

GROUP ACTIONS AND COVERINGS OF BRAUER GRAPH ALGEBRAS

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Abstract. We develop a theory of group actions and coverings on Brauer graphs that parallels the theory of group actions and coverings of algebras. In particular, we show that any Brauer graph can be covered by a tower of coverings of Brauer graphs such that the topmost covering has multiplicity function identically one, no loops, and no multiple edges. Furthermore, we classify the coverings of Brauer graph algebras that are again Brauer graph algebras.

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1. Introduction. In this paper, we introduce the theory of group actions and coverings on Brauer graphs that parallels the theory of group actions and coverings of algebras. In particular, we show that any Brauer graph can be covered by a tower of coverings, the topmost of which is a Brauer graph with no exceptional vertices, no loops, and no multiple edges (Theorem 7.7). This allows many homological questions related to Brauer graph algebras to be simplified by considering Brauer graphs with no exceptional vertices, no loops and no multiple edges. Specifically, we know from covering theory that, for a finite group G , the category of G -graded modules over a Brauer graph algebra is equivalent to the module category of the covering algebra associated with the group G . In a subsequent paper ([10]), we use this fact and Theorem 7.7 to compute a minimal set of generators for the Ext algebra of any Brauer graph algebra. In this current paper, we also classify the coverings of Brauer graph algebras that are again Brauer graph algebras (Theorem 7.1). One reason this result is of interest is that it is difficult, in general, to determine if a covering comes from a Brauer graph simply from knowledge of the quiver and relations.

Since their introduction by Riedtmann in [19], coverings of algebras have been extensively studied and they have rapidly proven to be of interest in the representation

theory of algebras (see, for example, [4, 7, 8, 9, 22] and references therein). In the case of self-injective algebras of finite or tame type – including Brauer graph algebras – this is also well demonstrated in the survey article [21]. The results in [8] and [9], relating the coverings of algebras given by quivers with relations on the one hand, and the group-gradings of algebras on the other hand, lead to an equivalence of categories; namely given a G -grading of a quiver with relations, then the category of finite-dimensional modules of the covering of the algebra is equivalent to the category of finite-dimensional G -graded modules.

Brauer graphs are a generalization of Brauer trees. A Brauer tree is a finite tree, with a cyclic ordering of the edges at each vertex and a multiplicity assigned to exactly one of its vertices, which is called the exceptional vertex. These trees were defined by Brauer in [5] in the study of block algebras of finite groups with cyclic defect groups. The entire algebra structure of a block with cyclic defect groups can be read directly from its Brauer tree. For example, in [12], Janusz gives a description of the non-projective, non-simple indecomposable modules of a block with cyclic defect groups in terms of paths in its Brauer tree, and in [11], Green defined ‘walking around the Brauer tree’, giving projective resolutions of certain modules of a block with cyclic defect groups.

It is well known (see for example [1] or [3]) that Brauer graph algebras are tame special biserial self-injective algebras and that those of finite representation type are the Brauer tree algebras. Furthermore, the derived equivalence classes of Brauer graph algebras have been extensively studied, beginning with Brauer tree algebras (for example, [18], and, for stable equivalence classes, see [17]), generalized Brauer tree algebras (see [16]) and finally Brauer graph algebras (see [13, 14, 15, 20] and [21] and the references therein).

We briefly summarize the results of the paper. In Section 2, we begin by recalling the definitions of a Brauer graph and a Brauer graph algebra together with some essential notation. In Section 3, we define a free Brauer action of a finite abelian group on a Brauer graph and construct the Brauer orbit graph. A free Brauer action of a finite abelian group G on a Brauer graph Γ induces a free action of G on the quiver of the associated Brauer graph algebra and the section culminates with Theorem 3.12, which relates the Brauer graph algebras of Γ and its orbit graph $\bar{\Gamma}$.

Section 4 introduces a Brauer weighting on a Brauer graph which is analogous to a weight function on a quiver. Given a Brauer weighting W to a finite abelian group G on a Brauer graph Δ , we define the Brauer covering graph Δ_W and show that there is a canonical action of G on Δ_W . Theorem 4.9 proves that this action is a free Brauer action and that $\Delta \cong \overline{\Delta_W}$ as Brauer graphs. We apply the results of Section 4 in Section 5 to Brauer graph algebras. If \mathcal{A}_Δ (respectively, \mathcal{A}_{Δ_W} , $\mathcal{A}_{\overline{\Delta_W}}$) is the Brauer graph algebra of Δ (respectively, Δ_W , $\overline{\Delta_W}$), then Theorem 5.6 shows that there are K -algebra isomorphisms

$$\mathcal{A}_\Delta \cong \mathcal{A}_{\overline{\Delta_W}} \cong \overline{\mathcal{A}_{\Delta_W}},$$

where $\overline{\mathcal{A}_{\Delta_W}}$ denotes the orbit algebra of \mathcal{A}_{Δ_W} .

The main result of Section 6 is Theorem 6.3, which shows that if G is a finite abelian group with free Brauer action on Γ , then there is a Brauer weighting W on $\bar{\Gamma}$ such that the Brauer graphs Γ and $\bar{\Gamma}_W$ are isomorphic, as are their corresponding Brauer graph algebras.

Applications of the theory presented are given in Section 7. The Appendix provides a brief survey of covering theory for path algebras and their quotients.

2. Notation. Let Γ be a finite connected graph with at least one edge. We denote by Γ_0 the set of vertices of Γ and by Γ_1 the set of edges of Γ . We equip Γ with a *multiplicity function* $m: \Gamma_0 \rightarrow \mathbb{N} \setminus \{0\}$ and, for each vertex in Γ , we fix a cyclic ordering of the edges incident with this vertex. We call such a graph a *Brauer graph*. Thus, a Brauer graph is a triple (Γ, σ, m) where σ denotes the cyclic ordering and m the multiplicity function. Note that in the Brauer tree case, the multiplicity function has value 1 at all but possibly one vertex, called the exceptional vertex. Although the convention in the literature is to denote a Brauer graph by Γ , where the choice of cyclic ordering and multiplicity function are suppressed, we usually write (Γ, σ, m) . In all examples a planar embedding of Γ is given and we choose the cyclic ordering to be the clockwise ordering of the edges around each vertex.

We say that an edge j in Γ is the *successor* of the edge i at the vertex μ if both i and j are incident with μ and edge j directly follows edge i in the cyclic ordering around μ . For each $\mu \in \Gamma_0$, let $\text{val}(\mu)$ denote the *valency* of μ , that is, the number of edges incident with μ where we count each loop as two edges. If $\text{val}(\mu) = 1$ with edge i incident with the vertex μ then we say that i is a successor of itself. If μ is a vertex with $\text{val}(\mu) = 1$ and $m(\mu) = 1$ so that i is the only edge incident with μ then we call i a *truncated edge* at the vertex μ .

Following [3] and [14], we let K be a field and introduce the Brauer graph algebra of a Brauer graph Γ . We associate to Γ a quiver \mathcal{Q}_Γ and a set of relations ρ_Γ in the path algebra $K\mathcal{Q}_\Gamma$, which we call the *Brauer graph relations*. Let I_Γ be the ideal of $K\mathcal{Q}_\Gamma$ which is generated by the set ρ_Γ . We define the *Brauer graph algebra* \mathcal{A}_Γ of Γ to be the quotient $\mathcal{A}_\Gamma = K\mathcal{Q}_\Gamma/I_\Gamma$.

If the Brauer graph Γ is $\mu \text{ --- } \nu$ with $m(\mu) = m(\nu) = 1$, then \mathcal{Q}_Γ is $\bullet \xrightarrow{x} \bullet$ and $\rho_\Gamma = \{x^2\}$ so the Brauer graph algebra is $K[x]/(x^2)$.

We now define \mathcal{Q}_Γ for a general Brauer graph (excluding the above case, so if edge i is truncated at vertex μ and the endpoints of i are μ and ν then $m(\nu) \text{ val}(\nu) \geq 2$). The vertices of \mathcal{Q}_Γ correspond to the edges of Γ , that is, for every edge $i \in \Gamma_1$ there is a corresponding vertex v_i in \mathcal{Q}_Γ . If edge j is the successor of edge i at the vertex μ and edge i is not a truncated edge at μ then there is an arrow from v_i to v_j in \mathcal{Q}_Γ . For each vertex μ and edge i incident with μ , let $i = i_1, i_2, \dots, i_{\text{val}(\mu)}$ be the edges incident with μ listed in the cyclic ordering, where the loops are listed twice and the other edges precisely once. We call this the *successor sequence of i at μ* . We set $i_{\text{val}(\mu)+1} = i$, noting that i is the successor of $i_{\text{val}(\mu)}$.

In case Γ has at least one loop, care must be taken. In such circumstances, for each vertex μ , we choose a distinguished edge, i_μ , incident with μ . If ℓ is a loop at μ , ℓ occurs twice in the successor sequence of i_μ . We distinguish the first and second occurrences of ℓ in this sequence and view the two occurrences as two edges in Γ_1 . Thus, Γ_1 is the set of all edges with the proviso that loops are listed twice and have different successors. In particular, if Γ is $\mu \text{ --- } i$ then, since i is viewed as two edges, say i and \hat{i} , the successor sequence of i at vertex μ is i, \hat{i} and the successor sequence of \hat{i} at μ is \hat{i}, i . These sequences imply that there are two arrows in the quiver \mathcal{Q}_Γ , and $\mathcal{Q}_\Gamma = \bullet \xrightarrow{i} \bullet \xrightarrow{\hat{i}} \bullet$.

