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Quasiconvexity and Density Topology

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Abstract. We prove that if $f\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ is quasiconvex and $U \subset \mathbb{R}^N$ is open in the density topology, then $\sup_U f = \operatorname{ess\,sup}_U f$, while $\inf_U f = \operatorname{ess\,inf}_U f$ if and only if the equality holds when $U = \mathbb{R}^N$. The first (second) property is typical of lsc (usc) functions, and, even when U is an ordinary open subset, there seems to be no record that they both hold for all quasiconvex functions.

This property ensures that the pointwise extrema of f on any nonempty density open subset can be arbitrarily closely approximated by values of f achieved on "large" subsets, which may be of relevance in a variety of situations. To support this claim, we use it to characterize the common points of continuity, or approximate continuity, of two quasiconvex functions that coincide away from a set of measure zero.

1 Introduction

To begin with matters of terminology, a *quasiconvex function f on $\mathbb{R}^N$* refers to an extended real-valued function whose lower level sets $\{x \in \mathbb{R}^N : f(x) < \alpha\}$ are convex for every $\alpha \in \mathbb{R}$. The same class is obtained if the level sets $\{x \in \mathbb{R}^N : f(x) \le \alpha\}$ are used instead. These functions were first introduced by de Finetti [5] in 1949,[1] although the nomenclature was only coined by Fenchel [4] a few years later.

A *null set* is a subset of $\mathbb{R}^N$ of Lebesgue measure 0 and Lebesgue measure, simply called measure, is denoted by μ_N. Without accompanying epithet, the words "open", "interior", "closure", "boundary", etc. and related symbols always refer to the euclidean topology of $\mathbb{R}^N$.

Recall also that the density topology on $\mathbb{R}^N$ is the topology whose open subsets are $\varnothing$ and the measurable subsets of $\mathbb{R}^N$ with density 1 at each point. They will henceforth be referred to as *density* open. Every open subset is density open. The (extended) real-valued functions on $\mathbb{R}^N$ that are (semi)continuous when $\mathbb{R}^N$ is equipped with the density topology and $\mathbb{R}$ with the euclidean topology are the so-called *approximately* (semi)continuous functions.

We shall only use elementary properties of the density topology. For convenience, a brief summary is given in the next section.

If $f\colon \mathbb{R}^N \to \overline{\mathbb{R}} := [-\infty, \infty]$ and $\alpha \in \mathbb{R}$, we set

$$F_\alpha := \{x \in \mathbb{R}^N : f(x) < \alpha\}. \tag{1.1}$$

This will only be used without further mention when the function of interest is called f, so no ambiguity will arise. As is customary, if f is measurable and $E \subset \mathbb{R}^N$ is a

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[1] The occasional claim that they were already investigated by von Neumann in 1928 is a gross exaggeration; see the historical article [7].

measurable subset, we define

$$\operatorname*{ess\,inf}_E f := \sup\{\alpha \in \mathbb{R} : \mu_N(E \cap F_\alpha) = 0\} \quad \text{and} \quad \operatorname*{ess\,sup}_E f := -\operatorname*{ess\,inf}_E(-f).$$

Since the sets F_α are linearly ordered by inclusion,

$$\operatorname*{ess\,inf}_E f = \inf\{\alpha \in \mathbb{R} : \mu_N(E \cap F_\alpha) > 0\}.$$

Now, if f is upper semicontinuous (usc for short) and $U \subset \mathbb{R}^N$ is an open subset, it is trivial that

$$\inf_U f = \operatorname*{ess\,inf}_U f. \tag{1.2}$$

Indeed, since the lower level sets F_α are open, the intersection $U \cap F_\alpha$ has positive measure whenever it is nonempty. More generally, (1.2) is true and equally straightforward if U is density open and f is approximately usc, but it fails if U has only positive measure, even if f is finite and continuous or has any amount of extra regularity.

Thus, heuristically at least, (1.2) for every density open subset $U \subset \mathbb{R}^N$ is best possible for any measurable function f. This property, which ensures that $\inf_U f$ can be arbitrarily closely approximated by values of f achieved on "large"subsets, is of possible relevance in a variety of technical situations. It may fail to hold if the function is modified at a single point, but elementary one-dimensional examples show that it is more general than upper semicontinuity, even approximate.

Likewise, if f is approximately lower semicontinuous, then

$$\sup_U f = \operatorname*{ess\,sup}_U f, \tag{1.3}$$

for every density open subset U of $\mathbb{R}^N$.

