Fourier's Integral.

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The purposes of the following note are these:—(1) To show the relation between Whittaker's Cardinal Function and Fourier's Repeated Integral; (2) to give a new derivation of Fourier's Integral Formula; and (3) to extend the notion of the Fourier Integral to the case in which the variables involved are complex.

§ 1. *The Cardinal Function.* Let $f(x)$ be a one-valued function of a variable x, and suppose we are given the values of $f(x)$ which correspond to a set of equidistant values of the argument, say $a, a+w, a-w, a+2w, a-2w, \ldots$. Then, supposing certain conditions of convergence satisfied, the function

$$C(x) = \sum_{r=-\infty}^{\infty} \frac{f(a+rw)\sin\frac{\pi}{w}(x-a-rw)}{\frac{\pi}{w}(x-a-rw)}$$

has been called by Prof. Whittaker* the "cardinal function" of the set of functions cotabular with $f(x)$, that is, the set of functions having the same values as $f(x)$ when the argument assumes the series of values $a, a-w, a+w, a-2w, a+2w, \ldots$. Prof. Whittaker has shown that the cardinal function $C(x)$ has the following properties:—

(i) $C(x)$ is cotabular with $f(x)$;

(ii) $C(x)$ has no singularities in the finite part of the x-plane;

(iii) When $C(x)$ is analysed into periodic constituents by Fourier's integral theorem, all constituents of period less than $2w$ are absent.

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§ 2. *Derivation of the Cardinal Function from Lagrange's Interpolation Formula.* We may arrive at the cardinal function very simply by considering Lagrange's formula of interpolation. Suppose we are given the values of a function corresponding to the values of the argument $a_1, a_2, a_3, \ldots a_{2r+1}$, then Lagrange's formula expresses the function in the form

$$\frac{(x-a_2)(x-a_3)\ldots(x-a_{2r+1})}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_{2r+1})} f(a_1)$$

$$+\frac{(x-a_1)(x-a_3)\ldots(x-a_{2r+1})}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_{2r+1})} f(a_2)$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{(x-a_1)(x-a_2)\ldots(x-a_{2r})}{(a_{2r+1}-a_1)(a_{2r+1}-a_2)\ldots(a_{2r+1}-a_{2r})} f(a_{2r+1}),$$

and this is evidently the polynomial in x of degree $2r$, which has the values

$$f(a_1),\ f(a_2),\ \ldots f(a_{2r+1}),$$

when the argument has the values

$$a_1,\ a_2\ \ldots\ a_{2r+1}.$$

Now replace $\quad a_1,\ a_2\ \ldots\ a_{2r+1}$

by $\quad a,\ a+w,\ a-w,\ \ldots\ a-rw,$

and take the limit of the above expression as r becomes infinite; the formula then evidently becomes

$$\frac{\sin\{\pi(x-a)/w\}}{\pi(x-a)/w} f(a)+\frac{\sin\{\pi(x-a-w)/w\}}{\pi(x-a-w)/w} f(a+w)$$

$$\frac{\sin\{\pi(x-a+w)/w\}}{\pi(x-a-w)/w} f(a-w)+\ldots,$$

that is, the cardinal function corresponding to the given function.

Hence *the cardinal function may be regarded as the limit of the above polynomial in x when r becomes infinite.* When we regard it from this point of view, we see the underlying reason for the absence of short period oscillations and of singularities in the finite part of the plane.

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§3. *The cardinal function and Fourier's integral.* Let us now find what the cardinal function becomes as the interval w decreases indefinitely. We have

$$C(x) = \frac{w}{\pi} \sum_{r=-\infty}^{\infty} f(a + rw) \frac{\sin\{\pi(x-a-rw)/w\}}{\pi(x-a-rw)/w}.$$

Now we may write

$$\frac{\sin\{\pi(x-a-rw)/w\}}{x-a-rw} = \int_0^{\pi/w} \cos n(x-a-rw)\,.\,dn,$$

so that

$$C(x) = \frac{w}{\pi} \sum_{r=-\infty}^{\infty} f(a+rw) \int_0^{\pi/w} \cos n(x-a-rw)\,.\,dn.$$

Let $a + rw = t$, a current variable, so that $w = dt$; then proceeding to the limit, it is readily shown that

$$C(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\,.\,dt \int_0^{\infty} \cos n(x-t)\,.\,dn,$$

which is Fourier's integral.