In order to define the Brauer graph relations ρ_Γ we need a *quantizing function* q . Let $\mathcal{X}_\Gamma = \{(i, \mu) \mid i \in \Gamma_1 \text{ is incident with } \mu \in \Gamma_0 \text{ and } i \text{ is not truncated at either of its endpoints}\}$ and let $q: \mathcal{X}_\Gamma \rightarrow K \setminus \{0\}$ be a set function. We denote $q((i, \mu))$ by $q_{i,\mu}$. With these additional data we call (Γ, σ, m, q) a *quantized Brauer graph*. We remark that if the Brauer graph Γ is $\mu \text{ --- } \nu$ then $\mathcal{X}_\Gamma = \emptyset$. Furthermore, if the

Brauer graph algebra is assumed to be symmetric and if the field is algebraically closed, then we can set $q = 1$ (see [3]).

There are three types of relations for (Γ, σ, m, q) . Note that we write our paths from left to right.

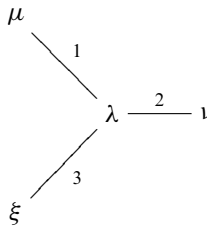
Relations of type one. For each vertex μ and edge i incident with μ , which is not truncated at the vertex μ , let $i = i_1, i_2, \dots, i_{\text{val}(\mu)}$ be the successor sequence of i at μ . From this we obtain a cycle $C_{i,\mu} = a_1 a_2 \dots a_{\text{val}(\mu)}$ in \mathcal{Q}_Γ , where the arrow a_r corresponds to the edge i_{r+1} being the successor of the edge i_r at the vertex μ . With this notation, for each edge $i \in \Gamma$ with endpoints μ and ν so that i is not truncated at either μ or ν , ρ_Γ contains either $q_{i,\mu} C_{i,\mu}^{m(\mu)} - q_{i,\nu} C_{i,\nu}^{m(\nu)}$ or $q_{i,\nu} C_{i,\nu}^{m(\nu)} - q_{i,\mu} C_{i,\mu}^{m(\mu)}$. We call this a type one relation. Note that since one of these relations is the negative of the other, the ideal I_Γ does not depend on this choice.

Relations of type two. The second type of relations occurs if i is a truncated edge at the vertex μ and the endpoints of i are μ and ν . Let $C_{i,\nu} = b_1 b_2 \dots b_{\text{val}(\nu)}$ be the cycle associated with edge i incident with vertex ν . In this case, we have a relation $C_{i,\nu}^{m(\nu)} b_1$.

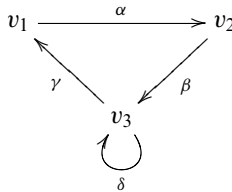
Relations of type three. The third type of relations are quadratic monomial relations of the form ab in $K\mathcal{Q}_\Gamma$ where ab is not a subpath of any $C_{i,\mu}$.

EXAMPLE 2.1.

- (1) The graph (Γ, σ, m)

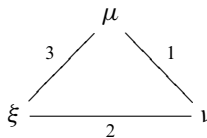


with $m(\lambda) = m(\mu) = m(\nu) = 1$ and $m(\xi) = 2$ has edge 1 truncated at vertex μ and edge 2 truncated at vertex ν . Then $\mathcal{X}_\Gamma = \{(3, \lambda), (3, \xi)\}$. Let $q: \mathcal{X}_\Gamma \rightarrow K \setminus \{0\}$ be a quantizing function. The Brauer graph algebra associated with the quantized Brauer graph (Γ, σ, m, q) has quiver

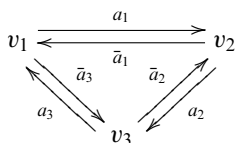


and $\rho_\Gamma = \{q_{3,\xi} \delta^2 - q_{3,\lambda} \gamma \alpha \beta, \alpha \beta \gamma \alpha, \beta \gamma \alpha \beta, \beta \delta, \delta \gamma\}$.

- (2) The graph (Γ, σ, m)



with $m(\mu) = 3$ and $m(\nu) = m(\xi) = 1$ has $\mathcal{X}_\Gamma = \{(1, \mu), (1, \nu), (2, \xi), (2, \nu), (3, \xi), (3, \mu)\}$. Let $q: \mathcal{X}_\Gamma \rightarrow K \setminus \{0\}$. The Brauer graph algebra of $(\Gamma, \mathfrak{o}, m, q)$ has quiver



$$\text{and } \rho_\Gamma = \{q_{1,\mu}(\bar{a}_3 a_3)^3 - q_{1,\nu} a_1 \bar{a}_1, q_{2,\xi} a_2 \bar{a}_2 - q_{2,\nu} \bar{a}_1 a_1, q_{3,\xi} \bar{a}_2 a_2 - q_{3,\mu} (a_3 \bar{a}_3)^3, a_1 a_2, a_2 a_3, a_3 a_1, \bar{a}_1 \bar{a}_3, \bar{a}_3 \bar{a}_2, \bar{a}_2 \bar{a}_1\}.$$

3. Group actions on Brauer graphs and Brauer graph algebras. We assume throughout this paper that G is a finite abelian group. We note here that G being abelian is a necessary condition for the main results of this paper, though is not explicitly required in the current section. However, since we require G to be abelian in Section 4 onwards, we keep this assumption throughout the paper.

Suppose that G is a finite abelian group which acts on a finite connected graph Σ . Let $\bar{\Sigma} = \Sigma/G$ be the orbit graph of Σ under G , and let $\bar{\mu} = \{\mu^g \mid g \in G\}$ and $\bar{i} = \{i^g \mid g \in G\}$ where $\mu \in \Sigma_0$ and $i \in \Sigma_1$. If the action of G on Σ is not faithful, then let N be the normal subgroup of G consisting of those elements that fix every vertex and every edge of Σ . Then G/N acts faithfully on Σ . Without loss of generality, we assume throughout this paper that our group actions are faithful.

Since a Brauer graph comes equipped with a cyclic ordering and multiplicity function we now define a Brauer action of a finite abelian group on a Brauer graph as follows.

DEFINITION 3.1. Let G be a finite abelian group and let $(\Gamma, \mathfrak{o}, m)$ be a Brauer graph. We say that there is a Brauer action of G on $(\Gamma, \mathfrak{o}, m)$ if

- (1) G acts (faithfully) on the graph Γ ,
- (2) the action of G on Γ is orientation preserving, that is, if j is the successor of i at the vertex μ , then, for all $g \in G$, j^g is the successor of i^g at the vertex μ^g , and
- (3) $m(\mu) = m(\mu^g)$ for all $g \in G, \mu \in \Gamma_0$.

If there is a Brauer action of a finite abelian group G on a Brauer graph $(\Gamma, \mathfrak{o}, m)$ then $\text{val}(\mu) = \text{val}(\mu^g)$ for all $g \in G$ and $\mu \in \Gamma_0$, since successors are preserved by the G -action. In particular, by property (3), $\text{val}(\mu)m(\mu) = \text{val}(\mu^g)m(\mu^g)$ for all $g \in G$ and $\mu \in \Gamma_0$.

We now define a free Brauer action.

DEFINITION 3.2. Let G be a finite abelian group and let $(\Gamma, \mathfrak{o}, m)$ be a Brauer graph. A Brauer action of G on $(\Gamma, \mathfrak{o}, m)$ is a free Brauer action if G acts freely on the edge set Γ_1 of Γ , that is, if $i \in \Gamma_1$ and $i^g = i$ then $g = \text{id}_G$, the identity in G .

We remark that if there is a Brauer action of G on $(\Gamma, \mathfrak{o}, m)$ and if there exists $g \in G$ such that $i^g = i$ for some $i \in \Gamma_1$ then, since the action of G is orientation preserving and Γ is connected, it follows that $j^g = j$ for all $j \in \Gamma_1$. Moreover, if there is a Brauer action of a non-trivial finite abelian group G on $(\Gamma, \mathfrak{o}, m)$ and if $|\Gamma_0| \geq 3$, then there

is a free Brauer action of G on (Γ, \circ, m) if and only if whenever $i \in \Gamma_1$ then there is some $g \in G$ with $i^g \neq i$. We emphasize that the group G need not act freely on the set of vertices of Γ for the Brauer action of G on (Γ, \circ, m) to be a free Brauer action.

The next results show that if there is a free Brauer action of G on the Brauer graph (Γ, \circ, m) , then there is a multiplicity function \bar{m} on $\bar{\Gamma}$ which is compatible with m and an induced cyclic ordering \bar{o} which makes $(\bar{\Gamma}, \bar{o}, \bar{m})$ a Brauer graph.

LEMMA 3.3. *Suppose there is a free Brauer action of G on (Γ, \circ, m) . Then $\text{val}(\bar{\mu}) \mid \text{val}(\mu)$ for all $\mu \in \Gamma_0$. Moreover, if $i \in \Gamma_1$ is incident with μ , then there exist $1 \leq k \leq \text{val}(\mu)$, $s \geq 0$ and $g \in G$ such that*

- (1) $\text{val}(\bar{\mu}) = k$;
- (2) the successor sequence of i at μ is

$$i = i_1, i_2, \dots, i_k, i_1^g, i_2^g, \dots, i_k^g, \dots, i_1^{g^s}, i_2^{g^s}, \dots, i_k^{g^s} = i_{\text{val}(\mu)};$$

- (3) $\text{val}(\mu) = \text{val}(\bar{\mu})(s + 1)$.