The main result of this note (Theorem 3.3) is that if f is quasiconvex, (1.3) always holds and (1.2) holds if and only if it holds when $U = \mathbb{R}^N$ (Theorem 3.3). Of course, (1.2) and (1.3) are trivial when f is approximately continuous (in particular, when U is open and f is continuous), but it is more surprising that they continue to hold when f is quasiconvex, without any continuity-like requirement. (Needless to say, quasiconvexity does not imply approximate continuity.) When f is an arbitrary convex function, not necessarily proper, an equivalent statement is given in Corollary 3.4.

In spite of the by now substantial literature involving quasiconvex functions, this arguably notable property seems to have remained unnoticed, even when U is an euclidean open subset. At any rate, prior connections between quasiconvexity in the sense of de Finetti and the density topology (or approximate continuity) appear to be nonexistent.

In Section 4, we use (1.2) and (1.3) to compare the points of (approximate) continuity of two real-valued quasiconvex functions f and g on $\mathbb{R}^N$ such that $f = g$ a.e., so that f and g have the same essential infimum $m := \operatorname{ess\,inf}_{\mathbb{R}^N} f = \operatorname{ess\,inf}_{\mathbb{R}^N} g$.

By a well known result of Crouzeix [3] (see also [2]), every real-valued quasiconvex function is Fréchet differentiable a.e. and so continuous a.e. An even sharper property is proved in Borwein and Wang [1] in the lsc case. Thus, f and g above are simultaneously continuous at the points of a large set, but this does not say whether f is continuous at a given point x where g is known to be continuous.

In Theorem 4.2, we show that this question and the same question for points of approximate continuity can be given simple, yet complete answers: a point x of approximate continuity of g is not a point of approximate continuity of f if and only if $m > -\infty$, $g(x) = m$, and $x \in F_m$ (see (1.1)), while a point x of continuity of g is not a point of continuity of f if and only if $m > -\infty, g(x) = m$, and $x \in \bigcup_{\alpha<m} \overline{F}_\alpha$. The similarity and the difference between these two results are better appreciated if it is noticed that $F_m = \bigcup_{\alpha<m} F_\alpha$.

Variants of the main results for quasiconvex functions on a Baire topological vector space X can be found in [13]. In that setting, the most significant by-product is a very simple characterization of the real-valued quasiconvex functions continuous at the points of a residual subset of X, in terms of basic topological properties of the sublevel sets.

2 Background

We begin with a brief review of the few properties of the density topology on $\mathbb{R}^N$ and related topics that will be used in this paper. Further information, notably the proof that the density topology *is* a topology, can be found in [6] or [11]. It was introduced in 1952 by Haupt and Pauc [8] in a more general setting, but many other expositions are limited to $N = 1$. For classical generalizations, see [12], [15].

First, recall that while the density of a set at a point x is often defined by using shrinking families of open cubes centered at x, an equivalent definition is obtained if cubes are replaced with euclidean balls. This is elementary but still requires a short argument; see, for instance, [10, p. 460]. While not a major point, this observation is convenient.

From the very definition of a density open subset, it follows that the density interior of a *measurable* subset $S \subset \mathbb{R}^N$ is the subset S_1 of S of those points at which S has density 1. By the Lebesgue density theorem, $S \backslash S_1$ is a null set. Thus, a null set has empty density interior and, conversely, a measurable set with empty density interior is a null set. (This converse is of course false with the euclidean topology.) In particular, a nonempty density open subset always has positive measure.

Every subset of $\mathbb{R}^N$, measurable or not, has a density interior, but a nonmeasurable subset with empty density interior is obviously not a null set. Such sets will never be involved in the sequel. Although we shall not use this here, we feel compelled to point out that every null set is density closed (and even discrete), because its complement is clearly density open.

A measurable subset $W \subset \mathbb{R}^N$ is a density neighborhood of a point x if and only if it contains a density open neighborhood of x. From the above, this happens if and only if W has density 1 at x, and then W has positive measure. Thus, the inverse image $f^{-1}(V)$ of an open subset $V \subset \mathbb{R}$ under a *measurable* function f is a density neighborhood of some point x if and only if $f^{-1}(V)$ has density 1 at x.

In the introduction, a function $f\colon \mathbb{R}^N \to \mathbb{R}$ was called approximately continuous if it is continuous when $\mathbb{R}^N$ is equipped with the density topology and $\mathbb{R}$ with the euclidean topology. A different definition is that every $x \in \mathbb{R}^N$ is contained in a measurable set E_x having density 1 at x such that $f_{|E_x}$ is continuous at x (for the euclidean topology). It is well known and not hard to prove, though not entirely trivial, that the two definitions are equivalent.

