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Compact Groups of Operators on Subproportional Quotients of l_1^m

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Abstract. It is proved that a "typical" n-dimensional quotient X_n of l_1^m with $n = m^\sigma$, $0 < \sigma < 1$, has the property

$$\text{Average} \int_G \|Tx\|_{X_n}\, dh_G(T) \geq \frac{c}{\sqrt{n \log^3 n}} \Bigl(n - \int_G |\operatorname{tr} T|\, dh_G(T) \Bigr),$$

for every compact group G of operators acting on X_n, where $d_G(T)$ stands for the normalized Haar measure on G and the average is taken over all extreme points of the unit ball of X_n. Several consequences of this estimate are presented.

1 Introduction

The fact that "typical" quotients of l_1^m play a special role in the local theory of Banach spaces was established by Gluskin in his ground breaking paper [G1] on the diameters of Minkowski compacts. Soon after, it was observed that such quotients are "rigid"—*i.e.*, they allow only a "few" well bounded operators, [S1], [G2], [S2], [M1], [B1], [M2]. On the other hand, it was shown by Bourgain that the techniques developed for "typical" quotients of l_1^m can be used in the context of general finite-dimensional Banach spaces, [B2], which lead to several interesting results both in the local and structural theory of Banach spaces, [MT1], [MT2], [MT3]. For more information on this subject the reader is refered to [MT4].

Several properties of finite-dimensional Banach spaces within the local theory of Banach spaces are described by means of some classes of compact groups of operators acting "well boundedly" on the spaces in question *cf. e.g.*, [GG], [BKPS]. In this paper we study the behavior of compact groups of operators acting on subproportional quotients of l_1^m, *i.e.*, n-dimensional quotients with $n = m^\sigma$, for some $0 < \sigma < 1$. We prove that "typical" such a quotient X_n has the property that for every compact group G of operators acting on it, the following estimate holds

$$\text{Average} \int_G \|Tx\|_{X_n}\, dh_G(T) \geq \frac{c}{\sqrt{n \log^3 n}} \Bigl(n - \int_G |\operatorname{tr} T|\, dh_G(T) \Bigr),$$

where the average is taken over all extreme points of the unit ball of X_n, h_G stands for the normalized Haar measure on G and $\operatorname{tr} T$ denotes the trace of T, Theorem 2.2 below. As a consequence we derive, Theorem 2.7, that for every sufficiently nontrivial compact group

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G of operators acting on such a quotient (*e.g.*, a group with no 1-dimensional invariant subspaces) we have

$$\text{Average} \int_G \|Tx\|_{X_n}\, dh_G(T) \geq c' \sqrt{n/\log^3 n}.$$

On the other hand, it is proved in Theorem 2.8, that if for some compact group G of operators acting on such X_n

$$\text{Average} \int_G \|Tx\|_{X_n}\, dh_G(T) \leq A$$

then there exists a linear subspace $H \subset X_n$ with $\dim H \geq \max\{0, n - CA\sqrt{n \log^3 n}\}$ such that $G|H$ is trivial, *i.e.*, $G|H$ either consist of $\pm \mathrm{Id}_H$ or Id_H only.

We shall use the standard notation as in [P], [T]. As it is a general practice in the context of random quotients of l_1^n, we shall consider only the spaces over reals, however the corresponding results for the complex case can be obtained along the same lines after a standard modification.

2 Main Results

We shall study finite-dimensional Banach spaces, which will be represented as $\mathbb{R}^n$ equipped with a suitable norm $\|\cdot\|$. In particular, by $\|\cdot\|_2$ we shall denote the standard Euclidean norm on $\mathbb{R}^n$. For a linear subspace $E \subset \mathbb{R}^n$, by $E^\perp$ and P_E we shall denote the orthogonal complement of E in $\mathbb{R}^n$ and the orthogonal projection in $\mathbb{R}^n$ onto E, respectively. The space of all linear operators acting on $\mathbb{R}^n$ will be denoted by $L(\mathbb{R}^n)$. For a finite-dimensional Banach space $X = (\mathbb{R}^n, \|\cdot\|_X)$ and a linear operator $T \in L(\mathbb{R}^n)$, the norm of T as an operator acting on X will be denoted by $\|T\|_X$. If X is $\mathbb{R}^n$ equipped with the standard Euclidean norm then $\|T\|_2$ will stand for the norm of T in X. For a compact group of operators $G \subset L(\mathbb{R}^n)$ by h_G we shall denote the normalized Haar measure on G. Finally, the trace of an operator $T \in L(\mathbb{R}^n)$ will be denoted by $\mathrm{tr}\, T$.

Recall, *cf.* [S2], [M1], [MT4], that an operator $T \in L(\mathbb{R}^n)$ is said to be (k, α)-mixing if and only if there exists a k-dimensional linear subspace $E \subset \mathbb{R}^n$ such that $\mathrm{dist}(Tx, E) = \|P_{E^\perp} Tx\|_2 \geq \alpha \|x\|_2$, for every $x \in E$. Furthermore, for $T \in L(\mathbb{R}^n)$ we define

$$\mathrm{Mix}_n(T) = \max\{\alpha k \mid T \text{ is } (k, \alpha)\text{-mixing}\}.$$

For a finite dimensional Banach space X by $\mathrm{Ex}(X)$ we shall denote the set of extreme points of the unit ball B_X of X and by $\mathrm{e}(X)$ its cardinality. Clearly, $\mathrm{e}(X) = m < \infty$ if and only if X is isometric to a quotient of l_1^m. The following theorem is a generalization of Theorem 1.4 in [S2].

Theorem 2.1 *There are constants $C > 1$ and $c > 0$ such that for every $n > 2$ there exists an n-dimensional Banach space $X_n = (\mathbb{R}^n, \|\cdot\|_{X_n})$ satisfying the properties:*

(i) $\mathrm{e}(X_n) = n^2$,

(ii) *for every* $x \in \mathbb{R}^n$

$$\frac{1}{2}\|x\|_2 \le \|x\|_{X_n} \le C\sqrt{\frac{n}{\log n}}\,\|x\|_2,$$

(iii)

$$\operatorname{card}\Big\{x \in \operatorname{Ex}(X_n) \mid \|Tx\|_{X_n} \ge \frac{c\,\operatorname{Mix}_n(T)}{\sqrt{n\log n}}\Big\} \ge \frac{\mathrm{e}(X_n)}{2},$$

for every $T \in L(\mathbb{R}^n)$.

The proof of this theorem is postponed to the next two sections. In fact, we shall prove a probabilistic version of it. Namely, Theorem 3.5 below states that "most of" n-dimensional quotients of l_1^m with $m = n^2$ satisfy the requirements of Theorem 2.1.

Remark The theorem remains valid if in (i) we require that $\mathrm{e}(X_n) = n^{1+\delta}$ for arbitrary fixed $\delta > 0$ (with both C and c depending on δ).

Remark Clearly, (ii) implies that the lower estimate for the norm in (iii) is up to a constant optimal.

Remark (ii) and (iii) yield that the Banach-Mazur distance $\mathrm{d}(X_n, l_n^2)$ is of order $\sqrt{n/\log n}$.