Proof. Let $\mu \in \Gamma_0$ and $i \in \Gamma_1$ be incident with μ . If $\text{val}(\mu) = 1$ then clearly $\text{val}(\bar{\mu}) = 1$ and we are done. So assume that $\text{val}(\mu) > 1$. Consider the successor sequence of i at μ , $i = i_1, i_2, \dots, i_{\text{val}(\mu)}$. Recall that $i_{\text{val}(\mu)+1} = i$ since i is the successor of $i_{\text{val}(\mu)}$. There is k minimal with $1 \leq k \leq \text{val}(\mu)$ and $i_{k+1} = i^g$ for some $g \in G$. If $1 \leq \alpha < \beta \leq k$ and $i_\alpha = i_{\beta+1}^h$ for some $h \in G$ then $\alpha = 1$ and $\beta = k$. To see this, if $\alpha = 1$ then $\beta = k$ from the choice of k . On the other hand, if $\alpha > 1$ and there exists h with $i_\alpha = i_{\beta+1}^h$, using Definition 3.1(2) we see that $i_1 = i_{\beta-\alpha+2}^h$. Therefore, $\beta - \alpha + 2 = k + 1$ by definition of k and hence $\beta - \alpha = k - 1$. But this contradicts $\alpha \geq 2$ and $\beta \leq k$. From this it follows that $\text{val}(\bar{\mu}) = k$.

Repeating this argument shows that the successor sequence of i at μ can be written as

$$i = i_1, i_2, \dots, i_k, i_1^g, i_2^g, \dots, i_k^g, \dots, i_1^{g^s}, i_2^{g^s}, \dots, i_k^{g^s} = i_{\text{val}(\mu)}$$

for some $s \geq 0$. Hence, $\text{val}(\bar{\mu})(s + 1) = \text{val}(\mu)$ and we are done. □

PROPOSITION 3.4. *Suppose there is a free Brauer action of G on (Γ, \circ, m) . Let \bar{m} be the function*

$$\bar{m}: \bar{\Gamma}_0 \rightarrow \mathbb{N} \setminus \{0\}, \quad \bar{\mu} \mapsto \text{val}(\mu)m(\mu) / \text{val}(\bar{\mu}).$$

Then there is a cyclic ordering \bar{o} induced by \circ such that $(\bar{\Gamma}, \bar{o}, \bar{m})$ is a Brauer graph.

Proof. We begin by showing that the cyclic ordering \circ induces a cyclic ordering \bar{o} . Let $\mu \in \Gamma_0$ and $i \in \Gamma_1$ be incident with μ . If $\text{val}(\mu) = 1$ so that $\text{val}(\bar{\mu}) = 1$, then the cyclic ordering on $\bar{\Gamma}$ is the only one possible. So assume that $\text{val}(\mu) > 1$. Consider the successor sequence of i at μ , $i = i_1, i_2, \dots, i_{\text{val}(\mu)}$. Let k be as in Lemma 3.3. We take the successor sequence for \bar{i} at $\bar{\mu}$ to be $\bar{i}_1, \bar{i}_2, \dots, \bar{i}_k$. This gives the desired cyclic ordering \bar{o} . In particular, if j is the successor of i at vertex μ then \bar{j} is the successor of \bar{i} at vertex $\bar{\mu}$.

The set function \bar{m} is well defined by Definition 3.1(3) and Lemma 3.3. □

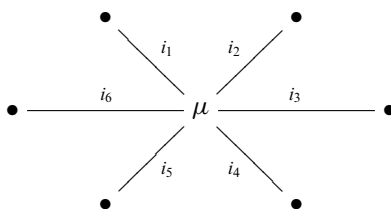
We call the Brauer graph $(\bar{\Gamma}, \bar{o}, \bar{m})$ in Proposition 3.4 the *Brauer orbit graph* of (Γ, \circ, m) associated with the action of G .

PROPOSITION 3.5. *Suppose there is a free Brauer action of G on (Γ, σ, m) . Then the following properties hold.*

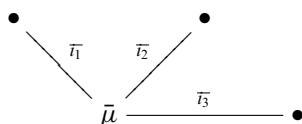
- (1) $\overline{m}(\overline{\mu}) \text{val}(\overline{\mu}) = m(\mu) \text{val}(\mu) = m(\mu^g) \text{val}(\mu^g)$ for all $\mu \in \Gamma_0$ and $g \in G$.
- (2) Edge $i \in \Gamma_1$ is truncated at vertex μ if and only if $\overline{i} \in \overline{\Gamma}_1$ is truncated at vertex $\overline{\mu}$.
- (3) Edge $i \in \Gamma_1$ is truncated at vertex μ if and only if $i^g \in \Gamma_1$ is truncated at vertex μ^g for all $g \in G$.

Proof. (1) is immediate from the definition of \overline{m} . For the proof of (2) we recall that the edge $i \in \Gamma_1$ is truncated at vertex μ if and only if $m(\mu) \text{val}(\mu) = 1$. Now, from (1), this holds if and only if $\overline{m}(\overline{\mu}) \text{val}(\overline{\mu}) = 1$, which is precisely the condition that $\overline{i} \in \overline{\Gamma}_1$ is truncated at vertex $\overline{\mu}$. Hence, property (2) holds. The proof of (3) is similar to that of (2), since $m(\mu) \text{val}(\mu) = 1$ if and only if $m(\mu^g) \text{val}(\mu^g) = 1$ from Definition 3.1(3) and the subsequent remark. □

EXAMPLE 3.6. Let (Γ, σ, m) be the Brauer graph



where each vertex has multiplicity 1. Let G be the cyclic group $G = \mathbb{Z}_2 = \langle g \mid g^2 = \text{id} \rangle$ so that G acts on Γ with $i_1^g = i_4, i_2^g = i_5, i_3^g = i_6$. Then necessarily $i_4^g = i_1, i_5^g = i_2, i_6^g = i_3$. This is a free Brauer action of G on Γ and $\overline{\Gamma}$ is the Brauer graph



where $\overline{\mu}$ has multiplicity 2 and the other vertices have multiplicity 1.

We now extend the concept of a Brauer action by defining a Brauer action on a quantized Brauer graph (Γ, σ, m, q) .

DEFINITION 3.7. A Brauer action of a finite abelian group G on the Brauer graph (Γ, σ, m) is a Brauer action of G on the quantized Brauer graph (Γ, σ, m, q) if

$$\frac{q_{i,\mu}}{q_{i,v}} = \frac{q_{i^g,\mu^g}}{q_{i^g,v^g}},$$

for all $g \in G$ and $\mu \overset{i}{\text{---}} v$ in Γ such that i is not truncated at either of its endpoints.

To understand the importance of the quotients above, we note that the relation of type one $q_{i,\mu} C_{i,\mu}^{m(\mu)} - q_{i,v} C_{i,v}^{m(v)}$ may be replaced by either $C_{i,\mu}^{m(\mu)} - \frac{q_{i,v}}{q_{i,\mu}} C_{i,v}^{m(v)}$ or $C_{i,v}^{m(v)} - \frac{q_{i,\mu}}{q_{i,v}} C_{i,\mu}^{m(\mu)}$ without altering the ideal I_Γ . These last two relations distinguish between the endpoints of the edge i . We formalize this as follows. Let $\mathcal{Y} = \{i \in \Gamma_1 \mid$

i is not truncated at either of its endpoints}. Let $\mathcal{E}_1: \mathcal{Y} \rightarrow \Gamma_0$ be a set function such that $\mathcal{E}_1(i)$ is an endpoint of i . Let $\mathcal{E}_2: \mathcal{Y} \rightarrow \Gamma_0$ be the set function with the property that $\mathcal{E}_1(i)$ and $\mathcal{E}_2(i)$ are the two endpoints of i . We note that i is a loop if and only if $\mathcal{E}_1(i) = \mathcal{E}_2(i)$. From the above remarks, we see that, for each $i \in \mathcal{Y}$, the relation of type one may be replaced by

$$C_{i, \mathcal{E}_1(i)}^{m(\mathcal{E}_1(i))} = \frac{q_{i, \mathcal{E}_2(i)}}{q_{i, \mathcal{E}_1(i)}} C_{i, \mathcal{E}_2(i)}^{m(\mathcal{E}_2(i))}.$$

LEMMA 3.8. *Suppose there is a free Brauer action of G on $(\Gamma, \mathfrak{o}, m, q)$. Then there is a choice of set functions \mathcal{E}_1 and \mathcal{E}_2 such that*

$$(\mathcal{E}_j(i))^g = \mathcal{E}_j(i^g)$$

for all $i \in \mathcal{Y}, g \in G, j = 1, 2$.

