x} \in \operatorname{ext}(F)$, $I_{\overline{x}} = \{i \in \mathbb{N} : \overline{x}^i = 1\}$ and consider any p in $\ell^1(N)$ satisfying (14). We first note that for all $x \in F$ and all $i \in \mathbb{N}$ we have

$$p^i x^i \leqslant p^i \overline{x}^i. \tag{15}$$

It follows that $\langle p, x\rangle \leqslant \langle p, \overline{x}\rangle$, for all $x \in F$, that is, $\langle p, \overline{x}\rangle = \sup\limits_{x\in F} \langle p, x\rangle$. Take now any sequence $\{x_n\}_{n\geqslant 1}$ in F such that $\lim\limits_{n\to+\infty} \langle p, x_n\rangle = \langle p, \overline{x}\rangle$. We show that

$$\lim_{n\to+\infty} x_n^i = \overline{x}^i \tag{16}$$

for all $i \geqslant 0$. Indeed, assume that for some i_0 (16) does not hold. Then there exist a subsequence $\{x_{n_k}^{i_0}\}_{k\geqslant 1}$ of $\{x_n^{i_0}\}_{n\geqslant 1}$, $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$

$$\left|x_{n_k}^{i_0} - \overline{x}^{i_0}\right| > \frac{\varepsilon}{|p^{i_0}|}.$$

Using (15) we infer that

$$p^{i_0}\, x_{n_k}^{i_0} < p^{i_0}\, \overline{x}^{i_0} - \varepsilon.$$

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Combining with (15) we get $\langle p, x_{n_k}\rangle < \langle p, \overline{x}\rangle - \varepsilon$, for all $k \geqslant k_0$. This contradicts the fact that $\langle p, x_n\rangle \to \langle p, \overline{x}\rangle$. It follows that (16) holds for all $i \geqslant 0$. Since the sequence $\{x_n\}_{n\geqslant 1}$ is bounded, we conclude from (16) that $x_n \xrightarrow{w} \overline{x}$. Hence the functional p is weakly exposing $\overline{x}$ in F. Since every weakly exposed point is obviously extreme, the proof of the claim is complete. □

Claim 2: $\mathrm{se}\,(F) = \emptyset$.

Proof of Claim 2: It clearly suffices to show that any point $\overline{x}$ in $\mathrm{we}\,(F)$ is not a point of continuity for F. To this end, take any $\overline{x} \in \mathrm{we}\,(F)$ and consider the sequence $\{x_n\}_{n\geqslant 1}$ in F with

$$x_n^i = \begin{cases} 1 & \text{if } i \in I_{\overline{x}} \cup \{n\} \\ 0 & \text{elsewhere} \end{cases}$$

where $I_{\overline{x}}$ is given by (13). Then it follows easily that $x_n \xrightarrow{w} \overline{x}$. On the other hand, for n sufficiently large, we have $\|x_n - \overline{x}\|_\infty = 1$. □

Consider now the weak*-lower semicontinuous convex continuous function $\psi_F^* : \ell^1(\mathbb{N}) \to \mathbb{R}$ defined by

$$\psi_F^*(p) := \sup_{x\in F} \langle p, x\rangle = \|p_+\|_1 \tag{17}$$

where $\|\cdot\|_1$ is the usual norm of $\ell^1(\mathbb{N})$ and

$$p_+^i = \begin{cases} p^i & \text{if } p^i > 0 \\ 0 & \text{if } p^i \leqslant 0. \end{cases}$$

Let us denote by D the set of all functionals $p = (p^i)_i$ in $\ell^1(\mathbb{N})$ satisfying (14) for some finite (possibly empty) subset I of $\mathbb{N}$. For every such functional p, consider the point $\overline{x} = (\overline{x}^i)_i$ of $c_0(\mathbb{N})$ defined by

$$\overline{x}^i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{if } i \in \mathbb{N}\backslash I. \end{cases}$$

Then $\overline{x} \in F$ and $I = I_{\overline{x}}$ (where $I_{\overline{x}}$ is given in (13)). It follows by Claim 1 that the functional p weakly exposes $\overline{x}$. Applying Lemma 5 (ii)$\to$(i) we conclude that $\overline{x}$ is the Gâteaux derivative of ψ_F^* at p.

We now show that D is dense in $\ell^1(\mathbb{N})$. Indeed, take any $q = (q^i)_i$ in $\ell^1(\mathbb{N})$ and any $\varepsilon > 0$. Then for some $n_0 \in \mathbb{N}$ we have:

$$\|q\|_1 \leqslant \sum_{i=0}^{n_0} |q^i| + \frac{\varepsilon}{2}.$$

Consider now the functional $p = (p^i)_i$ defined by

$$p^i = \begin{cases} q^i & \text{if } i \leqslant n_0 \text{ and } q^i \neq 0 \\ -\dfrac{\varepsilon}{2^{i+2}} & \text{elsewhere.} \end{cases}$$

It is easily seen that $p \in D$. Moreover,

$$\|q-p\|_1 = \sum_{i=0}^{+\infty} |q^i - p^i| \leqslant \sum_{i>n_0}^{+\infty} |q^i| + \sum_{i=0}^{+\infty} \frac{\varepsilon}{2^{i+2}} \leqslant \varepsilon.$$

We have shown that the function ψ_F^* is densely Gâteaux differentiable with derivatives in the predual space X. On the other hand, since by Claim 2 the set F has no strongly exposed points, it follows from [**1**, p. 450] that ψ_F^* is nowhere Fréchet differentiable.

Let us finally note that the function ψ_F^* is in fact generically Gâteaux differentiable. Indeed, it is easily seen that for every $p = (p^i)_i$ with $p^i \neq 0$ for all i, we have $\nabla^G \psi_F^*(p) = z^{**}$ where $z^{**} \in \ell^\infty(\mathbb{N})$ is given by

$$(z^{**})^i = \begin{cases} 1 & \text{if } p^i > 0 \\ 0 & \text{if } p^i < 0 \end{cases}$$

∎

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