In the sequel by $\mathcal{X}_n$ we shall denote the class of all n-dimensional Banach spaces satisfying the conditions (i)–(iii) of Theorem 2.1. As a consequence, for compact groups of operators we have

Theorem 2.2 *There are constants $C > 1$ and $c_1, c_2 > 0$ such that for every Banach space $X_n \in \mathcal{X}_n$ and every compact group G of operators acting on X_n one has*

$$\frac{1}{\mathrm{e}(X_n)} \sum_{x\in \operatorname{Ex}(X_n)} \int_G \|Tx\|_{X_n}\, dh_G(T) \ge \frac{c_1}{\sqrt{n\log n}} \int_G \operatorname{Mix}_n(T)\, dh_G(T)$$
$$\ge \frac{c_2}{\sqrt{n\log^3 n}} \Big(n - \int_G |\operatorname{tr} T|\, dh_G(T)\Big). \tag{2.1}$$

In particular,

$$\sup_{\|x\|_{X_n}=1} \int_G \|Tx\|_{X_n}\, dh_G(T) \ge \frac{c_1}{\sqrt{n\log n}} \int_G \operatorname{Mix}_n(T)\, dh_G(T)$$
$$\ge \frac{c_2}{\sqrt{n\log^3 n}} \Big(n - \int_G |\operatorname{tr} T|\, dh_G(T)\Big).$$

Proof Fix $X_n \in \mathcal{X}_n$ and an arbitrary compact group G of operators acting on X_n. Clearly, it is enough to prove (2.1). To this end observe that by the definition of $\mathcal{X}_n$ we have

$$\frac{1}{\mathrm{e}(X_n)} \sum_{x\in \operatorname{Ex}(X_n)} \int_G \|Tx\|_{X_n}\, dh_G(T) = \frac{1}{n^2} \int_G \sum_{x\in \operatorname{Ex}(X_n)} \|Tx\|_{X_n}\, dh_G(T)$$
$$\ge \int_G \frac{c\,\operatorname{Mix}_n(T)}{2\sqrt{n\log n}}\, dh_G(T),$$

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which proves the left hand side inequality. To prove the remaining part of (2.1) note that by Theorem 3.4 in [M2] there is a numerical constant $c' > 0$ such that for every $T \in G$

$$\mathrm{Mix}_n(T) + \mathrm{Mix}_n(T^{-1}) \geq \frac{c'(n - |\operatorname{tr} T|)}{\log n}.$$

Hence

$$\begin{aligned}\int_G \mathrm{Mix}_n(T)\, dh_G(T) &= \frac{1}{2}\int_G \big(\mathrm{Mix}_n(T) + \mathrm{Mix}_n(T^{-1})\big)\, dh_G(T)\\ &\geq \frac{c'}{2\log n}\int_G (n - |\operatorname{tr} T|)\, dh_G(T),\end{aligned}$$

which yields the second estimate. ∎

For $k, m \in \mathbb{N}$, $1 \leq k \leq m$ let $G_{m,k}$ denote the Grassmann manifold of k-dimensional subspaces of $\mathbb{R}^m$ equipped with the normalized Haar measure $\mu_{m,k}$. For a linear subspace $E \subset \mathbb{R}^m$ the quotient of l_1^m by E will be denoted by l_1^m/E. For $n \in \mathbb{N}$ and $c > 0$ define

$$\begin{aligned}\mathcal{Y}_{n,c} = \Big\{ & Z_n = l_1^{n^2}/E \mid E \in G_{n^2,n^2-n} \text{ such that}\\ & \frac{1}{\mathrm{e}(Z_n)} \sum_{x\in \mathrm{Ex}(Z)} \int_G \|Tx\|_{Z_n}\, dh_G(T) \geq \frac{c}{\sqrt{n\log^3 n}}\Big(n - \int_G |\operatorname{tr} T|\, dh_G(T)\Big)\\ & \text{for every compact group } G \text{ of operators acting on } Z_n\Big\}.\end{aligned}$$

In fact, the argument used to prove Theorem 2.1 yields as well (*cf.* Remark following Theorem 3.5)

Theorem 2.3 *There are numerical constants $c, c' > 0$ such that*

$$\mu_{n^2,n^2-n}(\mathcal{Y}_{n,c}) \geq 1 - e^{-c'n}$$

for every $n \in \mathbb{N}$.

For irreducible groups of operators we have

Theorem 2.4 *For every $X_n \in \mathcal{X}_n$ and for every group G of compact operators acting irreducible on X_n*

$$\frac{1}{\mathrm{e}(X_n)} \sum_{x\in \mathrm{Ex}(X_n)} \int_G \|Tx\|_{X_n}\, dh_G(T) \geq \frac{c}{800}\sqrt{\frac{n}{\log n}},$$

where c is the constant from Theorem 2.1.

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Proof By [M3, Theorem 3.1], for every compact group G of operators acting irreducible on X_n we have

$$h_G\Big\{T \in G \mid \mathrm{Mix}_n(T) \geq \frac{n}{80}\Big\} \geq \frac{1}{5}$$

and the theorem follows from Theorem 2.2, (2.1). ■

For an arbitrary fixed basis $\{x_i\}_{i=1}^n$ in an n-dimensional Banach space Y_n with dual functionals $\{x_i^*\}_{i=1}^n$ consider the compact group $G_{\{x_i\}}$ of operators on Y_n of the form

$$G_{\{x_i\}} = \Big\{T \in L(\mathbb{R}^n) \Bigm| T = \sum_{i=1}^n \varepsilon_i x_i^*(\cdot)x_i,\ \varepsilon_i \in \{-1, 1\} \text{ for } i = 1, 2, \ldots, n\Big\}.$$

Similarly as in the Theorem above, by [M3, Theorem 3.3], we have (*cf.* [BKPS], [B])

Theorem 2.5 *There exists a numerical constant $c > 0$ such that for every Banach space $X_n \in \mathfrak{X}_n$ and every basis $\{x_i\}_{i=1}^n$ in X_n one has*

$$\frac{1}{\mathrm{e}(X_n)} \sum_{x \in \mathrm{Ex}(X_n)} \int_{G(\{x_i\})} \|Tx\|_{X_n}\, dh_{G(\{x_i\})}(T) \geq c\sqrt{\frac{n}{\log n}}.$$

In particular,

$$\mathrm{ruc}(X_n) = \inf \sup_{\|x\|_{X_n}=1} \int_{G(\{x_i\})} \|Tx\|_{X_n}\, dh_{G(\{x_i\})}(T) \geq c\sqrt{\frac{n}{\log n}},$$

where infimum is taken over all bases in X_n.

Remark By the last Remark following Theorem 2.1 the Banach-Mazur distance of X_n to l_n^2 is of order $\sqrt{n/\log n}$. Hence the estimates in Theorems 2.4 and 2.5 are sharp up to a multiplicative constant.

Clearly, the right hand side inequality in Theorem 2.2, (2.1) is not sharp and cannot yield an optimal estimate. Before we shall be able to present its typical applications we need some basic facts concerning compact groups of operators acting on $\mathbb{R}^n$ (*cf. e.g.*, [M2]). For a linear subspace $E \subset \mathbb{R}^n$ and an operator $T \in L(\mathbb{R}^n)$ by $T|E$ we shall denote the restriction of T to the subspace E.

Fact 1 *Let G be a compact group of operators acting on $\mathbb{R}^n$. Then*

(1°) there is another scalar product $\langle\cdot,\cdot\rangle_1$ on $\mathbb{R}^n$ such that G is a group of isometries on $(\mathbb{R}^n, \|\cdot\|_1)$, where $\|x\|_1 = \langle x, x\rangle_1^{1/2}$ for $x \in \mathbb{R}^n$,

(2°) there is a decomposition of $\mathbb{R}^n$ into an $\|\ \ \|_1$-orthogonal sum of G-invariant subspaces $\mathbb{R}^n = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ with the properties:

(i) the group $G|E_i = \{T|E_i \mid T \in G\}$ acts irreducibly on E_i for $i = 1, 2, \ldots, k$,

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(ii) if $U \in L(E_i)$ commutes with every element of $G|E_i$ then

$$\langle Ux, x\rangle_1 = (\dim E_i)^{-1} \operatorname{tr} U \|x\|_1^2,$$

for every $x \in E_i$, $i = 1, 2, \ldots, k$.