Proof. It suffices to prove this for $j = 1$. Proposition 3.5(3) implies that \mathcal{Y} is a disjoint union of orbits. For each orbit, choose some edge i in the orbit and arbitrarily select an endpoint; define $\mathcal{E}_1(i)$ to be this endpoint. For all other i^g in the orbit of i , define $\mathcal{E}_1(i^g) = (\mathcal{E}_1(i))^g$. This yields a choice of \mathcal{E}_1 that satisfies the desired property. \square

If there is a free Brauer action of G on a quantized Brauer graph $(\Gamma, \mathfrak{o}, m, q)$, then we henceforth assume that the functions \mathcal{E}_1 and \mathcal{E}_2 satisfy $(\mathcal{E}_j(i))^g = \mathcal{E}_j(i^g)$, for all $i \in \mathcal{Y}, g \in G$, and $j = 1, 2$. We are now in a position to define a quantizing function \bar{q} for the Brauer orbit graph $(\bar{\Gamma}, \bar{\mathfrak{o}}, \bar{m}, \bar{q})$ and thus extend Proposition 3.4. Recall that $\mathcal{X}_{\bar{\Gamma}} = \{(\bar{i}, \bar{\mu}) \mid \bar{i} \in \bar{\Gamma}_1 \text{ is incident with } \bar{\mu} \in \bar{\Gamma}_0 \text{ and } \bar{i} \text{ is not truncated at either of its endpoints}\}$. Define $\bar{q}: \mathcal{X}_{\bar{\Gamma}} \rightarrow K \setminus \{0\}$ by

$$\bar{q}((\bar{i}, \bar{\mu})) = \frac{q_{i, \mu}}{q_{i, \mathcal{E}_1(i)}}.$$

We denote $\bar{q}((\bar{i}, \bar{\mu}))$ by $\bar{q}_{\bar{i}, \bar{\mu}}$ and note that, by the above discussion, \bar{q} is well defined. Summarizing, we have the following result.

PROPOSITION 3.9. *Suppose there is a free Brauer action of G on $(\Gamma, \mathfrak{o}, m, q)$. Then there is a quantizing function \bar{q} so that $(\bar{\Gamma}, \bar{\mathfrak{o}}, \bar{m}, \bar{q})$ is a quantized Brauer graph satisfying the property that if $i \in \mathcal{Y}$ with endpoints μ and ν , then*

$$\frac{q_{i, \mu}}{q_{i, \nu}} = \frac{\bar{q}_{\bar{i}, \bar{\mu}}}{\bar{q}_{\bar{i}, \bar{\nu}}}.$$

We call the quantized Brauer graph $(\bar{\Gamma}, \bar{\mathfrak{o}}, \bar{m}, \bar{q})$ in Proposition 3.9 the *quantized Brauer orbit graph* of $(\Gamma, \mathfrak{o}, m, q)$ associated with the action of G .

Now we turn our attention to the Brauer graph algebra associated with a Brauer graph on which there is a free Brauer action.

LEMMA 3.10. *Suppose there is a free Brauer action of G on the Brauer graph $(\Gamma, \mathfrak{o}, m)$. Then there is an induced free group action of G on the quiver \mathcal{Q}_{Γ} . Moreover, if $(\Gamma, \mathfrak{o}, m, q)$ is a quantized Brauer graph then the induced G -action satisfies*

$$x \in I_{\Gamma} \text{ if and only if } x^g \in I_{\Gamma} \text{ for all } g \in G.$$

Proof. Let v_i be a vertex in the quiver \mathcal{Q}_Γ so that v_i corresponds to the edge i in Γ . We define the action of G on the vertices of \mathcal{Q}_Γ by $(v_i)^g = v_{i^g}$ for $g \in G$. Let a be an arrow from v_i to v_j in the quiver \mathcal{Q}_Γ ; then i and j are two edges in Γ incident with the same vertex and such that j is the successor of i . By Definition 3.1(2), the edge j^g is the successor of the edge i^g for all $g \in G$. So, for $g \in G$, we define a^g to be the arrow from v_{i^g} to v_{j^g} in \mathcal{Q}_Γ . It is now straightforward to show, for all $g, h \in G$, that $((v_i)^g)^h = (v_{i^g})^h = v_{(i^g)^h} = v_{i^{(gh)}} = (v_i)^{(gh)}$ and, similarly, that $(a^g)^h = a^{(gh)}$.

To show that this is a free action on \mathcal{Q}_Γ , suppose that, for a vertex v_i in \mathcal{Q}_Γ (which corresponds to an edge i in Γ) and g in G , we have $(v_i)^g = v_i$. Then $v_{i^g} = v_i$ so by definition of \mathcal{Q}_Γ we have that $i^g = i$. Hence, by Definition 3.2 we have $g = \text{id}_G$. It follows that G acts freely on \mathcal{Q}_Γ .

Now suppose that we have a type one relation

$$C_{i, \mathcal{E}_1(i)}^{m(\mathcal{E}_1(i))} - \frac{q_{i, \mathcal{E}_2(i)}}{q_{i, \mathcal{E}_1(i)}} C_{i, \mathcal{E}_2(i)}^{m(\mathcal{E}_2(i))}.$$

From Definition 3.7 and Lemma 3.8 we have that

$$\left(C_{i, \mathcal{E}_1(i)}^{m(\mathcal{E}_1(i))} - \frac{q_{i, \mathcal{E}_2(i)}}{q_{i, \mathcal{E}_1(i)}} C_{i, \mathcal{E}_2(i)}^{m(\mathcal{E}_2(i))} \right)^g = C_{i^g, \mathcal{E}_1(i^g)}^{m(\mathcal{E}_1(i^g))} - \frac{q_{i^g, \mathcal{E}_2(i^g)}}{q_{i^g, \mathcal{E}_1(i^g)}} C_{i^g, \mathcal{E}_2(i^g)}^{m(\mathcal{E}_2(i^g))}$$

which is in I_Γ . It is easy to see that if x is a relation of type 2 or type 3, then x^g is also a relation of type 2 or type 3, respectively, for all $g \in G$. This completes the proof. \square

We remark that there is a concept of a free group action on a category, see [2, 6], which specializes to a free group action on a quiver, as is given in the previous lemma.

Suppose there is a free Brauer action of G on (Γ, σ, m) . The induced action of G on \mathcal{Q}_Γ given in Lemma 3.10, allows us to consider the orbit quiver $\overline{\mathcal{Q}}_\Gamma$, which is defined in the Appendix.

PROPOSITION 3.11. *Suppose there is a free Brauer action of G on (Γ, σ, m) . Then $\overline{\mathcal{Q}}_\Gamma \cong \mathcal{Q}_{\overline{\Gamma}}$ as quivers.*

Proof. We prove that $(\overline{\mathcal{Q}}_\Gamma)_0 \cong (\mathcal{Q}_{\overline{\Gamma}})_0$ and $(\overline{\mathcal{Q}}_\Gamma)_1 \cong (\mathcal{Q}_{\overline{\Gamma}})_1$.

The set $(\overline{\mathcal{Q}}_\Gamma)_0$ corresponds to the orbits of vertices of \mathcal{Q}_Γ under the action of G . The set $(\mathcal{Q}_{\overline{\Gamma}})_0$ is the set of vertices of $\mathcal{Q}_{\overline{\Gamma}}$ and so corresponds to the set of edges of $\overline{\Gamma}$, that is, to the orbits of edges of Γ under the action of G . By definition of \mathcal{Q}_Γ , the vertices of \mathcal{Q}_Γ correspond to the edges of Γ . From the definition of the action of G on \mathcal{Q}_Γ in Lemma 3.10, the two sets of orbits are in bijection. Thus, $(\overline{\mathcal{Q}}_\Gamma)_0 \cong (\mathcal{Q}_{\overline{\Gamma}})_0$.

The set $(\overline{\mathcal{Q}}_\Gamma)_1$ is the set of arrows of $\mathcal{Q}_{\overline{\Gamma}}$, and so corresponds to the orbits under the action of G of a vertex $\mu \in \Gamma$ with two incident edges i and j , where j is the successor of i . By Definition 3.1(2), the edges i^g and j^g are incident with the vertex μ^g and j^g is the successor of i^g . Thus, the orbit of this action on an arrow $v_i \xrightarrow{\alpha} v_j$ in \mathcal{Q}_Γ corresponds to an arrow $v_i \xrightarrow{\bar{\alpha}} v_j$ in $(\overline{\mathcal{Q}}_\Gamma)_1$. On the other hand, an element of $(\overline{\mathcal{Q}}_\Gamma)_1$ corresponds to an orbit $v_i \xrightarrow{\alpha} v_j$ under the action of G of an arrow $v_i \xrightarrow{\alpha} v_j$ in \mathcal{Q}_Γ . It is now easy to see that there is a bijection between elements $v_i \xrightarrow{\bar{\alpha}} v_j$ of $(\overline{\mathcal{Q}}_\Gamma)_1$ and elements $v_i \xrightarrow{\alpha} v_j$ of $(\mathcal{Q}_{\overline{\Gamma}})_1$. \square

Suppose there is a free Brauer action of G on the quantized Brauer graph $(\Gamma, \mathfrak{o}, m, q)$. Then, from the Appendix and Lemma 3.10, we have that G acts freely on \mathcal{Q}_Γ and $\overline{\mathcal{A}}_\Gamma = K\overline{\mathcal{Q}}_\Gamma/\overline{I}_\Gamma$.

THEOREM 3.12. *Let G be a finite abelian group with a free Brauer action on the quantized Brauer graph $(\Gamma, \mathfrak{o}, m, q)$. Let $(\overline{\Gamma}, \overline{\mathfrak{o}}, \overline{m}, \overline{q})$ be the associated quantized Brauer orbit graph. Let \mathcal{A}_Γ be the Brauer graph algebra associated with $(\Gamma, \mathfrak{o}, m, q)$ and $\mathcal{A}_{\overline{\Gamma}}$ the Brauer graph algebra associated with $(\overline{\Gamma}, \overline{\mathfrak{o}}, \overline{m}, \overline{q})$. Then*

$$\overline{\mathcal{A}}_\Gamma \cong \mathcal{A}_{\overline{\Gamma}}.$$

Proof. From the discussion above and Proposition 3.11, it suffices to show that $\overline{I}_\Gamma \cong I_{\overline{\Gamma}}$. Given our choice of \mathcal{E}_1 and \mathcal{E}_2 as in Lemma 3.8, we see that \mathcal{E}_1 and \mathcal{E}_2 induce set functions $\overline{\mathcal{E}}_1$ and $\overline{\mathcal{E}}_2$ for $\overline{\Gamma}$.