This shows that *the function represented by Fourier's Integral is essentially a cardinal function*, and we shall investigate some of the characteristic properties of the Fourier integral representation from this point of view.

§4. *Derivation of the Fourier integral formula*

$$\frac{1}{\pi} \int_0^{\infty} dn \int_{-\infty}^{\infty} f(t) \cos n(x-t)\,.\,dt = \tfrac{1}{2}\left[f(x+0) + f(x-0)\right].$$

Returning to the cardinal function, we may write

$$C(x) = \frac{w}{\pi} \sin\frac{\pi}{w}(x-a)\left[\frac{f(a)}{x-a} - \frac{f(a+w)}{x-a-w} + \frac{f(a+2w)}{x-a-2w} \cdots \right.$$
$$\left. - \frac{f(a-w)}{x-a+w} + \frac{f(a-2w)}{x-a+2w} \cdots \right].$$

Now suppose there is a finite discontinuity at the point a, and write $x = a + w/2$, so that

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$$C\left(a+\frac{w}{2}\right) = \frac{w}{\pi}\left[\frac{f(a)}{\frac{1}{2}w} - \frac{f(a+w)}{-\frac{1}{2}w} + \frac{f(a+2w)}{-\frac{3}{2}w} \cdots \right.$$
$$\left. - \frac{f(a-w)}{\frac{3}{2}w} + \frac{f(a-2w)}{\frac{5}{2}w} \cdots \right]$$

$$= \frac{1}{\pi}\left[2f(a) - \tfrac{2}{3}f(a-w) + \tfrac{2}{5}f(a-2w) \ldots \right.$$
$$\left. + 2f(a+w) - \tfrac{2}{3}f(a+2w) + \ldots\right].$$

Now let $w \to 0$, and this becomes

$$\frac{2}{\pi}\left[f(a+0) + f(a-0)\right]\left[1 - \tfrac{1}{3} + \tfrac{1}{5} - \ldots\right]$$
$$= \tfrac{1}{2}\left[f(a+0) + f(a-0)\right],$$

which is the value furnished by the Fourier integral at the discontinuity.

§ 5. *Discussion of the Fourier integral*

$$\frac{1}{\pi}\int_0^\infty dn \int_{-\infty}^{\infty} f(t) \,.\, \cos n\,(z-t) \,.\, dt,$$

z being a complex variable, n and t being, however, always real. The nature of the method will be seen best if we take an example, say $f(t) = \dfrac{1}{1+t^2}$. Let us find the cardinal function corresponding to the function $\dfrac{1}{1+z^2}$, and suppose that $a = 0$ and that w is real.

The cardinal function becomes

$$\frac{w}{\pi}\sin\frac{\pi z}{w}\left[\frac{1}{z} - \frac{1}{1+w^2}\left\{\frac{1}{z-w} + \frac{1}{z+w}\right\}\right.$$
$$\left. + \frac{1}{1+2^2 w^2}\left\{\frac{1}{z-2w} + \frac{1}{z+2w}\right\} - \frac{1}{1+3^2 w^2}\left\{\frac{1}{z-3w} + \frac{1}{z+3w}\right\} + \ldots\right].$$

In order to sum this series let us apply Mittag-Leffler's theorem to the function $\dfrac{1}{(1+z^2)\,.\,\sin\dfrac{\pi z}{w}}$. The only poles are $z = \pm i$, and $z = rw$, $(r = 0, \pm 1, \pm 2, \ldots)$.

The residue both at $z = i$ and $z = -i$ is $-\dfrac{1}{2\sinh \pi/w}$, and the residue at $z = rw$ is $(-1)^r \,.\, \dfrac{w}{\pi} \,.\, \dfrac{1}{1+r^2 w^2}$.