Aside from the density topology and approximately continuous functions, we shall also use several properties of convex subsets of $\mathbb{R}^N$, some of which, but not all, are explicitly spelled out in standard texts. A basic fact is that if a convex subset $C \subset \mathbb{R}^N$ has empty interior, it is contained in an affine hyperplane ([14]). Then, elementary considerations yield the following. For every convex subset $C \subset \mathbb{R}^N$ the statements: (i) C has empty interior, (ii) C is a null set, (iii) $\overline{C}$ is a null set, and (iv) $\overline{C}$ has empty interior, are all equivalent.

Another useful property is that if $C \subset \mathbb{R}^N$ is closed and convex, at least one supporting hyperplane passes through each point of its boundary ∂C. Furthermore, every convex subset $C \subset \mathbb{R}^N$ is measurable, because C is the union of its interior C° with a subset of ∂C, and ∂C is always a null set. Indeed, the distance function to C is convex and finite, hence a.e. differentiable ([14, Theorem 25.5]). Since it is readily checked that it is not differentiable at any point of ∂C, the latter is a null set.

Notice that the measurability of convex sets implies at once that all quasiconvex functions are measurable.

3 Main Result

We need two preliminary lemmas.

Lemma 3.1 *Let $C \subset \mathbb{R}^N$ be convex and $U \subset \mathbb{R}^N$ be density open.*

(i) *If C has nonempty interior and $U \cap C \neq \varnothing$, then $\mu_N(U \cap C) > 0$.*
(ii) *If $U \cap (\mathbb{R}^N \backslash C) \neq \varnothing$, then $\mu_N(U \cap (\mathbb{R}^N \backslash C)) > 0$.*

Proof (i) Choose $x_0 \in U \cap C$ along with an open ball $B \subset C$ such that $x_0 \notin B$. The hypothesis $C^\circ \neq \varnothing$ ensures that B exists. Indeed, choose $B \subset C^\circ$. If x_0 is not the center of B, shrink the radius of B until $x_0 \notin B$. If x_0 is the center of B, just replace B by an open ball contained in B that does not contain x_0.

The set $K := \bigcup_{\lambda \in (0,1)} (\lambda B + (1-\lambda)x_0)$ is a truncated open convex cone with apex at x_0 (and spherical "end") contained in C°. Let $B(x_0, r)$ denote the open ball with center x_0 and radius $r > 0$. Clearly, the ratio

$$\kappa := \frac{\mu_N(K \cap B(x_0, r))}{\mu_N(B(x_0, r))} \in (0, 1)$$

is independent of $r > 0$ small enough. On the other hand, since U has density 1 at x_0,

$$\frac{\mu_N(U \cap B(x_0, r))}{\mu_N(B(x_0, r))} > 1 - \kappa$$

if $r > 0$ is small enough. This implies that $\mu_N(U \cap K \cap B(x_0, r)) > 0$, for otherwise the intersection of $U \cap B(x_0, r)$ and $K \cap B(x_0, r)$ (that is, $U \cap K \cap B(x_0, r)$) is a null

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set, so that

$$\mu_N\big((U \cup K) \cap B(x_0, r)\big) = \mu_N\big(U \cap B(x_0, r)\big) + \mu_N\big(K \cap B(x_0, r)\big) > \mu_N\big(B(x_0, r)\big),$$

which is absurd. Since $K \cap B(x_0, r) \subset K \subset C$, it follows that $\mu_N(U \cap C) > 0$.

(ii) Choose $x_0 \in U \cap (\mathbb{R}^N \backslash C)$. We claim that $\mathbb{R}^N \backslash C$ contains (at least) half of any open ball centered at x_0 with small enough radius. Since this is obvious if x_0 lies in the interior of $\mathbb{R}^N \backslash C$, we assume that $x_0 \in \partial(\mathbb{R}^N \backslash C) = \partial C$. There is at least one affine hyperplane H supporting $\overline{C}$ at x_0. Therefore, H splits every open ball $B(x_0, r)$ into two open halves, one of which does not intersect $\overline{C}$ and is therefore contained in $\mathbb{R}^N \backslash C$.