Lemma 2.6 *For every $r \geq 2$ and every irreducible compact group G of isometries acting on $\mathbb{R}^r$*

$$\int_G |\operatorname{tr} T| \, dh_G(T) \leq \frac{r}{\sqrt{2}}.$$

Proof Fix an irreducible compact group G of isometries acting on $\mathbb{R}^r$. For every $e \in \mathbb{R}^r$ with $\|e\|_2 = 1$ write $T_e = \langle e, \cdot\rangle e$ and set

$$U_e = \int_G T^{-1} T_e T \, dh_G(T).$$

Since $\operatorname{tr} U_e = \operatorname{tr} T_e = 1$ and U_e commutes with every $T \in G$, by Fact 1, we infer that $\langle U_e x, x\rangle = 1/r$ for every $x \in \mathbb{R}^r$ with $\|x\|_2 = 1$. Hence, for every $e \in \mathbb{R}^r$ with $\|e\|_2 = 1$

$$\frac{1}{r} = \langle U_e e, e\rangle = \int_G \langle T^{-1} T_e T e, e\rangle \, dh_G(T)$$
$$= \int_G \langle \langle e, Te\rangle e, Te\rangle \, dh_G(T) = \int_G \langle e, Te\rangle^2 \, dh_G(T).$$

Thus, by the Hölder inequality, for every $e \in \mathbb{R}^r$ with $\|e\|_2 = 1$ we have

$$\int_G |\langle e, Te\rangle| \, dh_G(T) \leq \frac{1}{\sqrt{r}}.$$

Therefore

$$\int_G |\operatorname{tr} T| \, dh_G(T) \leq \sum_{i=1}^r \int |\langle e_i, Te_i\rangle| \, dh_G(T) \leq \sqrt{r} \leq \frac{r}{\sqrt{2}}. \qquad \blacksquare$$

As a consequence of Theorem 2.2 (2.1) we have

Theorem 2.7 *For every Banach space $X_n \in \mathfrak{X}_n$ and for every compact group G of operators acting on X_n one has*

$$\frac{1}{\mathrm{e}(X_n)} \sum_{x \in \mathrm{Ex}(X_n)} \int_G \|Tx\|_{X_n} \, dh_G(T) \geq \frac{c_2 \dim E}{4\sqrt{n \log^3 n}}$$

for every G-invariant subspace $E \subset X_n$ which admits no 1-dimensional G-invariant subspaces, where c_2 is the constant from Theorem 2.2.

Proof Let $\langle \cdot, \cdot \rangle_1$ be the scalar product which makes G to be a group of isometries of $(\mathbb{R}^n, \| \cdot \|_1)$ and let $F = E^\perp$ be the $\langle \cdot, \cdot \rangle_1$-orthogonal complement of E. Note that for every G-invariant subspace $H \subset \mathbb{R}^n$ we have $| \operatorname{tr} T|H| \leq \dim H$. Therefore

$$\begin{aligned}n - \int_G |\operatorname{tr} T|\, dh_G(T) &= \dim F - \int_G |\operatorname{tr} T|F|\, dh_G(T) + \dim E - \int_G |\operatorname{tr} T|E|\, dh_G(T) \\ &\geq \dim E - \int_G |\operatorname{tr} T|E|\, dh_G(T).\end{aligned} \tag{2.2}$$

Thus, in view of Theorem 2.2, it suffices to show that

$$\int_G |\operatorname{tr} T|E|\, dh_G(T) \leq \frac{\dim E}{\sqrt{2}}.$$

To this end let $E = E_1 \oplus E_2 \oplus, \dots, \oplus E_k$ be a decomposition of E into $\langle \cdot, \cdot \rangle_1$-orthogonal sum of G-invariant G-irreducible subspaces. Since $\dim E_i \geq 2$ for $i = 1, 2, \dots, k$, by the previous lemma we have

$$\begin{aligned}\int_G |\operatorname{tr} T|E|\, dh_G(T) &= \sum_{i=1}^k \int_G |\operatorname{tr} T|E_i|\, dh_G(T) \\ &= \sum_{i=1}^k \int_{G_{E_i}} |\operatorname{tr} T|\, dh_i(T) \leq \frac{1}{\sqrt{2}} \sum_{i=1}^k \dim E_i \\ &= \frac{\dim E}{\sqrt{2}},\end{aligned}$$

where h_i for $i = 1, 2, \dots, k$ denotes the normalized Haar measure on $G|E_i$. ∎

Theorem 2.8 *There exists a constant $C_0 > 0$ such that for every $X_n \in \mathfrak{X}_n$ and every compact group G of operators acting on X_n satisfying*

$$\frac{1}{\mathrm{e}(X_n)} \sum_{x \in \mathrm{Ex}(X_n)} \int_G \|Tx\|_{X_n}\, dh_G(T) \leq A$$

there is a linear subspace $H \subset X_n$ with $\dim H \geq \max\{0, n - C_0 A \sqrt{n \log^3 n}\}$ such that $G|H$ acts trivially on H (i.e., $G|H$ consists of either Id_H or $\pm \mathrm{Id}_H$).

Proof Fix X_n and G satisfying the assumption of the theorem and let $X_n = E_1 \oplus E_2 \oplus \cdots \oplus E_m$ be a decomposition of X_n into an $\| \cdot \|_1$-orthogonal sum of G-irreducible invariant subspaces, where $\| \cdot \|_1$ is a suitable Euclidean norm, *cf.* Fact 1. Without any loss of generality we may assume that

$$\dim E_1 \geq \dim E_2 \geq \cdots \geq \dim E_{m-k_1-k_2} > \dim E_{m-k_1-k_2+1} = \cdots = \dim E_m = 1,$$

where for $j = m - k_1 - k_2 + 1, m - k_1 - k_2 + 2, \ldots, m - k_2$ we have $G|E_j = \pm \operatorname{Id}_{E_j}$ and $G|E_j = \operatorname{Id}_{E_j}$ for $j = m - k_2 + 1, m - k_2 + 2, \ldots, m$. Observe that by Theorem 2.2 and by (2.2), for every G-invariant subspace $E \subset X_n$ we have

$$A \geq \frac{c_2}{\sqrt{n \log^3 n}} \left(\dim E - \int_G |\operatorname{tr} T|E| \, dh_G(T) \right). \tag{2.3}$$

Set

$$F = \operatorname{lin}\{E_j \mid j = m - k_1 - k_2 + 1, m - k_1 - k_2 + 2, \ldots, m - k_2\}.$$

It is not difficult to see that F admits a unique decomposition

$$F = F_1 \oplus F_2 \oplus \cdots \oplus F_{m_0}$$

of minimal length m_0 such that $G|F_j = \pm \operatorname{Id}_{F_j}$ for every $j = 1, 2, \ldots, m_0$. For every $T \in G$ and for $j = 1, 2, \ldots, m_0$ define $\varepsilon_j(T)$ by the equality $T|F_j = \varepsilon_j(T) \operatorname{Id}_{F_j}$. The minimality of m_0 yields $\int_G \varepsilon_{j_1}(T) \varepsilon_{j_2}(T) \, dh_G(T) = 0$ for $j_1, j_2 = 1, 2, \ldots, m_0$, $j_1 \neq j_2$. In order to simplify notations assume that $\dim F_1 = \max\{\dim F_j \mid j = 1, 2, \ldots, m_0\}$. By the Jensen inequality

$$\begin{aligned} \left(\int_G |\operatorname{tr} T|F| \, dh_G(T) \right)^2 &\leq \int_G |\operatorname{tr} T|F|^2 \, dh_G(T) \\ &= \int_G \left(\sum_{j=1}^{m_0} \varepsilon_j(T) \dim F_j \right)^2 dh_G(T) = \sum_{j=1}^{m_0} (\dim F_j)^2 \\ &\leq k_1 \dim F_1 \leq n \dim F_1. \end{aligned} \tag{2.4}$$

Let $E_0 = \operatorname{lin}\{E_i \mid i = 1, 2, \ldots, m - k_1 - k_2\}$ and put $k_0 = \dim E_0$. By Theorem 2.7 we have

$$k_0 = \dim E_0 \leq \frac{4A}{c_2} \sqrt{n \log^3 n}.$$

Therefore, if $k_1 = \dim F \leq (8A\sqrt{n \log^3 n})/c_2$ then $k_2 \geq n - (12A\sqrt{n \log^3 n})/c_2$ and we are done. Thus, it remains to consider the case when $k_1 > (8A\sqrt{n \log^3 n})/c_2$. This case splits into two disjoint sub-cases

(A) $k_2 < (2A\sqrt{n \log^3 n})/c_2$,

(B) $k_2 \geq (2A\sqrt{n \log^3 n})/c_2$.