First, we consider a relation x of type one in I_Γ . For this, suppose i is an edge in Γ_1 which is not truncated at either of its endpoints and let $\mathcal{E}_1(i) = \mu$ and $\mathcal{E}_2(i) = \nu$. Let $i = i_1, i_2, \dots, i_{\text{val}(\mu)}$ (respectively $i = j_1, j_2, \dots, j_{\text{val}(\nu)}$) be the successor sequence for i at vertex μ (respectively i at vertex ν) with associated cycle $C_{i,\mu} = a_1 a_2 \dots a_{\text{val}(\mu)}$ (respectively $C_{i,\nu} = b_1 b_2 \dots b_{\text{val}(\nu)}$) in \mathcal{Q}_Γ . Thus, $x = C_{i,\mu}^{m(\mu)} - \frac{q_{i,\nu}}{q_{i,\mu}} C_{i,\nu}^{m(\nu)}$. By Lemma 3.3, there exist $1 \leq k \leq \text{val}(\mu)$, $1 \leq l \leq \text{val}(\nu)$, $g, h \in G$ and $s, t \geq 0$ such that the successor sequence for i at μ is

$$i_1, i_2, \dots, i_k, i_1^g, i_2^g, \dots, i_k^g, \dots, i_1^{g^s}, i_2^{g^s}, \dots, i_k^{g^s}$$

and the successor sequence for i at ν is

$$j_1, j_2, \dots, j_l, j_1^h, j_2^h, \dots, j_l^h, \dots, j_1^{h^t}, j_2^{h^t}, \dots, j_l^{h^t}.$$

It now follows that $\bar{x} \in \overline{I}_\Gamma$ is given by

$$\bar{x} = (\bar{a}_1 \bar{a}_2 \dots \bar{a}_k)^{m(\mu)(s+1)} - \frac{q_{i,\nu}}{q_{i,\mu}} (\bar{b}_1 \bar{b}_2 \dots \bar{b}_l)^{m(\nu)(t+1)}.$$

From Lemma 3.3 we have $\text{val}(\mu) = \text{val}(\bar{\mu})(s + 1)$ and $\text{val}(\nu) = \text{val}(\bar{\nu})(t + 1)$, and, from Proposition 3.5(1), we have $\overline{m}(\bar{\mu}) \text{val}(\bar{\mu}) = m(\mu) \text{val}(\mu)$. Thus, $m(\mu)(s + 1) = \overline{m}(\bar{\mu})$ and $m(\nu)(t + 1) = \overline{m}(\bar{\nu})$. Finally by Proposition 3.9,

$$\bar{x} = (\bar{a}_1 \bar{a}_2 \dots \bar{a}_k)^{\overline{m}(\bar{\mu})} - \frac{\overline{q}_{i,\bar{\nu}}}{\overline{q}_{i,\bar{\mu}}} (\bar{b}_1 \bar{b}_2 \dots \bar{b}_l)^{\overline{m}(\bar{\nu})},$$

which is the relation of type one in $K\overline{\mathcal{Q}}_{\overline{\Gamma}}$ corresponding to \bar{i} under the isomorphism of Proposition 3.11.

The remaining relations of types two and three are monomial and a similar argument holds for these cases. □

4. Weightings on Brauer graphs. Throughout this section, $(\Delta, \mathfrak{o}, m)$ will denote a Brauer graph and G will continue to be a finite abelian group. For each $\mu \in \Delta_0$, we define \mathcal{Z}_μ to be the set $\mathcal{Z}_\mu = \{(i, j) \mid i, j \in \Delta_1, j \text{ is the successor of } i \text{ at vertex } \mu\}$. Let

\mathcal{Z}_Δ be the disjoint union

$$\mathcal{Z}_\Delta = \dot{\bigcup}_{\mu \in \Delta_0} \mathcal{Z}_\mu .$$

DEFINITION 4.1. A set function $W: \mathcal{Z}_\Delta \rightarrow G$ is called a *successor weighting* of the Brauer graph (Δ, σ, m) . For $\mu \in \Delta_0$ we define the *order of μ* , denoted $\text{ord}(\mu)$, to be the order in G of the element

$$\omega_\mu = \prod_{(i,j) \in \mathcal{Z}_\mu} W(i,j).$$

Let $W: \mathcal{Z}_\Delta \rightarrow G$ be a successor weighting of (Δ, σ, m) . The aim of this section is to construct a new Brauer graph $(\Delta_W, \sigma_W, m_W)$ and a free Brauer action of G on $(\Delta_W, \sigma_W, m_W)$ such that the Brauer orbit graph $(\overline{\Delta_W}, \overline{\sigma_W}, \overline{m_W})$ associated with W is isomorphic to (Δ, σ, m) . We begin with the construction of Δ_W .

Suppose μ is a vertex in Δ . Let H_μ denote the subgroup of G generated by the element ω_μ , and $H_\mu \setminus G$ the set of cosets of H_μ in G . We define an equivalence relation on the set $\{(i, L) \mid i \in \Delta_1 \text{ is incident with } \mu, L \in H_\mu \setminus G\}$ to be the equivalence relation generated by

$$(i, L) \sim (j, L') \text{ if } j \text{ is the successor of } i \text{ at vertex } \mu \text{ and } L' = LW(i, j), \text{ for } L, L' \in H_\mu \setminus G.$$

Let \mathcal{D}_μ be the set of equivalence classes under this equivalence relation; we denote the equivalence class of (i, L) by $[i, L]$.

We recall that if an edge i is a loop at the vertex μ , then, by the remarks in Section 2, i is viewed as two edges, say i and \hat{i} , having different successors. It can easily occur that (i, L) is not equivalent to (\hat{i}, L) , for all cosets $L \in H_\mu \setminus G$.

Suppose $i \in \Delta_1$ is incident with vertex μ in Δ and let $i = i_1, i_2, \dots, i_{\text{val}(\mu)}$ be the successor sequence of i at vertex μ with our usual convention that $i_{\text{val}(\mu)+1} = i_1$. Define $\omega(i_1, i_r) = W(i_1, i_2)W(i_2, i_3) \cdots W(i_{r-1}, i_r)$ for $1 \leq r \leq \text{val}(\mu) + 1$. Then $\omega_\mu = \omega(i_1, i_{\text{val}(\mu)+1})$. In the rest of the paper and in order to emphasize the vertex μ , we denote a coset of H_μ in G by $H_{\mu,s}$ (for some $1 \leq s \leq |G|/\text{ord}(\mu)$).

LEMMA 4.2. Let $W: \mathcal{Z}_\Delta \rightarrow G$ be a successor weighting of (Δ, σ, m) , $i \in \Delta_1$ be incident with $\mu \in \Delta_0$ and $i = i_1, i_2, \dots, i_{\text{val}(\mu)}$ be the successor sequence of i at μ . Let $H_{\mu,s}$ be a coset of H_μ in G and j an edge in Δ . Then, $[i, H_{\mu,s}] = [j, H_{v,t}]$ if and only if $v = \mu, j = i_r$, and $H_{\mu,t} = H_{\mu,s}\omega(i_1, i_r)\omega_\mu^\theta$ for some $1 \leq r \leq \text{val}(\mu)$ and $0 \leq \theta < \text{ord}(\mu)$.

Proof. Suppose first that $[i, H_{\mu,s}] = [j, H_{v,t}]$. By the definition of the equivalence relation, $\mu = v$, so that j is incident with μ in Δ . Hence, $j = i_r$ for some $1 \leq r \leq \text{val}(\mu)$. Now $(i_1, H_{\mu,s})$ is equivalent to $(i_2, H_{\mu,s}W(i_1, i_2))$, so inductively $(i_1, H_{\mu,s})$ is equivalent to $(i_r, H_{\mu,s}W(i_1, i_2) \cdots W(i_{r-1}, i_r)) = (i_r, H_{\mu,s}\omega(i_1, i_r))$. However, $H_\mu = \langle \omega_\mu \rangle$ so $H_{\mu,s} = H_{\mu,s}\omega_\mu^\theta$ for all $0 \leq \theta < \text{ord}(\mu)$. Thus, $H_{\mu,t} = H_{\mu,s}\omega(i_1, i_r)\omega_\mu^\theta$ for some $0 \leq \theta < \text{ord}(\mu)$.

Conversely, suppose that $j = i_r$ and $H_{\mu,t} = H_{\mu,s}\omega(i_1, i_r)\omega_\mu^\theta$ for some $1 \leq r \leq \text{val}(\mu), 0 \leq \theta < \text{ord}(\mu)$. A similar argument to that given above shows that $(i, H_{\mu,s})$ is equivalent to $(i_r, H_{\mu,t})$ and hence $[i, H_{\mu,s}] = [i_r, H_{\mu,t}]$. \square

DEFINITION 4.3. We define Δ_W to be the graph with vertex set $\{\mu_d \mid \mu \in \Delta_0, d \in \mathcal{D}_\mu\}$ and edge set $\{i_g \mid g \in G, i \in \Delta_1\}$. If i is an edge in Δ_1 with endpoints μ and v in

Δ_0 then the edge i_g in $(\Delta_W)_1$ has endpoints $\mu_{[i, H_{\mu,s}]}$ and $\nu_{[i, H_{\nu,t}]}$, where $H_{\mu,s}$ (respectively $H_{\nu,t}$) is the unique coset of H_μ in G (respectively H_ν in G) containing g .