Since U has density 1 at x_0, it follows that $\mu_N(U \cap B(x_0, r)) > \frac{1}{2}\mu_N(B(x_0, r))$ if $r > 0$ is small enough. From the above, half of $B(x_0, r)$ is contained in $\mathbb{R}^N \backslash C$ and the other half cannot contain a set of measure greater than $\frac{1}{2}\mu_N(B(x_0, r))$. As a result, the half-ball contained in $\mathbb{R}^N \backslash C$ must intersect U along a set of positive measure, so that $\mu_N(U \cap (\mathbb{R}^N \backslash C)) > 0$. ■

Lemma 3.2 *If $f \colon \mathbb{R}^N \to \mathbb{R}$ is measurable, the following statements are equivalent:*

(i) $\sup_U f = \operatorname{ess\,sup}_U f$ *for every density open subset* $U \subset \mathbb{R}^N$;

(ii) *for every $x_0 \in \mathbb{R}^N$, every density open subset $U \subset \mathbb{R}^N$ containing x_0, and every $\varepsilon > 0$,*

$$\mu_N\big(\{x \in U : f(x) \geq f(x_0) - \varepsilon\}\big) > 0.$$

Likewise, the following statements are equivalent:

(i′) $\inf_U f = \operatorname{ess\,inf}_U f$ *for every density open subset* $U \subset \mathbb{R}^N$;

(ii′) *for every $x_0 \in \mathbb{R}^N$, every density open subset $U \subset \mathbb{R}^N$ containing x_0 and every $\varepsilon > 0$,*

$$\mu_N\big(\{x \in U : f(x) < f(x_0) + \varepsilon\}\big) > 0.$$

Proof (i) $\Rightarrow$ (ii) Suppose that (i) holds and, by contradiction, assume that there are $x_0 \in \mathbb{R}^N$, a density open subset $U \subset \mathbb{R}^N$ containing x_0, and some $\varepsilon > 0$ such that

$$\mu_N(\{x \in U : f(x) \geq f(x_0) - \varepsilon\}) = 0.$$

Then $\operatorname{ess\,sup}_U f \leq f(x_0) - \varepsilon < f(x_0) \leq \sup_U f$, which contradicts (i).

(ii) $\Rightarrow$ (i) Let $U \subset \mathbb{R}^N$ be a density open subset. We argue by contradiction, thereby assuming that $\sup_U f > \operatorname{ess\,sup}_U f$. If so, U is not empty (otherwise, both suprema are $-\infty$) and $\operatorname{ess\,sup}_U f < \infty$. Thus, the assumption $\sup_U f > \operatorname{ess\,sup}_U f$ implies the existence of $x_0 \in U$ such that $\operatorname{ess\,sup}_U f < f(x_0) \leq \sup_U f$. Choose $\varepsilon > 0$ small enough that $\operatorname{ess\,sup}_U f < f(x_0) - \varepsilon$. By (ii),

$$\mu_N(\{x \in U : f(x) \geq f(x_0) - \varepsilon\}) > 0,$$

so that $\operatorname{ess\,sup}_U f \geq f(x_0) - \varepsilon$, which is a contradiction.

That (i′) $\Leftrightarrow$ (ii′) follows after replacing f by $-f$ above, and noticing that the equivalence between (i) and (ii), remains true if the inequality in $\{x \in U : f(x) \geq f(x_0) - \varepsilon\}$ is replaced by the corresponding strict inequality. ■

We now prove the main result announced in the introduction.

Theorem 3.3 *Let $f\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ be quasiconvex.*

(i) $\sup_U f = \operatorname{ess\,sup}_U f$ *for every density open subset* $U \subset \mathbb{R}^N$.
(ii) $\inf_U f = \operatorname{ess\,inf}_U f$ *for every density open subset* $U \subset \mathbb{R}^N$ *if and only if this is true when* $U = \mathbb{R}^N$.

Proof The extended real-valued case can be deduced from the real-valued one by changing f into $\arctan f$. This does not affect quasiconvexity, and it is easily checked that arctan commutes with $\operatorname{ess\,sup}_U$ and $\operatorname{ess\,inf}_U$. Accordingly, in the remainder of the proof, f is real-valued.

(i) We show that Lemma 3.2(ii) holds and use the equivalence with (i) of that lemma.

Pick $x_0 \in \mathbb{R}^N$, a density open subset $U \subset \mathbb{R}^N$ containing x_0, and $\varepsilon > 0$. The set $\{x \in U : f(x) \geq f(x_0) - \varepsilon\}$ is the intersection $U \cap (\mathbb{R}^N \backslash F_{f(x_0)-\varepsilon})$ (see (1.1)). Since $U \cap (\mathbb{R}^N \backslash F_{f(x_0)-\varepsilon}) \neq \varnothing$ (it contains x_0), it follows from Lemma 3.1(ii) that $\mu_N(U \cap (\mathbb{R}^N \backslash F_{f(x_0)-\varepsilon})) > 0$.