To establish the theorem in the sub-case (A) it suffices to note that $k_1 = \dim F \geq n - (6A\sqrt{n \log^3 n})/c_2$. Hence, by (2.3)

$$\int_G |\operatorname{tr} T|F| \, dh_G(T) \geq n - \frac{7A}{c_2} \sqrt{n \log^3 n} \tag{2.5}$$

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and by combining (2.5) and (2.4) we get

$$\dim F_1 \geq n - \frac{14A}{c_2}\sqrt{n\log^3 n}. \tag{2.6}$$

The last step is to prove that, in fact, the sub-case (B) cannot occur. Indeed, to see this assume that (B) holds. Observe that $\int_G |\mathrm{tr}\, T|F|\, dh_G(T) \leq k_1/2$ and $k_1 > (8A\sqrt{n\log^3 n})/c_2$ contradict (2.3). On the other hand, if $\int_G |\mathrm{tr}\, T|F|\, dh_G(T) > k_1/2$ then, by (2.4), $\dim F_1 > k_1/4 \geq (2A\sqrt{n\log^3 n})/c_2$. Let $\tilde{F}$ and $\tilde{H}$ be arbitrary $(2A\sqrt{n\log^3 n})/c_2$-dimensional subspaces of F_1 and $\mathrm{lin}\{E_j \mid j = m-k_2+1, m-k_2+2, \dots, m\}$ respectively. Then

$$\int_G |\mathrm{tr}\, T|\tilde{F}\oplus\tilde{H}|\, dh_G(T) = \frac{2A\sqrt{n\log^3 n}}{c_2} = \frac{\dim \tilde{F}\oplus\tilde{H}}{2},$$

which yields a contradiction with (2.3) and completes the proof of the theorem. ■

3 Technical Theorem

We begin with some notations. Fix a probability space $(\Omega, \mathbf{P})$. For every $n \in \mathbb{N}$ we shall consider a sequence of independent Gaussian vectors $\{g_{n,1}, g_{n,2}, \dots, g_{n,n^2}\}$ in $\mathbb{R}^n$, each with the distribution $\mathrm{N}(0, 1, \mathbb{R}^n)$, *i.e.*, with the density $(n/2\pi)^{n/2}\exp(-n\|x\|_2^2/2)$ with respect to the standard Lebesgue measure in $\mathbb{R}^n$. In order to simplify the notations, for a given $n \in \mathbb{N}$ we shall write $m = n^2$. We shall need the following well known properties of Gaussian vectors in $\mathbb{R}^n$. Let g be such a vector with the distribution $\mathrm{N}(0, 1, \mathbb{R}^n)$.

Fact 2

(i) *For every linear subspace E of $\mathbb{R}^n$ with $\dim E = k$, the random vector $\sqrt{n/k}\, P_E g$ is Gaussian with the distribution* $\mathrm{N}(0, 1, E)$.

(ii) *For every pair of linear orthogonal subspaces E_1 and E_2 in $\mathbb{R}^n$ the random vectors $P_{E_1} g$ and $P_{E_2} g$ are independent.*

(iii) *There is an universal constant $c' > 0$ such that*

$$\mathbf{P}\{\omega \in \Omega \mid 1/2 \leq \|g\|_2 \leq 2\} > 1 - e^{-c'n}.$$

(iv) *For every measurable subset $A \subset \mathbb{R}^n$*

$$\mathbf{P}\{\omega \in \Omega \mid g \in A\} \leq \left(\frac{n}{2\pi}\right)^{n/2} \mathrm{vol}\, A,$$

where vol A *denotes the n-dimensional Lebesgue measure of A.*

For each $n \in \mathbb{N}$ and $\omega \in \Omega$ we set

$$B_{n,\omega} = \mathrm{absconv}\{g_{n,1}, g_{n,2}, \dots, g_{n,m}\}.$$

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For each $n \in \mathbb{N}$, by Ω_n' denote the set $\{\omega \in \Omega \mid B_{n,\omega}$ is a convex body in $\mathbb{R}^n\}$. Clearly, $\mathbf{P}(\Omega_n') = 1$. For $\omega \in \Omega_n'$, let $\|\cdot\|_{n,\omega}$ be the norm on $\mathbb{R}^n$ with $B_{n,\omega}$ as its unit ball. In this way for each $n \in \mathbb{N}$, we have defined a random family of Banach spaces $X_{n,\omega} = (\mathbb{R}^n, \|\cdot\|_{n,\omega})$. Observe that each Banach space $X_{n,\omega}$ is canonically isometrically isomorphic to a quotient of l_1^m via the quotient map $Q_{n,\omega}$ defined by the equality

$$Q_{n,\omega}(e_i) = g_{n,i}(\omega) \quad \text{for } i = 1, 2, \dots, m,$$

where $\{e_i\}_{i=1}^m$ denotes the standard unit vector basis in $l_1^m = (\mathbb{R}^m, \|\cdot\|_1)$. It is not difficult to verify that for each $n \in \mathbb{N}$ the distribution of kernels of of $Q_{n,\omega}$ is rotational invariant in $\mathbb{R}^m$. Therefore, by the uniqueness of the Haar measure $\mu_{m,m-n}$ we have

$$\mu_{m,m-n}\{E \in G_{m,m-n} \mid E \in \mathcal{B}\} = \mathbf{P}\{\omega \in \Omega \mid \ker Q_{n,\omega} \in \mathcal{B}\} \tag{3.1}$$

for every Borel subset $\mathcal{B} \subset G_{m,m-n}$.

For each $n \in \mathbb{N}$ set

$$\Omega_n'' = \{\omega \in \Omega_n' \mid 1/2 \le \|g_{n,i}\|_2 \le 2 \text{ for every } i \le m\}$$

and define

$$\Omega_n''' = \left\{\omega \in \Omega_n'' \;\middle|\; \frac{1}{4}\sqrt{\frac{\log n}{n}} B_n^2 \subset B_{n,\omega}\right\}.$$

The following proposition is well known to specialists

Proposition 3.1 *There exists a constant $\tilde{c} > 0$ such that*

$$\mathbf{P}(\Omega_n''') > 1 - \exp(\tilde{c}n).$$

Proof Note that for a real Gaussian variable g with the distribution $\mathrm{N}(0, 1, \mathbb{R})$ we have

$$\mathbf{P}\left(\left\{\omega \in \Omega \;\middle|\; |g| \ge \frac{1}{2}\sqrt{\log n}\right\}\right) \ge \frac{1}{4\sqrt{n}},$$

and observe that by Fact 2, (i), for every $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$ and every $i \le m$ the Gaussian variable $\sqrt{n}\langle x, g_{n,i}\rangle$ has the $\mathrm{N}(0, 1, \mathbb{R})$ distribution. Thus, by the independence of the Gaussian variables $g_{n,i}$ we infer that

$$\mathbf{P}\left(\left\{\omega \in \Omega \;\middle|\; \sup_{i \le m}\{|\langle x, g_{n,i}\rangle|\} < \frac{1}{2}\sqrt{\frac{\log n}{n}}\right\}\right) < \left(1 - \frac{1}{4\sqrt{n}}\right)^m, \tag{3.2}$$

for every $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$.

Let $\mathcal{N}_n$ be the $(\sqrt{\log n/n}/8)$-net in the unit sphere S^{n-1} of $\mathbb{R}^n$ with $\operatorname{card}\mathcal{N}_n \le (40\sqrt{n/\log n})^n$, *cf. e.g.*[P]. Define

$$\mathcal{A}_{\mathcal{N}_n} = \left\{\omega \in \Omega_n'' \;\middle|\; \sup_{i \le m}\{|\langle x, g_{n,i}\rangle|\} \ge \frac{1}{2}\sqrt{\frac{\log n}{n}} \text{ for every } x \in \mathcal{N}_n\right\}.$$

Clearly,

$$\mathbf{P}(\mathcal{A}_{\mathcal{N}_n}) \geq \mathbf{P}(\Omega_n'') - \left(1 - \frac{1}{4\sqrt{n}}\right)^m \left(40\sqrt{\frac{n}{\log n}}\right)^n. \tag{3.3}$$

Now, fix arbitrary $x_0 \in S^{n-1}$ and $\omega \in \mathcal{A}_{\mathcal{N}_n}$ and choose $x \in \mathcal{N}_n$ with

$$\|x_0 - x\|_2 \leq \frac{1}{8}\sqrt{\frac{n}{\log n}}.$$

Since

$$\begin{aligned}\sup_{i \leq m}\{|\langle x_0, g_{n,i}(\omega)\rangle|\} &\geq \sup_{i \leq m}\{|\langle x, g_{n,i}(\omega)\rangle|\} - \|x_0 - x\|_2 \sup_{i \leq m}\{\|g_{n,i}\|_2\} \\ &\geq \frac{1}{4}\sqrt{\frac{\log n}{n}},\end{aligned}$$

we infer that for every $x \in S^{n-1}$ and $\omega \in \mathcal{A}_{\mathcal{N}_n}$ we have

$$\sup_{i \leq m}\{|\langle x, g_{n,i}(\omega)\rangle|\} \geq \frac{1}{4}\sqrt{\frac{\log n}{n}}.$$

By the standard separation argument, this yields that for $\omega \in \mathcal{A}_{\mathcal{N}_n}$, the ball $B_{n,\omega}$ contains $(\sqrt{\log n / n}\,/4)B_n^2$ and therefore $\mathcal{A}_{\mathcal{N}_n} \subset \Omega_n'''$. The proof is completed by combining Fact 2 (iii) with (3.3). ■

Remark Note that

$$d(X_{n,\omega}, l_2^n) \leq 8\sqrt{\frac{n}{\log n}}$$

for $\omega \in \Omega_n'''$.