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Hence we have

$$\frac{1}{(1+z^2)\,.\,\sin\frac{\pi z}{w}} = -\frac{1}{2\sinh\frac{\pi}{w}}\left\{\frac{1}{z-i}+\frac{1}{z+i}\right\}$$

$$+\frac{w}{\pi}\left[\frac{1}{z}-\frac{1}{1+w^2}\left\{\frac{1}{z-w}+\frac{1}{w}\right\}-\frac{1}{1+w^2}\left\{\frac{1}{z+w}-\frac{1}{w}\right\}\right.$$

$$\left.+\frac{1}{1+2^2w^2}\left\{\frac{1}{z-2w}+\frac{1}{2w}\right\}+\frac{1}{1+2^2w^2}\left\{\frac{1}{z+2w}-\frac{1}{2w}\right\}\cdots\right]+\phi(z),$$

when $\phi(z)$ is an undetermined function. It may be readily shown in the present case that $\phi(z)$ is identically zero.

We thus have, on multiplying by $\sin\frac{\pi z}{w}$ and rearranging the terms,

$$\frac{w}{\pi}\sin\frac{\pi z}{w}\left[\frac{1}{z}-\frac{1}{1+w^2}\left\{\frac{1}{z-w}+\frac{1}{z+w}\right\}\right.$$

$$\left.+\frac{1}{1+2^2w^2}\left\{\frac{1}{z-2w}+\frac{1}{z+2w}\right\}\cdots\cdots\cdots\right]$$

$$=\frac{1}{1+z^2}+\frac{z\sin\frac{\pi z}{w}}{(1+z^2)\,.\,\sinh\frac{\pi}{w}}.$$

The limit of this as $w \to 0$ is the function represented by the Fourier integral

$$\frac{1}{\pi}\int_0^\infty dn\int_{-\infty}^\infty\frac{\cos n(z-t)}{1+t^2}\,.\,dt.$$

We see from the above result that the cardinal function

$$\frac{1}{1+z^2}+\frac{z}{1+z^2}\,\frac{\sin\frac{\pi z}{w}}{\sinh\frac{\pi}{w}}$$

only agrees with the tabulated function $\frac{1}{1+z^2}$ when $\sin\frac{\pi z}{w} = 0$, *i.e.* when $x = rw$, and $y = 0$.

When $x = rw$, the cardinal function differs from the tabulated function by the term

$$\frac{iz}{1+z^2}\,.\,\frac{\cos\frac{\pi x}{w}\,.\,\sinh\frac{\pi y}{w}}{\sinh\frac{\pi}{w}}.$$

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Now $\underset{w=0}{L} \dfrac{\sinh\dfrac{\pi y}{w}}{\sinh\dfrac{\pi}{w}}$ becomes, on putting $\dfrac{\pi}{w} = m$,

$$\underset{m=\infty}{L} \frac{e^{my} - e^{-my}}{e^m - e^{-m}}.$$

If y be positive, this becomes

$$\underset{m=\infty}{L} \quad e^{m(y-1)},$$

and we see that it vanishes so long as $y < 1$. Also, if y be negative, it vanishes so long as $y > -1$.

When, however, $|y| > 1$, the expression

$$\underset{w=0}{L} \frac{\cos\dfrac{\pi x}{w} \, . \, \sinh\dfrac{\pi y}{w}}{\sinh\dfrac{\pi}{w}}$$

becomes quite indeterminate.

It follows then that the integral

$$\frac{1}{\pi}\int_0^\infty dn \int_{-\infty}^\infty \frac{\cos n(z-t)}{1+t^2} \, . \, dt$$

does represent the function $\dfrac{1}{1+z^2}$ within the strip of the z-plane bounded by the lines $y^2 - 1 = 0$, but is quite indeterminate outside this strip.

§ 6. *Discussion of the Fourier integral when all the quantities n, z, t, may be complex.*

The cardinal function corresponding to the function $f(z)$ becomes, when we take the origin at the point a in the complex plane,

$$\frac{w}{\pi} \sum_{r=-\infty}^{\infty} f(rw) \, . \, \frac{\sin\dfrac{\pi}{w}(z - rw)}{z - rw}.$$

If w be a complex quantity, say $se^{i\theta}$, then we see that the quantities $f(rw)$ now give the values of the function $f(z)$ which lie along a straight line passing through the origin and inclined at an angle θ to the real axis. The tabular interval is s.