(ii) It is obvious that $\inf_{\mathbb{R}^N} f = \operatorname{ess\,inf}_{\mathbb{R}^N} f$ is necessary. Conversely, assuming this, we show that Lemma 3.2(ii′) holds and use the equivalence with (i′) of that lemma.

Pick $x_0 \in \mathbb{R}^N$, a density open subset $U \subset \mathbb{R}^N$ containing x_0, and $\varepsilon > 0$. The set $\{x \in U : f(x) < f(x_0) + \varepsilon\}$ is the intersection $U \cap F_{f(x_0)+\varepsilon}$. Since $\operatorname{ess\,inf}_{\mathbb{R}^N} f = \inf_{\mathbb{R}^N} f \leq f(x_0) < f(x_0) + \varepsilon$, the set $F_{f(x_0)+\varepsilon}$ has positive measure, and hence nonempty interior since it is convex. Therefore, $\mu_N(U \cap F_{f(x_0)+\varepsilon}) > 0$ by Lemma 3.1(i). ■

For convex functions (defined as functions with convex epigraphs and hence not necessarily proper), Theorem 3.3 can be phrased differently. Recall that the domain $\operatorname{dom} f$ of a convex function f is the set of points where $f < \infty$. It includes the points where $f = -\infty$, if any.

Corollary 3.4 *Let $f\colon \mathbb{R}^N \to [-\infty, \infty]$ be convex. Then:*

(i) $\sup_U f = \operatorname{ess\,sup}_U f$ *for every density open subset* $U \subset \mathbb{R}^N$;
(ii) $\inf_U f = \operatorname{ess\,inf}_U f$ *for every density open subset* $U \subset \mathbb{R}^N$ *if and only if either* $f = \infty$ *everywhere, or* $\operatorname{dom} f$ *has nonempty interior.*

Proof Since there is no need to discuss the case when $f = \infty$ everywhere (trivial convex function), we henceforth assume that f is not trivial. By Theorem 3.3, it suffices to prove that $\inf_{\mathbb{R}^N} f = \operatorname{ess\,inf}_{\mathbb{R}^N} f$ if and only if $\operatorname{dom} f$ has nonempty interior.

We begin with necessity. If $\inf_{\mathbb{R}^N} f = \operatorname{ess\,inf}_{\mathbb{R}^N} f$, then $\operatorname{dom} f$ has nonempty interior, for otherwise $\operatorname{dom} f$ (convex) is a null set, so that $\operatorname{ess\,inf}_{\mathbb{R}^N} f = \infty$, while $\inf_{\mathbb{R}^N} f < \infty$, since f is not trivial. The proof of sufficiency requires a little work. For convenience, we set $D := \operatorname{dom} f$ and from now on assume that $D^\circ \neq \varnothing$.

That $\inf_{\mathbb{R}^N} f = \operatorname{ess\,inf}_{\mathbb{R}^N} f = -\infty$ is trivial if $f^{-1}(-\infty)$ is a (convex) set of positive measure. On the other hand, if $f^{-1}(-\infty)$ is a null set, then $f^{-1}(-\infty) = \varnothing$, for otherwise $f = -\infty$ on D° by [14, Theorem 7.2], which is not a null set. Thus, from now on, f is proper.

Since $f = \infty$ outside D and D has positive measure, $\operatorname{ess\,inf}_{\mathbb{R}^N} f = \operatorname{ess\,inf}_D f$. Also, it is plain that $\inf_{\mathbb{R}^N} f = \inf_D f$. To complete the proof, it suffices to show that

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$f(x) \geq \operatorname{ess\,inf}_D f$ for every $x \in D$. This is obvious if $x \in D^\circ$, since a proper convex function is continuous on the interior of its domain.

Let then $x \in D \cap \partial D$. Given $y \in D^\circ$, the segment $(x, y]$ is entirely contained in D° ([14, Theorem 6.1]) and a (finite) convex function on an interval is upper semicontinuous on the closure of that interval ([9, p. 16]). Thus,

$$f(x) \geq \lim \sup_{z \to x, z \in (x,y]} f(z).$$

As observed earlier, $f(z) \geq \operatorname{ess\,inf}_D f$, since $z \in D^\circ$, so that $f(x) \geq \operatorname{ess\,inf}_D f$. ∎

4 Common Points of Continuity of Equivalent Quasiconvex Functions

The equivalence referred to in the section head is equality almost everywhere. A point of (approximate) continuity of $f\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ is defined as a point $x \in f^{-1}(\mathbb{R})$ such that f is (approximately) continuous at x. Such points are the points x of (approximate) continuity of $\arctan f$ such that $\arctan f(x) \neq \pm\frac{\pi}{2}$. Thus, as in the proof of Theorem 3.3, we may and will confine attention to real-valued functions.