Remark The same line of argument shows that in order to ensure that most of the unit balls $B_{n,\omega}$ contain $(c\sqrt{\log n / n})B_n^2$ for sufficiently small $c > 0$, it is enough to consider $m = n^{1+\varepsilon}$, with $\varepsilon = \varepsilon(c) > 0$, independent Gaussian variables in $\mathbb{R}^n$, while to obtain that most of $B_{n,\omega}$'s contain $(c/\sqrt{n})B_n^2$, for sufficiently small $c > 0$, it is enough to consider $m = Cn\log n$ with $C = C(c)$.

Proposition 3.2 *For $n \in \mathbb{N}$ let*

$$\Omega_n^0 = \{\omega \in \Omega_n''' \mid \mathrm{e}(X_{n,\omega}) = m\}.$$

Then there exists a numerical constant $c_0 > 0$ such that $\mathbf{P}(\Omega_n^0) \geq 1 - e^{-c_0 n}$.

Proof By the definition of the spaces $X_{n,\omega}$ we have that $\mathrm{e}(X_{n,\omega}) \leq m$ for every $\omega \in \Omega$. Clearly, if for some $\omega \in \Omega$ the cardinality of extreme points of the unit ball of $X_{n,\omega}$ is less than m then there exists a positive integer $j_0 \leq m$ such that $g_{n,j_0}(\omega) \in \operatorname{absconv}\{g_{n,i}(\omega) \mid i \neq j_0\}$. Set

$$B^{j_0}_{n,\omega} = \operatorname{absconv}\{g_{n,i}(\omega) \mid i \neq j_0\}.$$

By Fact 2 (iii) and the independence of random vectors $g_{n,I}$ for $i = 1, 2, \ldots, m$ we infer that

$$\mathbf{P}\{\omega \in \Omega'''_n \mid g_{n,j_0} \in B^{j_0}_{n,\omega}\} \leq \left(\frac{n}{2\pi}\right)^{n/2} \max \operatorname{vol} B^{j_0}_{n,\omega},$$

where maximum is taken over all $\omega \in \Omega'''_n$. On the other hand, by Theorem 1 in [BP] and the Santaló inequality we obtain that $\operatorname{vol} B^{j_0}_{n,\omega} \leq (c\sqrt{\log n}\,/n)^n$ for some numerical constant $c > 0$ and every $\omega \in \Omega'''_n$. Therefore

$$\mathbf{P}(\Omega^0_n) \geq \mathbf{P}(\Omega'''_n) - m(c\sqrt{\log n}\,/n)^n$$

and the proof is completed by applying the previous proposition. ∎

We shall need

Lemma 3.3 *For every $n \in \mathbb{N}$ and $\delta > 0$, let*

$$\tilde{\mathcal{A}}^\delta_n = \left\{\omega \in \Omega^0_n \;\middle|\; \operatorname{card}\left\{i \leq m \;\middle|\; |\langle g_{n,i}, e\rangle| < \frac{\delta}{\sqrt{n}}\right\} > \frac{m}{2} \right.$$
$$\left. \text{for some } e \in \mathbb{R}^n \text{ with } \|e\|_2 = 1\right\}.$$

Then there exists a numerical constant $c'' > 0$ such that $\mathbf{P}(\tilde{\mathcal{A}}^\delta_n) < e^{-c''n}$ for every $0 < \delta \leq 1/32$.

Proof Since $\tilde{\mathcal{A}}^{\delta_1}_n \subset \tilde{\mathcal{A}}^{\delta_2}_n$ for $\delta_1 < \delta_2$ it suffices to prove the proposition for $\delta = 1/32$.

Fix $n \in \mathbb{N}$, $\delta > 0$ and $e \in \mathbb{R}^n$ with $\|e\|_2 = 1$. Let

$$\mathcal{A}^\delta_e = \left\{\omega \in \Omega^0_n \;\middle|\; \operatorname{card}\left\{i \leq m \;\middle|\; |\langle g_{n,i}, e\rangle| < \frac{2\delta}{\sqrt{n}}\right\} > \frac{m}{2}\right\}.$$

Since, by Fact 2 (i), the random variables $\sqrt{n}\langle g_{n,i}, e\rangle$'s have the $\mathrm{N}(0, 1, \mathbb{R})$ distribution we infer that for every $i \leq m$

$$\mathbf{P}\left(\left\{\omega \in \Omega \;\middle|\; |\langle g_{n,i}, e\rangle| < \frac{2\delta}{\sqrt{n}}\right\}\right) = \mathbf{P}(\{\omega \in \Omega \mid |\sqrt{n}\langle g_{n,i}, e\rangle| < 2\delta\}) \leq 2\delta.$$

By the independence of the random vectors $g_{n,i}$ and by the binomial formulae, we get

$$\mathbf{P}(\mathcal{A}^\delta_e) < \sum_{j=m/2}^{m} \binom{m}{j} (2\delta)^j < 2^m (2\delta)^{m/2}, \tag{3.4}$$

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for $\delta < 1/2$.

Let $\mathcal{N}(n,\delta)$ be the $\delta/4\sqrt{n}$-net in the unit sphere S^{n-1} of $\mathbb{R}^n$ with

$$\operatorname{card} \mathcal{N}(n,\delta) \le (20\sqrt{n}/\delta)^n.$$

Set

$$\mathcal{A}_{\mathcal{N}(n,\delta)} = \bigcup_{e \in \mathcal{N}(n,\delta)} \mathcal{A}_e^{\delta}.$$

Assume that for some $\omega \in \Omega_n^0$, $i \le m$ and $e_1 \in \mathbb{R}^n$ with $\|e_1\|_2 = 1$ we have $|\langle g_{n,i}(\omega), e_1\rangle| < \delta/\sqrt{n}$, and choose $e \in \mathcal{N}(n,\delta)$ with $\|e_1 - e\|_2 \le \delta/4\sqrt{n}$. Then

$$|\langle g_{n,i}(\omega), e\rangle| \le |\langle g_{n,i}(\omega), e_1\rangle| + \|e_1 - e\|_2\, \|g_{n,i}(\omega)\|_2 < \frac{2\delta}{\sqrt{n}}.$$

This proves that $\tilde{\mathcal{A}}_n^{\delta} \subset \mathcal{A}_{\mathcal{N}(n,\delta)}$. To complete the proof it is enough to note that for $\delta = 1/32$ we get

$$\mathbf{P}(\tilde{\mathcal{A}}_n^{\delta}) \le \mathbf{P}(\mathcal{A}_{\mathcal{N}(n,\delta)}) \le 2^m (2\delta)^{m/2} (20\sqrt{n}/\delta)^n = 2^{-m}(640\sqrt{n})^n. \qquad \blacksquare$$

As a consequence, we have

Proposition 3.4 *For every $n \in \mathbb{N}$, let*

$$\tilde{\mathcal{A}}_n = \left\{ \omega \in \Omega_n^0 \;\middle|\; \operatorname{card}\left\{ i \le m \;\middle|\; \|Tg_{n,i}\|_{n,\omega} < \frac{1}{100\sqrt{n}} \right\} > \frac{m}{2} \right.$$
$$\left. \text{for some } T \in L(\mathbb{R}^n) \text{ with } \|T\|_2 \ge 1 \right\}.$$

Then $\mathbf{P}(\tilde{\mathcal{A}}_n) < e^{-c''n}$, where c'' is the numerical constant from the previous lemma.