If i is a loop in Δ incident with vertex μ , then consider i as two edges i and \hat{i} . Let $g \in G$. The edge i_g has endpoints $\mu_{[i, H_{\mu,s}]}$ and $\mu_{[\hat{i}, H_{\mu,s}]}$, and the edge \hat{i}_g has endpoints $\mu_{[\hat{i}, H_{\mu,s}]}$ and $\mu_{[i, H_{\mu,s}]}$, where $H_{\mu,s}$ is the unique coset of H_μ in G containing g .

Suppose that j is the successor of i at the vertex μ in Δ , let $g \in G$, and let $H_{\mu,s}$ be the coset of H_μ in G containing g . Then, both i_g and $j_{gW(i,j)}$ are incident with vertex $\mu_{[i, H_{\mu,s}]}$ in Δ_W .

PROPOSITION 4.4. *Let $W: \mathcal{Z}_\Delta \rightarrow G$ be a successor weighting of $(\Delta, \mathfrak{o}, m)$. For each $\mu \in \Delta_0$ and $d \in \mathcal{D}_\mu$ we have $\text{val}(\mu_d) = \text{ord}(\mu) \text{val}(\mu)$.*

Proof. Let $\mu \in \Delta_0$ and $d = [i, H_{\mu,s}] \in \mathcal{D}_\mu$. Suppose that $h \in G$ and j_h is incident with μ_d in Δ_W . Since one endpoint of j_h is μ_d it follows from Definition 4.3 that $\mu_{[i, H_{\mu,s}]} = \mu_{[j, H_{\mu,t}]}$, where $h \in H_{\mu,t}$. Thus, $[i, H_{\mu,s}] = [j, H_{\mu,t}]$ and, by Lemma 4.2, j is in the successor sequence $i = i_1, i_2, \dots, i_{\text{val}(\mu)}$ of i at vertex μ in Δ . For $g' \in G$ we have $hg' \in H_{\mu,t}$ if and only if $g' = \omega_\mu^\theta$ for some $0 \leq \theta < \text{ord}(\mu)$. Thus, there are $\text{val}(\mu)$ choices for j and $\text{ord}(\mu)$ choices for h . Hence, $\text{val}(\mu_d) = \text{ord}(\mu) \text{val}(\mu)$. \square

In order to construct a multiplicity function m_W for Δ_W , we need to place an additional condition on our successor weightings.

DEFINITION 4.5.

- (1) A successor weighting $W: \mathcal{Z}_\Delta \rightarrow G$ of the Brauer graph $(\Delta, \mathfrak{o}, m)$ is called a *Brauer weighting* if $\text{ord}(\mu) \mid m(\mu)$ for all $\mu \in \Delta_0$.
- (2) If $W: \mathcal{Z}_\Delta \rightarrow G$ is a Brauer weighting, we define the function m_W by

$$m_W: (\Delta_W)_0 \rightarrow \mathbb{N} \setminus \{0\}, \quad \mu_d \mapsto m(\mu) / \text{ord}(\mu).$$

We remark that $\text{ord}(\mu) \mid m(\mu)$ if and only if $\omega_\mu^{m(\mu)} = \text{id}_G$.

PROPOSITION 4.6. *Let $W: \mathcal{Z}_\Delta \rightarrow G$ be a Brauer weighting of $(\Delta, \mathfrak{o}, m)$. Suppose that j is the successor of i at the vertex μ in Δ , let $g \in G$, and let $H_{\mu,s}$ be the coset of H_μ in G containing g . Defining $j_{gW(i,j)}$ to be the successor of i_g at vertex $\mu_{[i, H_{\mu,s}]}$ induces a cyclic ordering \mathfrak{o}_W so that $(\Delta_W, \mathfrak{o}_W, m_W)$ is a Brauer graph.*

Proof. Let $i = i_1, j = i_2, i_3, \dots, i_{\text{val}(\mu)}$ be the successor sequence of i at the vertex μ in Δ . From Lemma 4.2 and Proposition 4.4, the successor sequence of i_g at vertex $\mu_{[i, H_{\mu,s}]}$ is

$$\begin{aligned} & (i_1)_g, (i_2)_{g\omega_\mu(i_1, i_2)}, (i_3)_{g\omega_\mu(i_1, i_3)}, \dots, (i_{\text{val}(\mu)})_{g\omega_\mu(i_1, i_{\text{val}(\mu)})}, \\ & (i_1)_{g\omega_\mu}, (i_2)_{g\omega_\mu(i_1, i_2)\omega_\mu}, \dots, (i_{\text{val}(\mu)})_{g\omega_\mu(i_1, i_{\text{val}(\mu)})\omega_\mu}, \dots, \\ & (i_1)_{g\omega_\mu^{\text{ord}(\mu)-1}}, (i_2)_{g\omega_\mu(i_1, i_2)\omega_\mu^{\text{ord}(\mu)-1}}, \dots, (i_{\text{val}(\mu)})_{g\omega_\mu(i_1, i_{\text{val}(\mu)})\omega_\mu^{\text{ord}(\mu)-1}}. \end{aligned}$$

This explicitly describes the cyclic ordering \mathfrak{o}_W . \square

LEMMA 4.7. *There is a canonical group action of G on Δ_W given by*

$$(\mu_{[i, H_{\mu,s}]})^h = \mu_{[i, H_{\mu,sh}]} \text{ and } (i_g)^h = i_{gh}$$

for all $h \in G$.

Proof. We prove first that the action of G on a vertex of Δ_W is well defined. Suppose $[i, H_{\mu,s}] = [j, H_{\mu,t}]$ and that j is the successor of i at the vertex μ . By definition, $H_{\mu,s}W(i, j) = H_{\mu,t}$ so that $H_{\mu,s}W(i, j)h = H_{\mu,t}h$ for all $h \in G$. Since G is abelian, it follows that $H_{\mu,s}h \cdot W(i, j) = H_{\mu,t}h$ and hence $[i, H_{\mu,s}h] = [j, H_{\mu,t}h]$. Thus, the action of G respects the generators of the equivalence relation and hence the action of G on $(\Delta_W)_0$ is well defined.

We leave it to the reader to show that if i_g has endpoints $\mu_{[i, H_{\mu,s}]}$ and $\nu_{[i, H_{\nu,t}]}$ then $(i_g)^h$ has endpoints $(\mu_{[i, H_{\mu,s}]})^h$ and $(\nu_{[i, H_{\nu,t}]})^h$, where $h \in G$. □

DEFINITION 4.8. A map $\varphi: (\Gamma, \sigma, m) \rightarrow (\Gamma', \sigma', m')$ is an *isomorphism of Brauer graphs* if $\varphi: \Gamma \rightarrow \Gamma'$ is a graph isomorphism such that

- (1) if j is the successor of i at vertex μ in Γ , then $\varphi(j)$ is the successor of $\varphi(i)$ at vertex $\varphi(\mu)$ in Γ' , and
- (2) $m(\mu) = m'(\varphi(\mu))$ for all $\mu \in \Gamma_0$.

In this case, we say that (Γ, σ, m) and (Γ', σ', m') are *isomorphic as Brauer graphs*.

It is easy to see that if $\varphi: (\Gamma, \sigma, m) \rightarrow (\Gamma', \sigma', m')$ is an isomorphism of Brauer graphs, then φ induces an isomorphism of the associated quivers \mathcal{Q}_Γ and $\mathcal{Q}_{\Gamma'}$, and hence induces a K -algebra isomorphism of the path algebras $K\mathcal{Q}_\Gamma$ and $K\mathcal{Q}_{\Gamma'}$.

THEOREM 4.9. *Let (Δ, σ, m) be a Brauer graph, G be a finite abelian group, $W: \mathcal{Z}_\Delta \rightarrow G$ be a Brauer weighting and $(\Delta_W, \sigma_W, m_W)$ be the Brauer graph associated with W . Then the canonical action of G on $(\Delta_W, \sigma_W, m_W)$ is a free Brauer action. Moreover, if $(\overline{\Delta_W}, \overline{\sigma_W}, \overline{m_W})$ is the Brauer orbit graph under this action, then (Δ, σ, m) and $(\overline{\Delta_W}, \overline{\sigma_W}, \overline{m_W})$ are isomorphic as Brauer graphs.*

Proof. We first show that the action is a free Brauer action on $(\Delta_W, \sigma_W, m_W)$. Let $i \in \Delta_1$ and $g, h \in G$. Since $(i_g)^h = i_{gh}$, the action of G on Δ_W is free on the edge set $(\Delta_W)_1$. If j is the successor of i at the vertex μ in Δ and $g \in H_{\mu,s}$, then $j_{gW(i,j)}$ is the successor of i_g at the vertex $\mu_{[i, H_{\mu,s}]}$ in Δ_W and $j_{ghW(i,j)}$ is the successor of i_{gh} at the vertex $\mu_{[i, H_{\mu,s}h]}$. Since G is abelian, $ghW(i, j) = gW(i, j)h$, and we see that $(j_{gW(i,j)})^h$ is the successor of $(i_g)^h$. Now we show that $m_W(\mu_d) = m_W(\mu_d^g)$ for all $g \in G$, where $\mu_d \in (\Delta_W)_0$. By Definition 4.5, $m_W(\mu_d) = m(\mu) / \text{ord}(\mu)$. However, $\mu_d^g = \mu_{d'}$ for some $d' \in \mathcal{Z}_\mu$ since $H_{\mu,s}g = H_{\mu,t}$ for some $1 \leq t \leq |G| / \text{ord}(\mu)$. Hence, $m_W(\mu_d^g) = m(\mu) / \text{ord}(\mu)$ and $m_W(\mu_d) = m_W(\mu_d^g)$. Thus, the canonical action of G on $(\Delta_W, \sigma_W, m_W)$ is a free Brauer action.