Lemma 4.1 *Let $f, g\colon \mathbb{R}^N \to \mathbb{R}$ be quasiconvex functions such that $f = g$ a.e., so that $\operatorname{ess\,inf}_{\mathbb{R}^N} f = \operatorname{ess\,inf}_{\mathbb{R}^N} g := m$ ($\geq -\infty$).*

(i) *If also $\inf_{\mathbb{R}^N} f = m$ and $\inf_{\mathbb{R}^N} g = m$ (not a restriction if $m = -\infty$), then f and g have the same points of continuity, the same points of approximate continuity, and achieve a common value at such points.*

(ii) *$\max\{f, m\}$ and $\max\{g, m\}$ have the same points of continuity, the same points of approximate continuity, and achieve a common value at such points.*

Proof (i) Let x denote a point of continuity of f, so that for every $\varepsilon > 0$, there is an open neighborhood U of x such that $f(U) \subset [f(x) - \varepsilon, f(x) + \varepsilon]$. Thus, $\inf_U f \geq f(x) - \varepsilon$ and $\sup_U f \leq f(x) + \varepsilon$. By Theorem 3.3(i) and (ii), this is the same as $\operatorname{ess\,inf}_U f \geq f(x) - \varepsilon$ and $\operatorname{ess\,sup}_U f \leq f(x) + \varepsilon$.

Since $f = g$ a.e., the essential extrema are unchanged when f is replaced by g so that $\operatorname{ess\,inf}_U g \geq f(x) - \varepsilon$ and $\operatorname{ess\,sup}_U g \leq f(x) + \varepsilon$. By using once again Theorem 3.3(i) and (ii), it follows that $\inf_U g \geq f(x) - \varepsilon$ and $\sup_U g \leq f(x) + \varepsilon$, whence $g(U) \subset [f(x) - \varepsilon, f(x) + \varepsilon]$. In particular, $g(x) \in [f(x) - \varepsilon, f(x) + \varepsilon]$. Since $\varepsilon > 0$ is arbitrary, it follows that $g(x) = f(x)$ and hence that $g(U) \subset [g(x) - \varepsilon, g(x) + \varepsilon]$, which proves the continuity of g at x.

In summary, the points of continuity of f are points of continuity of g and $g = f$ at such points. By exchanging the roles of f and g, the converse is true.

The exact same argument can be repeated for the points of approximate continuity, since Theorem 3.3 is applicable when U is density open.

(ii) Just use (i) with $\max\{f, m\}$ and $\max\{g, m\}$, respectively. Neither quasiconvexity nor a.e. equality is affected and

$$\operatorname{ess\,inf}_{\mathbb{R}^N} \max\{f, m\} = \operatorname{ess\,inf}_{\mathbb{R}^N} \max\{g, m\} = m,$$

so that $\inf_{\mathbb{R}^N} \max\{f, m\} = m = \inf_{\mathbb{R}^N} \max\{g, m\}$ is obvious. ∎

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Lemma 4.1(ii) will now be instrumental in identifying simple necessary and sufficient conditions ensuring that a given point of (approximate) continuity of one function, say g, is not a point of (approximate) continuity of f.

Theorem 4.2 *Let $f, g \colon \mathbb{R}^N \to \mathbb{R}$ be quasiconvex functions such that $f = g$ a.e. so that* $\operatorname{ess\,inf}_{\mathbb{R}^N} f = \operatorname{ess\,inf}_{\mathbb{R}^N} g := m \ (\geq -\infty)$.

(i) *If $x \in \mathbb{R}^N$ is a point of approximate continuity of g, then $g(x) \geq m$. Furthermore, x is a point of approximate continuity of g, but not one of f, if and only if $m > -\infty, g(x) = m$, and $x \in F_m$, a set of measure 0.*

(ii) *If $x \in \mathbb{R}^N$ is a point of continuity of g, then $g(x) \geq m$. Furthermore, x is a point of continuity of g, but not one of f, if and only if $m > -\infty, g(x) = m$, and $x \in \bigcup_{\alpha<m} \overline{F}_\alpha \subset \overline{F}_m$, a set of measure 0.*

Proof With no loss of generality, assume $m > -\infty$, since, otherwise, everything follows at once from Lemma 4.1(i). We first justify the statement that F_m and $\bigcup_{\alpha<m} \overline{F}_\alpha$ are null sets. Notice that $F_m = \bigcup_{\alpha<m,\alpha\in\mathbb{Q}} F_\alpha$ and that each F_α with $\alpha < m$ is a null set by definition of m. Thus, F_m is a null set and therefore $\overline{F}_m$ is also a null set since F_m is convex (see Section 2). Thus, $\bigcup_{\alpha<m} \overline{F}_\alpha \subset \overline{F}_m$ is a null set.