Proof Fix $n \in \mathbb{N}$ and $T \in L(\mathbb{R}^n)$ with $\|T\|_2 \ge 1$ and assume that for some $\omega \in \Omega_n^0$ and $i \le m$ we have

$$\|Tg_{n,i}\|_{n,\omega} < \frac{1}{100\sqrt{n}}.$$

Write T in the polar decomposition form, *i.e.*,

$$Tx = \sum_{j=1}^{n} \lambda_j < u_j, x > v_j,$$

where $\{u_j\}_{j=1}^n$ and $\{v_j\}_{j=1}^n$ are orthonormal bases in $\mathbb{R}^n$ and $\|T\|_2 = \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$. Since $\|x\|_{n,\omega} \ge \frac{1}{2}\|x\|_2$ for every $x \in \mathbb{R}^n$ and $\omega \in \Omega_n^0$, we infer that

$$\frac{1}{100\sqrt{n}} > \|Tg_{n,i}\|_{n,\omega} \ge \frac{1}{2}\|Tg_{n,i}\|_2 \ge \frac{\lambda_1}{2}|\langle u_1, g_{n,i}\rangle| \ge \frac{1}{2}|\langle u_1, g_{n,i}\rangle|.$$

Hence $|\langle u_1, g_{n,i}\rangle| < 1/50\sqrt{n}$, which clearly implies that $\tilde{\mathcal{A}}_n \subset \tilde{\mathcal{A}}_n^{1/50}$ for every $n \in \mathbb{N}$ which, by Lemma 3.3, yields the required estimate. $\blacksquare$

The following is a "random" version of Theorem 2.1.

Theorem 3.5 *There are numerical constants $c, c''' > 0$ such that*

$$\mathbf{P}(\Omega_n^0 \setminus \mathcal{A}_n) \geq 1 - e^{-c'''n},$$

where for $n \in \mathbb{N}$ the set $\mathcal{A}_n$ is defined by

$$\mathcal{A}_n = \Big\{\omega \in \Omega_n^0 \;\Big|\; \operatorname{card}\Big\{i \leq m \;\Big|\; \|Tg_{n,i}\|_{n,\omega} < \frac{c\operatorname{Mix}_n(T)}{\sqrt{n\log n}}\Big\} > \frac{m}{2} \text{ for some } T \in L(\mathbb{R}^n)\Big\}.$$

Remark Observe that Theorem 2.3 follows directly from (3.1) and Theorems 2.1, 2.2 and 3.5.

Due to the homogenuity of the operator norm it is enough to prove the theorem for $T \in L(\mathbb{R}^n)$ with $\operatorname{Mix}_n(T) = 1$. Clearly, $\operatorname{Mix}_n(T) = 1$ implies that $T \in \operatorname{Mix}_n(k, 1/k)$ for some $k \leq n/2$. Set $\overline{\operatorname{Mix}}_n(l, \alpha) = \{T \in \operatorname{Mix}_n(l, \alpha) \mid \|T\|_2 \leq 1\}$ for $l \leq n/2$ and $\alpha \in \mathbb{R}^+$. Define, for $n \in \mathbb{N}$, $k \leq n/2$ and $c > 0$

$$\mathcal{B}_{n,k}^c = \Big\{\omega \in \Omega_n^0 \;\Big|\; \operatorname{card}\Big\{i \leq m \;\Big|\; \|Tg_{n,i}\|_{n,\omega} < \frac{c}{\sqrt{n\log n}}\Big\} > \frac{m}{2} \text{ for some } T \in \overline{\operatorname{Mix}}_n(k, 1/k)\Big\}.$$

Proposition 3.6 *There are numerical constants $c, \tilde{c} > 0$ such that*

$$\mathbf{P}\Big(\bigcup_{k=1}^{n/2} \mathcal{B}_{n,k}^c\Big) \leq e^{-\tilde{c}n}.$$

Clearly, Theorem 3.5 is a direct consequence of Propositions 3.1, 3.2, 3.4 and 3.6 so it remains to prove the last one.

4 Proof of Proposition 3.6

Lemma 4.1 *There is a constant $C_1 > 0$ such that for every $n \in \mathbb{N}$, every $c < C_1^{-1}$, every pair E_1, E_2 of k-dimensional orthogonal subspaces in $\mathbb{R}^n$ and every linear operator $T \colon \mathbb{R}^n \to E_2$ satisfying the inequality $\|Tx\|_2 \geq \|x\|_2/2k$ for every $x \in E_1$, we have*

$$\mathbf{P}(\mathcal{A}_{n,E_1,E_2,T}^c) < 2^m (C_1 c)^{km/2},$$

where

$$\mathcal{A}_{n,E_1,E_2,T}^c = \Big\{\omega \in \Omega_n^0 \;\Big|\; \operatorname{card}\Big\{i \leq m \;\Big|\; Tg_{n,i} \in \frac{c}{\sqrt{n\log n}} P_{E_2} B_{n,\omega}\Big\} > \frac{m}{2}\Big\}.$$

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Proof Fix $i_0 \le m$ and let $g = P_{E_1} g_{n,i_0}$. Then

$$\begin{aligned} &\Big\{\omega \in \Omega_n^0 \;\Big|\; T g_{n,i_0} \in \frac{c}{\sqrt{n \log n}} P_{E_2} B_{n,\omega}\Big\} \\ &\subset \Big\{\omega \in \Omega_n^0 \;\Big|\; T g \in \frac{c}{\sqrt{n \log n}} P_{E_2} B_{n,\omega} - T P_{E_1^\perp} g_{n,i_0}\Big\} \\ &\subset \Big\{\omega \in \Omega_n^0 \;\Big|\; g \in \frac{c}{\sqrt{n \log n}} (T|E_1)^{-1} P_{E_2} B_{n,\omega} - (T|E_1)^{-1} T P_{E_1^\perp} g_{n,i_0}\Big\}. \end{aligned} \tag{4.1}$$

Observe that since $P_{E_2} B_{n,\omega} = \operatorname{absconv}\{P_{E_2} g_{n,i} \mid i \le m\}$, by Fact 2 (ii), we infer that $P_{E_2} B_{n,\omega}$ as well as $(T|E_1)^{-1} T P_{E_1^\perp} g_{n,i_0}$ is independent of g. By Fact 2 (i) and (iv), we have

$$\begin{aligned} &\mathbf{P}\Big(\Big\{\omega \in \Omega_n^0 \;\Big|\; g \in \frac{c}{\sqrt{n \log n}} (T|E_1)^{-1} P_{E_2} B_{n,\omega} - (T|E_1)^{-1} T P_{E_1^\perp} g_{n,i_0}\Big\}\Big) \\ &\le \Big(\frac{c}{\sqrt{2\pi \log n}}\Big)^k \operatorname{vol}\{(T|E_1)^{-1} P_{E_2} B_{n,\omega}\} \end{aligned} \tag{4.2}$$

in order to estimate the volume in question, note that $\|(T|E_1)^{-1}\|_2 \le 2k$ and that for $\omega \in \Omega_n^0$ we have $\|g_{n,i}\|_2 \le 2$ for every $i \le m$. Therefore $(T|E_1)^{-1} P_{E_2} B_{n,\omega}$ is an absolute convex of m vectors, each of them of the length not greater than $4k$. Hence, by *e.g.*, [BP, Theorem 1] and Santaló inequality, we have

$$\operatorname{vol}\{(T|E_1)^{-1} P_{E_2} B_{n,\omega}\} \le \big(C_0 \sqrt{\log(m/k)}\big)^k, \tag{4.3}$$

where $C_0 > 0$ is a suitable numerical constant. Combining (4.1), (4.2) and (4.3) we get

$$\mathbf{P}\Big(\Big\{\omega \in \Omega_n^0 \;\Big|\; T g_{n,i_0} \in \frac{c}{\sqrt{n \log n}} P_{E_2} B_{n,\omega}\Big\}\Big) < (C_1 c)^k,$$

for some numerical constant $C_1 > 0$. Hence, by the binomial formulae, for $c < C_1^{-1}$

$$\mathcal{A}^c_{n,E_1,E_2,T} < 2^m (C_1 c)^{km/2},$$

which completes the proof of the lemma. ■

Lemma 4.2 *For every E_1, E_2 pair of k-dimensional orthogonal subspaces in $\mathbb{R}^n$ and every $c > 0$ set*

$$\mathcal{T}_{E_1,E_2} = \{T \colon \mathbb{R}^n \to E_2 \mid \|T\|_2 \le 1 \text{ and } \|Tx\|_2 \ge \|x\|_2/2k \text{ for every } x \in E_1\}$$

and

$$\mathcal{A}^c_{E_1,E_2} = \Big\{\omega \in \Omega_n^0 \;\Big|\; \operatorname{card}\Big\{i \le m \;\Big|\; T g_{n,i} \in \frac{c}{2\sqrt{n \log n}} P_{E_2} B_{n,\omega}\Big\} > \frac{m}{2} \quad \text{for some } T \in \mathcal{T}_{E_1,E_2}\Big\}.$$

Then $\mathbf{P}(\mathcal{A}^c_{E_1,E_2}) < 2^m (C_1 c)^{km/2} (100 n^3)^{2nk}$ for every $0 < c < C_1^{-1}$ and sufficiently large $n \in \mathbb{N}$, where C_1 is the numerical constant from the previous Lemma.