We form the Brauer orbit graph $(\overline{\Delta_W}, \overline{\sigma_W}, \overline{m_W})$ under this free Brauer action. The next step is to show that $\overline{\Delta_W} \cong \Delta$ as graphs. Suppose μ is a vertex in Δ and $d \in \mathcal{D}_\mu$. If $\mu_d = \nu_e$ for some $\nu \in \Delta_0$ and $e \in \mathcal{D}_\nu$ then, from Lemma 4.2, we have that $\mu = \nu$, so the orbit of the vertex $\mu_d \in (\Delta_W)_0$ is contained in the set $\{\mu_{d'} \mid d' \in \mathcal{D}_\mu\}$. Now let $d' \in \mathcal{D}_\mu$ and suppose $d = [i, H_{\mu,s}]$, $d' = [j, H_{\mu,t}]$ for some $i, j \in \Delta_1$ incident with μ . Let $i = i_1, i_2, \dots, i_{\text{val}(\mu)}$ be the successor sequence of i at μ . By Lemma 4.2, $j = i_r$ and $H_{\mu,t} = H_{\mu,s}\omega(i_1, i_r)\omega_\mu^\theta$ for some $1 \leq r \leq \text{val}(\mu)$, $0 \leq \theta < \text{ord}(\mu)$. Thus, $(\mu_d)^{\omega(i_1, i_r)\omega_\mu^\theta} = (\mu_{[i_1, H_{\mu,s}]})^{\omega(i_1, i_r)\omega_\mu^\theta} = \mu_{[i_1, H_{\mu,s}\omega(i_1, i_r)\omega_\mu^\theta]} = \mu_{[i_r, H_{\mu,t}]} = \mu_{d'}$. Hence, $\mu_{d'}$ is in the orbit of μ_d , and so the orbit of μ_d is $\{\mu_{d'} \mid d' \in \mathcal{D}_\mu\}$. Thus, the vertices $\overline{\mu_d}$ in $(\overline{\Delta_W})_0$ are in one-to-one correspondence with the vertices μ in Δ_0 .

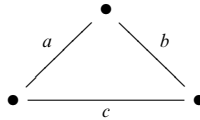
Now, suppose that i is an edge in Δ and $g \in G$. We show that the orbit of the edge i_g in $(\Delta_W)_1$ is the set $\{i_h \mid h \in G\}$. Let j be an edge in Δ and $h \in G$ such that j_h is in the orbit of i_g . Then there is some $g' \in G$ with $j_h = (i_g)^{g'} = i_{gg'}$ so that $j = i$. Since $(i_g)^{g^{-1}h} = i_h$, it follows that the orbit of i_g is precisely the set $\{i_h \mid h \in G\}$. Hence, the

edges $\overline{i_g}$ in $(\overline{\Delta_W})_1$ are in one-to-one correspondence with the edges i in Δ_1 . It is now straightforward to show that there is an isomorphism of graphs $\varphi: \overline{\Delta_W} \rightarrow \Delta$ given by $\varphi(\overline{\mu_d}) = \mu$ and $\varphi(\overline{i_g}) = i$.

It is clear that the induced cyclic ordering $\overline{\sigma_W}$ is the cyclic ordering σ under this isomorphism φ . Finally, from Propositions 3.5(1) and 4.4, we have that $\overline{m_W}(\overline{\mu_d}) \text{val}(\overline{\mu_d}) = m_W(\mu_d) \text{val}(\mu_d) = m_W(\mu_d) \text{ord}(\mu) \text{val}(\mu)$. From Definition 4.5, $m_W(\mu_d) = m(\mu) / \text{ord}(\mu)$ so it follows that $\overline{m_W}(\overline{\mu_d}) \text{val}(\overline{\mu_d}) = m(\mu) \text{val}(\mu)$. It remains to show that $\text{val}(\overline{\mu_d}) = \text{val}(\mu)$, for then we have that $\overline{m_W}(\overline{\mu_d}) = m(\mu)$ and hence $\overline{m_W} = m\varphi$ as required. To see this, let $\mu \in \Delta_0$, $d = [i, H_{\mu,s}]$ and $g \in G$, where i is incident with μ , $g \in H_{\mu,s}$ and $1 \leq s \leq |G| / \text{ord} \mu$. If the element $(i_r)_h$ is in the orbit of $(i_{r'})_{h'}$, for some $h, h' \in G$, then $r = r'$, and so, from the successor sequence of i_g at μ_d given in the proof of Proposition 4.6, we see that $\text{val}(\overline{\mu_d}) \geq \text{val}(\mu)$. However, noting that $(i_r)_{g\omega(i,i_r)\omega_\mu^p} = ((i_r)_{g\omega(i,i_r)})^{\omega_\mu^p}$ is in the orbit of $(i_r)_{g\omega(i,i_r)}$, we conclude that $\text{val}(\overline{\mu_d}) \leq \text{val}(\mu)$ and the proof is complete. \square

DEFINITION 4.10. Let $W: \mathcal{Z}_\Delta \rightarrow G$ be a Brauer weighting of (Δ, σ, m) and let $(\Delta_W, \sigma_W, m_W)$ be the Brauer graph associated with W . We call $(\Delta_W, \sigma_W, m_W)$ the Brauer covering graph of (Δ, σ, m) associated with W .

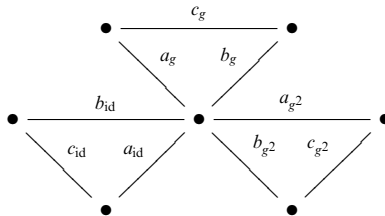
EXAMPLE 4.11. Let G be the cyclic group $G = \mathbb{Z}_3 = \langle g \mid g^3 = \text{id} \rangle$ and let Δ be the graph



with weighting W given by

$$W(a, b) = \text{id}, W(b, a) = g, W(b, c) = \text{id}, W(c, b) = \text{id}, W(c, a) = \text{id}, W(a, c) = \text{id}.$$

Following the above construction, Δ_W is



Moreover, $\overline{\Delta_W} = \Delta$.

In the next section, we investigate coverings of Brauer graph algebras associated with a Brauer covering graph.

5. Brauer graph algebras arising from weightings on Brauer graphs. Let (Δ, σ, m) be a Brauer graph, (Δ, σ, m, q) be a quantization of (Δ, σ, m) and G be a finite abelian group. Let $W: \mathcal{Z}_\Delta \rightarrow G$ be a Brauer weighting and $(\Delta_W, \sigma_W, m_W)$ be the Brauer graph associated with W . The quantizing function $q: \mathcal{X}_\Delta \rightarrow K \setminus \{0\}$ induces a quantizing

function $q_W: \mathcal{X}_{\Delta_W} \rightarrow K \setminus \{0\}$ given by $q_W((i_g, \mu_d)) = q_{i,\mu}$. It is clear that this map is well defined since if $(i_g, \mu_d) \in \mathcal{X}_{\Delta_W}$ then $(i, \mu) \in \mathcal{X}_{\Delta}$. We call $(\Delta_W, \circ_W, m_W, q_W)$ the *quantized Brauer covering graph* (associated with W).

Let $W: \mathcal{Z}_{\Delta} \rightarrow G$ be a Brauer weighting for (Δ, \circ, m, q) and let $(\Delta_W, \circ_W, m_W, q_W)$ be the quantized Brauer covering graph associated with W . The weighting W induces a weight function $W^*: (\mathcal{Q}_{\Delta})_1 \rightarrow G$ of the arrows of \mathcal{Q}_{Δ} as follows. If j is the successor of i at the vertex μ in Δ and $v_i \xrightarrow{a} v_j$ is the associated arrow in $(\mathcal{Q}_{\Delta})_1$, then we define $W^*(a) = W(i, j)$. For a path $p = a_1 a_2 \cdots a_{\sigma}$ in \mathcal{Q}_{Δ} , we define $W^*(p) = W^*(a_1)W^*(a_2) \cdots W^*(a_{\sigma})$. Furthermore, W^* induces a G -grading on $K\mathcal{Q}_{\Delta}$ by $p = a_1 a_2 \cdots a_{\sigma}$ being homogeneous of degree $W^*(p)$.

Recall that if $i \in \Delta_1$ is incident with $\mu \in \Delta_0$ with successor sequence $i = i_1, i_2, \dots, i_{\text{val}(\mu)}$ then $C_{i,\mu} = a_1 a_2 \cdots a_{\text{val}(\mu)}$ in \mathcal{Q}_{Δ} , where the arrow a_r corresponds to the edge i_{r+1} being the successor of the edge i_r and $i_{\text{val}(\mu)+1} = i_1$.