(i) By contradiction, assume that x is a point of approximate continuity of g and that $g(x) < m$. Pick $\alpha \in \mathbb{R}$ such that $g(x) < \alpha < m$. By definition of m, the set $G_\alpha := \{y \in \mathbb{R}^N : g(y) < \alpha\}$ is a null set. On the other hand, since $x \in G_\alpha = g^{-1}((-\infty, \alpha))$, the approximate continuity of g at x implies that G_α is a density neighborhood of x so that it has positive measure. This contradiction proves that $g(x) \geq m$, as claimed.

Next, let x be a point of approximate continuity of g and hence one of $\max\{g, m\}$. By Lemma 4.1(ii), x is a point of approximate continuity of $\max\{f, m\}$ and $\max\{g(x), m\} = \max\{f(x), m\}$. Therefore, if $g(x) > m$ *or* $f(x) > m$, then $g(x) > m$ *and* $f(x) > m$. To see that x is a point of approximate continuity of f, choose $\varepsilon > 0$ small enough that $m < f(x) - \varepsilon$ and let $I_\varepsilon := (f(x) - \varepsilon, f(x) + \varepsilon)$. Then $(\max\{f, m\})^{-1}(I_\varepsilon)$ is a density neighborhood W_ε of x. From the choice of ε, it is obvious that $W_\varepsilon = f^{-1}(I_\varepsilon)$. Since this is true for every $\varepsilon > 0$ small enough, it follows that f is approximately continuous at x.

From the above, if x is a point of approximate continuity of g, but not one of f, then $g(x) = m$ and $f(x) \leq m$. As was seen earlier (with g instead of f), x is not a point of approximate continuity of f if $f(x) < m$. It remains to prove that the converse is true, *i.e.,* that if $f(x) = m$, then f is approximately continuous at x.

It suffices to show that if $\alpha < m < \beta$ and $I := (\alpha, \beta)$, then $f^{-1}(I)$ is a density neighborhood of x, *i.e.,* that $f^{-1}(I)$ has density 1 at x (since I is an interval and quasiconvex functions are measurable, $f^{-1}(I)$ is measurable). Now, $\max\{f, m\}$ *is* approximately continuous at x, whence $(\max\{f, m\})^{-1}(I)$ does have density 1 at x. Since $m < \beta$, we may split $(\max\{f, m\})^{-1}(I) = F_m \cup E$ with $E := \{y \in \mathbb{R}^N : m \leq f(y) < \beta\}$, and we already know that F_m is a null set. Therefore, E and $(\max\{f, m\})^{-1}(I)$ have the same density at every point of $\mathbb{R}^N$. In particular, E has density 1 at x, so that $f^{-1}(I) \supset E$ has density 1 at x, as claimed.

(ii) That $g(x) \geq m$ follows at once from (i). The proof that x is a point of continuity of f if it is one of g and either $g(x) > m$ or $f(x) > m$ proceeds as above, by

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merely changing the terminology in the obvious way. Thus, it only remains to show that if x is a point of continuity of g such that $g(x) = m$, it is not a point of continuity of f if and only if $x \in \bigcup_{\alpha<m} \overline{F}_\alpha$.

If $f(x) < m$, *i.e.*, $x \in F_m$, then by (i) x is not a point of approximate continuity of f, so it is not a point of continuity of f and $x \in \bigcup_{\alpha<m} \overline{F}_\alpha$, since $F_m = \bigcup_{\alpha<m} F_\alpha$. Next, if $f(x) \geq m$ and x is not a point of continuity of f, then $f(x) = m$ from the above. To see that $x \in \bigcup_{\alpha<m} \overline{F}_\alpha$, suppose by contradiction that $x \notin \bigcup_{\alpha<m} \overline{F}_\alpha$. Let (x_n) be a sequence tending to x and let $\varepsilon > 0$ be given. If n is large enough, then $f(x_n) \geq m - \varepsilon$, for otherwise there is a subsequence (x_{n_k}) such that $x_{n_k} \in F_{m-\varepsilon}$ so that $x \in \overline{F}_{m-\varepsilon}$, which is not the case. Therefore, $\liminf_{n\to\infty} f(x_n) \geq m - \varepsilon$.