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Proof Fix any $c > 0$ satisfying the requirement of the previous lemma and let $\mathcal{N}_{E_1,E_2}$ be a $1/16n^2$-net in $\mathcal{T}_{E_1,E_2}$ with respect to the operator norm in $(\mathbb{R}^n, \|\cdot\|_2)$ with the minimal cardinality, and let $\mathcal{N}_{\mathbb{R}^n}$ and $\mathcal{N}_{E_2}$ be $1/32n^3$-nets in the unit balls of $\mathbb{R}^n$ and E_2 with $\operatorname{card} \mathcal{N}_{\mathbb{R}^n} \le (100n^3)^n$ and $\operatorname{card} \mathcal{N}_{E_2} \le (100n^3)^k$ respectively. Obviously, the set of operators of the form

$$T = \sum_{j=1}^{k} \langle x_j, \cdot \rangle y_j$$

with $x_j \in \mathcal{N}_{\mathbb{R}^n}$ and $y_j \in \mathcal{N}_{E_2}$ for $j = 1, 2, \ldots, k$ is a $k/16n^3$-net for $\mathcal{T}_{E_1,E_2}$ with cardinality not greater than $(100n^3)^{nk+k^2} < (100n^3)^{2nk}$. By a well known argument, this yields that

$$\operatorname{card} \mathcal{N}_{E_1,E_2} < (100n^3)^{2nk}. \tag{4.4}$$

Now, let

$$\tilde{\mathcal{A}}^c_{E_1,E_2} = \bigcup_{T \in \mathcal{N}_{E_1,E_2}} \mathcal{A}^c_{n,E_1,E_2,T}, \tag{4.5}$$

where $\mathcal{A}^c_{n,E_1,E_2,T}$ for $T \in \mathcal{N}_{E_1,E_2}$ are the sets defined in the lemma above. Assume that for some $\omega \in \Omega^0_n$, $i_0 \le m$ and some $T \in \mathcal{T}_{E_1,E_2}$ we have

$$\|Tg_{n,i_0}\|_{P_{E_2}B_{n,\omega}} < \frac{c}{2\sqrt{n \log n}},$$

where $\|\cdot\|_{P_{E_2}B_{n,\omega}}$ the norm on E_2 induced by $P_{E_2}B_{n,\omega}$. Choose $T_1 \in \mathcal{N}_{E_1,E_2}$ with $\|T - T_1\|_2 \le 1/16n^2$. Since $(\sqrt{\log n / n}/4)B^2_n \subset B_{n,\omega}$, for $\omega \in \Omega^0_n$, we infer that

$$\begin{aligned}
\|T_1 g_{n,i_0}(\omega)\|_{P_{E_2}B_{n,\omega}} &\le \|(T_1 - T)g_{n,i_0}(\omega)\|_{P_{E_2}B_{n,\omega}} + \|Tg_{n,i_0}(\omega)\|_{P_{E_2}B_{n,\omega}} \\
&\le 4\sqrt{\frac{n}{\log n}}\, \|(T_1 - T)\|_2\, \|g_{n,i_0}(\omega)\|_2 + \|Tg_{n,i_0}(\omega)\|_{P_{E_2}B_{n,\omega}} \\
&\le \frac{1}{2n\sqrt{n \log n}} + \frac{c}{2n\sqrt{n \log n}} < \frac{c}{\sqrt{n \log n}},
\end{aligned}$$

for n large enough. This implies that $\mathcal{A}^c_{E_1,E_2} \subset \tilde{\mathcal{A}}^c_{E_1,E_2}$ for sufficiently large n and the proof is completed by combining (4.4), (4.5) and Lemma 4.1. ■

Lemma 4.3 *For every $k \le n/2$ and $c > 0$ set*

$$\begin{aligned}
\mathcal{T}_{n,k} = \{T \in L(\mathbb{R}^n) \mid \|T\|_2 \le 1 \text{ rank } T = k \textit{ and there is} \\
\textit{a } k\textit{-dimensional subspace } E_1 \subset \mathbb{R}^n \textit{ orthogonal to } T(\mathbb{R}^n) \textit{ with} \\
\|Tx\|_2 \ge \|x\|_2/k \textit{ for every } x \in E_1\},
\end{aligned}$$

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and let

$$\mathcal{A}^c_{n,k} = \Big\{ \omega \in \Omega^0_n \;\Big|\; \operatorname{card}\Big\{ i \le m \;\Big|\; Tg_{n,i} \in \frac{c}{4\sqrt{n\log n}} P_{T(\mathbb{R}^n)} B_{n,\omega} \Big\} > \frac{m}{2} \quad \text{for some } T \in \mathcal{T}_{n,k} \Big\}.$$

Then there is a constant $C_2 > 0$ such that

$$\mathbf{P}(\mathcal{A}^c_{n,k}) < (C_2 c)^{km/2} (1600 n^5)^{2nk}$$

for $c > 0$ small enough and sufficiently large n.

Proof For $k < n$, denote by $G_{k,n}$ the Grassmann manifold of k-dimensional subspaces of $\mathbb{R}^n$ with the metric $\mathrm{d}(E_1, E_2) = \|P_{E_1} - P_{E_2}\|_2$ and let $\mathcal{N}_{n,k}$ be the $1/16n^2$-net in $G_{k,n}$ with

$$\operatorname{card} \mathcal{N}_{n,k} \le C_3^m (16n^2)^{k(n-k)}, \tag{4.6}$$

where C_3 is an universal constant, *cf.* [S1]. For every $F \in \mathcal{N}_{n,k}$ by $G_{k,n,F}$ denote the Grassmann manifold of k-dimensional subspaces of $F^{\perp}$ and let $\mathcal{N}_{n,k,F}$ be the $1/16n^2$-net in $G_{k,n,F}$ with

$$\operatorname{card} \mathcal{N}_{n,k,F} \le C_3^m (16n^2)^{k(n-2k)}. \tag{4.7}$$

Claim For every $T \in \mathcal{T}_{n,k}$ there are $F_2 \in \mathcal{N}_{n,k}$ and $F_1 \in \mathcal{N}_{n,k,F_2}$ such that $P_{F_2} T \in \mathcal{T}_{F_1,F_2}$ and $\|P_{E_2} - P_{F_2}\|_2 \le (16n^2)^{-1}$, where $E_2 = T(\mathbb{R}^n)$.