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Let us consider then

$$\underset{s=0}{L}\ \frac{s\,e^{i\theta}}{\pi}\ \Sigma f(r\,s\,e^{i\theta})\,.\ \frac{\sin\frac{\pi}{s}\,e^{-i\theta}\,(z-r\,s\,e^{i\theta})}{z-r\,s\,e^{i\theta}},$$

$$=\underset{s=0}{L}\ \frac{s}{\pi}\ \Sigma f(rse^{i\theta})\int_0^{\pi/s}\cos n\,e^{-i\theta}\,(z-r\,s\,e^{i\theta})\,.\,d\,n$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}d\,t\,f\,(t\,e^{i\theta})\int_0^{\infty}\cos n\,e^{-i\theta}\,(z-t\,e^{i\theta})\,.\,d\,n,$$

or, replacing $t\,e^{i\theta}$ by t and $n\,e^{-i\theta}$ by n

$$=\frac{1}{\pi}\int_0^{\infty e^{-i\theta}}d\,n\int_{-\infty\,e^{i\theta}}^{\infty\,e^{i\theta}}f(t)\,.\cos n\,(z-t)\,.\,d\,t.$$

It will be seen that while the integration with respect to t proceeds along a straight line passing through the origin of gradient $\tan\theta$, that with respect to n is taken along a straight line also passing through the origin, but of gradient $-\tan\theta$.

If we again take $f(t)=\frac{1}{1+t^2}$, we may find the region of convergence of this integral by a process very similar to that which we have already adopted. The cardinal function corresponding to the function $\frac{1}{1+z^2}$ is, as we have seen,

$$\frac{1}{1+z^2}+\frac{z\sin\frac{\pi z}{w}}{(1+z^2)\sinh\frac{\pi}{w}},$$

and in this we must put $w=s\,e^{i\theta}$. We have then to consider the term

$$\frac{\sin\frac{\pi\,z\,e^{-i\theta}}{s}}{\sinh\frac{\pi}{s}\,e^{-i\theta}}.$$

Now $z\,e^{-i\theta}=x'+i\,y'$ where x' is measured along the w-line and y' perpendicular to it. Hence, as before, the cardinal function

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differs from the tabular function at the tabulated values by the term

$$\frac{i \cos\frac{\pi x'}{s} \cdot \sinh\frac{\pi y'}{s}}{\sinh\frac{\pi}{s} e^{-i\theta}}.$$

Taking the limit of this expression as $s \to 0$, we find that it vanishes so long as

$$| y' | < \cos\theta,$$

so that the integral in this case converges within the strip of the complex plane bounded by the lines

$$(y - \tan\theta\, x)^2 - 1 = 0.$$

If we evaluate

$$\int_{-\infty}^{\infty} \frac{\cos n e^{-i\theta} (z - t e^{i\theta})}{1 + t^2 . e^{2i\theta}} dt$$

as we may do by integrating the expression $\dfrac{e^{int}}{1 + t^2 e^{2i\theta}}$ round an infinitely large semicircle lying on the upper half of the t-plane and having the real axis as diameter, then we find

$$\frac{1}{1+z^2} = \int_0^{\infty} e^{-n e^{-i\theta}} \cos n e^{-i\theta} z . d n e^{-i\theta}$$

$$= \int_0^{\infty e^{-i\theta}} e^{-n} \cos n z . dn.$$

and this holds in the above region.

By a slight extension of the above mode of reasoning we may establish the following general result :—

If $f(z)$ be a function of z, and if it be decomposed by the general form of Fourier's integral

$$\frac{1}{\pi} \int_0^{\infty e^{-i\theta}} dn \int_{-\infty e^{i\theta}}^{\infty e^{i\theta}} f(t) \cos n (z - t) . dt$$

into periodic constituents, the periodic elements all lying on a straight line of gradient $\tan\theta$ and passing through the origin, then if a pair of straight lines be drawn parallel to and equidistant from this line and enclosing no poles of $f(z)$, the integral (provided it exists) will converge in the strip so formed.

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