Since x is a point of continuity of g, it is one of $\max\{g, m\}$. Thus, from Lemma 4.1(ii), $\max\{f, m\}$ is continuous at x and so $\lim_{n\to\infty} \max\{f(x_n), m\} = \max\{f(x), m\} = m$ since $f(x) = m$. It follows that $\limsup_{n\to\infty} f(x_n) \leq m$. In summary,

$$m - \varepsilon \leq \liminf_{n\to\infty} f(x_n) \leq \limsup_{n\to\infty} f(x_n) \leq m.$$

Since $\varepsilon > 0$ is arbitrary, $\liminf_{n\to\infty} f(x_n) = \limsup_{n\to\infty} f(x_n) = m = f(x)$. Thus, f is continuous at x, which is a contradiction.

To complete the proof, suppose that $x \in \bigcup_{\alpha<m} \overline{F}_\alpha$. If $f(x) < m$, we already know that f is not continuous at x, so we may assume $f(x) \geq m$. Let $\alpha < m$ be such that $x \in \overline{F}_\alpha$ and let $(x_n) \subset F_\alpha$ be a sequence tending to x. Since $f(x_n) < \alpha < m \leq f(x)$, it is obvious that $f(x_n)$ does not tend to $f(x)$. This proves that x is not a point of continuity of f. ∎

Remark 4.3 The above proof shows that in Theorem 4.2(ii), the condition $x \in \bigcup_{\alpha<m} \overline{F}_\alpha$ is equivalent to the seemingly stronger condition $x \in F'_m \cap (\bigcup_{\alpha<m} \overline{F}_\alpha)$, where $F'_m := \{x \in \mathbb{R}^N : f(x) \leq m\}$. Otherwise, a contradiction arises from f not being continuous at x when $x \in \bigcup_{\alpha<m} \overline{F}_\alpha$ and f being continuous at x when $f(x) > m$.

For completeness, we give an example when $\bigcup_{\alpha<m} \overline{F}_\alpha \neq \overline{F}_m$.

Example 4.4 In $\mathbb{R}^2$ with $x = (x_1, x_2)$, let $f(x) = |x_1|$ if $x_2 \geq 0$ or if $x_2 < 0, x_1 \neq 0$ and let $f(0, x_2) = x_2$ if $x_2 < 0$. Then f is quasiconvex, $m = 0$, and $\bigcup_{\alpha<0} \overline{F}_\alpha = \{0\} \times (-\infty, 0)$, but $\overline{F}_0 = \{0\} \times (-\infty, 0]$. Observe that f is continuous at $(0, 0)$. This is no longer true if f is modified by setting $f(0, x_2) = -1$ if $x_2 < 0$, but f is still approximately continuous at $(0, 0)$.

The next corollary generalizes Lemma 4.1(i). The proof is mostly a rephrasing of Theorem 4.2. The only extra technicality is to show that if f and g are (approximately) continuous at the same point x, they must coincide at that point. Since $f = g$ a.e., this is obvious, but we spell out the argument in the approximately continuous case: If $f(x) \neq g(x)$, there is a density neighborhood W of x such that $f(y) \neq g(y)$ for every $y \in W$. Since W has positive measure (Section 2), a contradiction arises with $f = g$ a.e.

Corollary 4.5 *Let $f, g\colon \mathbb{R}^N \to \mathbb{R}$ be quasiconvex functions such that $f = g$ a.e., so that $\operatorname{ess\,inf}_{\mathbb{R}^N} f = \operatorname{ess\,inf}_{\mathbb{R}^N} g := m$ $(\geq -\infty)$.*

(i) *Every point of approximate continuity of g is a point of approximate continuity of f if and only if g has no point of approximate continuity x such that $g(x) = m$ and $f(x) < m$ (always true if $m = -\infty$). If so, $f(x) = g(x)$ at every point x of approximate continuity of g.*

(ii) *Every point of continuity of g is a point of continuity of f if and only if g has no point of continuity x such that $g(x) = m$ and $x = \lim_{n\to\infty} x_n$ where (x_n) is a sequence such that $f(x_n) < \alpha < m$ for some $\alpha \in \mathbb{R}$ and every $n \in \mathbb{N}$ (always true if $m = -\infty$). If so, $f(x) = g(x)$ for every point of continuity x of g.*

Clearly, $\inf_{\mathbb{R}^N} f = m \geq -\infty$ is only an especially simple special case when the conditions given in Corollary 4.5(i) and (ii) hold. If also $\inf_{\mathbb{R}^N} g = m$, the roles of f and g can be exchanged in Corollary 4.5 so that f and g have the same points of continuity and Lemma 4.1(i) is recovered.

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