Proof of the Claim Fix $T \in \mathcal{T}_{n,k}$ and set $E_2 = T(\mathbb{R}^n)$. Let E_1 be a k-dimensional subspace orthogonal to E_2 such that $\|Tx\|_2 \ge \|x\|_2/k$ for every $x \in E_1$. Choose $F_2 \in \mathcal{N}_{n,k}$ with $\|P_{E_2} - P_{F_2}\|_2 \le (16n^2)^{-1}$ and let $\tilde{E} = P_{F_2^{\perp}} E_1$. Clearly, $P_{F_2^{\perp}}|E_1$ is a one-to-one mapping and $\|(P_{F_2^{\perp}}|E_1)^{-1} y\|_2 \ge \|y\|_2$ for every $y \in \tilde{E}$. On the other hand, setting $x = (P_{F_2^{\perp}}|E_1)^{-1} y$ for $y \in \tilde{E}$ we have

$$\|x - y\|_2 = \|P_{E_2^{\perp}} x - P_{F_2^{\perp}} x\|_2 \le \|x\|_2/16n^2 \le \|y\|_2/15n^2. \tag{4.8}$$

Now, choose $F_1 \in \mathcal{N}_{n,k,F_2}$ with $\|P_{F_1} - P_{\tilde{E}}\|_2 \le (16n^2)^{-1}$. Let $z \in F_1$, $y = P_{\tilde{E}} z$ and $x = (P_{F_2^{\perp}}|E_1)^{-1} y$. Since $P_{E_2} T = T$ and $\|z\|_2 \ge \|y\|_2$ then, by (4.8),

$$\begin{aligned}
\|P_{F_2} Tz\|_2 &\ge \|Tz\|_2 - \|(P_{E_2} - P_{F_2}) Tz\|_2 \ge \|Tz\|_2 - \|z\|_2/16n^2 \\
&\ge \|Ty\|_2 - \|T(P_{\tilde{E}} - P_{F_1}) z\|_2 - \|z\|_2/16n^2 \ge \|Ty\|_2 - \|z\|_2/8n^2 \\
&\ge \|Tx\|_2 - \|T(y - x)\|_2 - \|z\|_2/8n^2 \ge \|Tx\|_2 - \|y\|_2/15n^2 - \|z\|_2/8n^2 \\
&\ge \|Tx\|_2 - \|z\|_2/4n^2.
\end{aligned} \tag{4.9}$$

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On the other hand, since $x \in E_1$ and also $\|y\|_2 \geq \frac{15}{16}\|z\|_2$ we have

$$\|Tx\|_2 \geq \|x\|_2/k \geq \|y\|_2/k \geq \frac{15}{16k}\|z\|_2. \tag{4.10}$$

Combining (4.9) and (4.10) we get

$$\|P_{F_2}Tz\|_2 \geq \frac{15}{16k}\|z\|_2 - \|z\|_2/4n^2 \geq \|z\|_2/2k,$$

which completes the proof of the Claim.

Returning to the proof of the lemma, pick $T \in \mathcal{T}_{n,k}$ and let E_2, F_2 and F_1 be as in the Claim. Assume that for some $\omega \in \Omega_n^0$ and some $i_0 \leq m$ we have

$$Tg_{n,i_0} \in \frac{c}{4\sqrt{n \log n}} P_{E_2}B_{n,\omega}$$

and observe that since $P_{F_2}B_{n,\omega} = \operatorname{absconv}\{P_{F_2}g_{n,i} \mid i \leq m\}$ and

$$\|P_{F_2}g_{n,i} - P_{F_2}P_{E_2}g_{n,i}\|_2 = \|P_{F_2}(P_{F_2} - P_{E_2})g_{n,i}\|_2 \leq \frac{1}{8n^2},$$

for $i = 1, 2, \dots, m$ then, by the definition of the set Ω_n^0, we have

$$P_{F_2}P_{E_2}B_{n,\omega} \subset P_{F_2}B_{n,\omega} + \frac{1}{8n^2}P_{F_2}B_n^2 \subset 2P_{F_2}B_{n,\omega}.$$

Therefore

$$P_{F_2}Tg_{n,i_0} \in \frac{c}{4\sqrt{n \log n}} P_{F_2}P_{E_2}B_{n,\omega} \subset \frac{c}{2\sqrt{n \log n}} P_{F_2}B_{n,\omega}.$$

By the Claim, this implies that

$$\mathcal{A}_{n,k}^c \subset \bigcup_{F_2 \in \mathcal{N}_{n,k}} \bigcup_{F_1 \in \mathcal{N}_{n,k,F_2}} \mathcal{A}_{F_1,F_2}^c.$$

The proof is completed by combining Lemma 4.2 with (4.6) and (4.7) and setting $C_2 = (\max\{2, C_1, C_3\})^9$. ■

Proof of Proposition 3.6 Let $T \in \overline{\mathrm{Mix}}_n(k, 1/k)$ for some $n, k \in \mathbb{N}$. By the definition, there is a k-dimensional subspace E_1 with the property $\|P_{E_1^\perp}Tx\|_2 \geq \|x\|_2/k$ for every $x \in E_1$. Set $E_2 = P_{E_1^\perp}TE_1$. Clearly, $\tilde{T} = P_{E_2}T \in \mathcal{T}_{n,k}$. Therefore, for $c_0 = c/4$ and sufficiently large n we have $\mathcal{B}_{n,k}^{c_0} \subset \mathcal{A}_{n,k}^c$. Therefore, by Lemma 4.3, for sufficiently large n and $c > 0$ small enough we have

$$\mathbf{P}\Big(\bigcup_{k=1}^{n/2} \mathcal{B}_{n,k}^{c_0}\Big) \leq \mathbf{P}\Big(\bigcup_{k=1}^{n/2} \mathcal{A}_{n,k}^c\Big) \leq \sum_{k=1}^{n/2} (C_2c)^{km/2}(1600n^5)^{2nk}$$

which yields the required estimate. ■

References

[B] K. Ball, *Normed spaces with a weak-Gordon-Lewis property.* In: Functional Analysis, Springer Lecture Notes in Math. **1470**(1991), 36–47.

[BKPS] P. Billard, S. Kwapień, A. Pełczyński and Ch. Samuel, *Biorthogonal systems of random unconditional convergence in Banach spaces.* Longhorn Notes, The University of Texas at Austin, 1986, 13–35.

[BP] K. Ball and A. Pajor, *Convex bodies with few faces.* Proc. Amer. Math. Soc. **110**(1990), 225–231.

[B1] J. Bourgain, *Real isomorphic complex spaces need not to be complex isomorphic.* Proc. Amer. Math. Soc. **96**(1986), 221–226.

[B2] ———, *On finite dimensional homogenous Banach spaces.* In: Geometric Aspects of Functional Analysis, Israel Seminar (eds. J. Lindenstrauss and V. D. Milman), Springer Lecture Notes in Math. **1317**(1988), 232–239.

[G1] E. D. Gluskin, *The diameter of Minkowski compactum roughly equals to n.* (Russian) Funktsional. Anal. i Priložen. **15**(1981), 72–73 (English transl.: Funct. Anal. Appl. **15**(1981), 57–58).

[G2] ———, *Finite-dimensional analogues of spaces without a basis.* (Russian) Dokl. Acad. Nauk SSSR **261**(1981), 1046–1050.

[GG] D. J. H. Garling and Y. Gordon, *Relations between some constants associated with finite-dimensional Banach spaces.* Israel J. Math. **9**(1971), 346–361.

[M1] P. Mankiewicz, *Finite dimensional Banach spaces with symmetry constant of order $\sqrt{n}$.* Studia Math. **79**(1984), 193–200.

[M2] ———, *Subspace mixing properties of operators in $\mathbb{R}^n$ with applications to Gluskin spaces.* Studia Math. **88**(1988), 51–67.

[M3] ———, *Compact groups of operators on proportional quotients of l_1^n.* Israel J. Math. **109**(1999), 75–91.

[MT1] ———, P. Mankiewicz and N. Tomczak-Jaegermann, *A solution of the finite-dimensional homogeneous Banach spaces problem.* Israel J. Math. **75**(1991), 129–159.

[MT2] ———, *Schauder bases in quotients of subspaces of $l_2(X)$.* Amer. J. Math. **116**(1994), 1341–1363.

[MT3] ———, *Structural properties of weak cotype 2 spaces.* Canad. J. Math. **48**(1996), 607–624.

[MT4] ———, *Quotients of finite-dimensional Banach spaces; random phenomena.* Handbook of geometry of Banach spaces, Elsevier Publ., to appear.

[P] G. Pisier, *Volumes of Convex Bodies and Banach Spaces Geometry.* Cambridge Univ. Press, 1989.

[S1] S. J. Szarek, *The finite dimensional basis problem with an appendix on nets of Grassmann manifolds.* Acta Math. **151**(1983), 153–179.

[S2] ———, *On the existence and uniqueness of complex structure and spaces with "few" operators.* Trans. Amer. Math. Soc. **293**(1986), 339–353.

[T] N. Tomczak-Jaegermann, *Banach-Mazur Distances and Finite Dimensional Operator Ideals.* Pitman Monographs Surveys Pure Appl. Math. **38**, Longman Scientific & Technical, Harlow, and John Wiley, New York, 1989.

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