LEMMA 5.1. *If $i \in \Delta_1$ is incident with $\mu \in \Delta_0$ then $W^*(C_{i,\mu}^{m(\mu)}) = \text{id}_G$.*

Proof. We have $W^*(C_{i,\mu}) = W(i_1, i_2)W(i_2, i_3) \cdots W(i_{\text{val}(\mu)-1}, i_{\text{val}(\mu)})W(i_{\text{val}(\mu)}, i_1) = \omega_{\mu}$ and $\omega_{\mu}^{m(\mu)} = \text{id}_G$. □

COROLLARY 5.2. *The ideal I_{Δ} can be generated by elements of $K\mathcal{Q}_{\Delta}$ which are homogeneous in the G -grading induced by W^* .*

Proof. The relations of type one in I_{Δ} are of the form $q_{i,\mu} C_{i,\mu}^{m(\mu)} - q_{i,v} C_{i,v}^{m(v)}$. By Lemma 5.1, these are homogeneous of degree id_G . The relations of types two and three are homogeneous since they are paths. □

The weight function $W^*: (\mathcal{Q}_{\Delta})_1 \rightarrow G$ gives rise to a covering quiver \mathcal{Q}_{W^*} with vertex set $(\mathcal{Q}_{\Delta})_0 \times G = \{v_g \mid v \in (\mathcal{Q}_{\Delta})_0, g \in G\}$ and arrow set $(\mathcal{Q}_{\Delta})_1 \times G = \{a_g \mid a \in (\mathcal{Q}_{\Delta})_1, g \in G\}$ such that if $v \xrightarrow{a} w$ is an arrow in \mathcal{Q}_{Δ} then $v_g \xrightarrow{a_g} w_g W^*(a)$ is an arrow in \mathcal{Q}_{W^*} . Define the map $\pi: \mathcal{Q}_{W^*} \rightarrow \mathcal{Q}_{\Delta}$ by $\pi(v_g) = v$ and $\pi(a_g) = a$. We extend π to paths $a_1 a_2 \cdots a_{\sigma}$ in $K\mathcal{Q}_{W^*}$ by setting $\pi(a_1 a_2 \cdots a_{\sigma}) = \pi(a_1)\pi(a_2) \cdots \pi(a_{\sigma})$. Hence, we may linearly extend π to a map $K\mathcal{Q}_{W^*} \rightarrow K\mathcal{Q}_{\Delta}$, which we also denote by π . Define $I_{W^*} = \pi^{-1}(I_{\Delta})$. Then $K\mathcal{Q}_{W^*}/I_{W^*}$ is the covering algebra of $\mathcal{A}_{\Delta} = K\mathcal{Q}_{\Delta}/I_{\Delta}$ and we have the following theorem, whose proof is obtained from a careful analysis of the definitions and is left to the reader.

THEOREM 5.3. *Let (Δ, \circ, m, q) be a quantized Brauer graph, G a finite abelian group and $W: \mathcal{Z}_{\Delta} \rightarrow G$ a Brauer weighting. Let $(\Delta_W, \circ_W, m_W, q_W)$ be the quantized Brauer covering graph and $W^*: (\mathcal{Q}_{\Delta})_1 \rightarrow G$ the induced weight function. Then $\mathcal{A}_{\Delta_W} \cong K\mathcal{Q}_{W^*}/I_{W^*}$.*

The next result extends Theorem 4.9 and shows that the canonical action of G on (Δ_W, \circ_W, m_W) is a free Brauer action on $(\Delta_W, \circ_W, m_W, q_W)$.

PROPOSITION 5.4. *Let (Δ, \circ, m, q) be a quantized Brauer graph and $W: \mathcal{Z}_{\Delta} \rightarrow G$ a Brauer weighting. Then the canonical action of G on Δ_W is a free Brauer action on the quantized Brauer covering graph $(\Delta_W, \circ_W, m_W, q_W)$.*

Proof. Suppose that $i \in \Delta_1$ has endpoints μ and $v, g \in G$, and the cosets $H_{\mu,s}$ and $H_{v,t}$ both contain g . Then the endpoints of i_g are μ_d and v_e , where $d = [i, H_{\mu,s}]$ and $e = [i, H_{v,t}]$. By Theorem 4.9 there is a free Brauer action of G on (Δ_W, \circ_W, m_W) . It

remains to show that

$$\frac{q_W((i_g, \mu_d))}{q_W((i_g, v_e))} = \frac{q_W((i_g, \mu_d)^h)}{q_W((i_g, v_e)^h)}$$

for all $h \in G$. Let $h \in G$. Then $(i_g, \mu_d)^h = ((i_g)^h, (\mu_d)^h) = (i_{gh}, \mu_{d'})$, where $d' = [i_g, H_{\mu,s}h]$ and $(i_g, v_e)^h = ((i_g)^h, (v_e)^h) = (i_{gh}, v_{e'})$, where $e' = [i_g, H_{v,t}h]$. It follows from the definition of q_W that

$$\frac{q_W((i_g, \mu_d))}{q_W((i_g, v_e))} = \frac{q_{i,\mu}}{q_{i,v}} = \frac{q_W((i_g, \mu_d)^h)}{q_W((i_g, v_e)^h)}.$$

This completes the proof. □

DEFINITION 5.5. A map $\varphi: (\Gamma, \mathfrak{o}, m, q) \rightarrow (\Gamma', \mathfrak{o}', m', q')$ is an *isomorphism of quantized Brauer graphs* if $\varphi: (\Gamma, \mathfrak{o}, m) \rightarrow (\Gamma', \mathfrak{o}', m')$ is an isomorphism of Brauer graphs, and the isomorphism from $K\mathcal{Q}_\Gamma$ to $K\mathcal{Q}_{\Gamma'}$ induced by φ , restricts to an isomorphism from the ideal of relations I_Γ to $I_{\Gamma'}$. In this case, we say that $(\Gamma, \mathfrak{o}, m, q)$ and $(\Gamma', \mathfrak{o}', m', q')$ are *isomorphic as quantized Brauer graphs*.

We are now in a position to prove the main result of this section.

THEOREM 5.6. *Let $(\Delta, \mathfrak{o}, m, q)$ be a quantized Brauer graph, G be a finite abelian group and $W: \mathcal{Z}_\Delta \rightarrow G$ be a Brauer weighting. Let $(\Delta_W, \mathfrak{o}_W, m_W, q_W)$ be the quantized Brauer covering graph with the canonical action of G and $(\overline{\Delta_W}, \overline{\mathfrak{o}_W}, \overline{m_W}, \overline{q_W})$ be the associated quantized Brauer orbit graph. Then $(\Delta, \mathfrak{o}, m, q)$ and $(\overline{\Delta_W}, \overline{\mathfrak{o}_W}, \overline{m_W}, \overline{q_W})$ are isomorphic as quantized Brauer graphs and there are algebra isomorphisms*

$$\mathcal{A}_\Delta \cong \mathcal{A}_{\overline{\Delta_W}} \cong \overline{\mathcal{A}_{\Delta_W}}.$$

Proof. The second isomorphism holds by Theorem 3.12. By Theorem 4.9, $(\Delta, \mathfrak{o}, m)$ and $(\overline{\Delta_W}, \overline{\mathfrak{o}_W}, \overline{m_W})$ are isomorphic as Brauer graphs. We need to show that the induced isomorphism from $K\mathcal{Q}_{\overline{\Delta_W}}$ to $K\mathcal{Q}_\Delta$ restricts to an isomorphism from $I_{\overline{\Delta_W}}$ to I_Δ . First, we consider relations of type one. Let i be an edge in Δ such that i is not truncated at either of its endpoints μ and v , and let $g \in G$ with $g \in H_{\mu,s}$ and $g \in H_{v,t}$. By Proposition 3.9 we have

$$\frac{\overline{q_W}(\overline{(i_g, \mu_d)})}{\overline{q_W}(\overline{(i_g, v_e)})} = \frac{q_W((i_g, \mu_d))}{q_W((i_g, v_e))}$$

and, by definition of the quantizing function q_W , we have $q_W((i_g, \mu_d)) = q_{i,\mu}$ and $q_W((i_g, v_e)) = q_{i,v}$. Thus,

$$\frac{\overline{q_W}(\overline{(i_g, \mu_d)})}{\overline{q_W}(\overline{(i_g, v_e)})} = \frac{q_{i,\mu}}{q_{i,v}}.$$

Hence, a relation of type one in $I_{\overline{\Delta_W}}$ restricts to a relation of type one in I_Δ . We leave it to the reader to verify the corresponding statement for relations of types two and three. Hence, $(\Delta, \mathfrak{o}, m, q)$ and $(\overline{\Delta_W}, \overline{\mathfrak{o}_W}, \overline{m_W}, \overline{q_W})$ are isomorphic as quantized Brauer graphs and $\mathcal{A}_\Delta \cong \mathcal{A}_{\overline{\Delta_W}}$. □

6. From actions to orbits to coverings. Throughout this section, we assume that G is a finite abelian group with free Brauer action on the quantized Brauer graph $(\Gamma, \mathfrak{o}, m, q)$. We denote the quantized Brauer orbit graph associated with the action of G by $(\bar{\Gamma}, \bar{\mathfrak{o}}, \bar{m}, \bar{q})$. We show that there is a Brauer weighting $W: \mathcal{Z}_{\bar{\Gamma}} \rightarrow G$ such that the quantized Brauer covering graph $((\bar{\Gamma})_W, (\bar{\mathfrak{o}})_W, (\bar{m})_W, (\bar{q})_W)$ is isomorphic to $(\Gamma, \mathfrak{o}, m, q)$. In particular, the Brauer graph algebras \mathcal{A}_Γ and $\mathcal{A}_{(\bar{\Gamma})_W}$ are isomorphic algebras.

We begin with the construction of $W: \mathcal{Z}_{\bar{\Gamma}} \rightarrow G$. If i (respectively μ) is an edge (respectively a vertex) in Γ then, as before, we denote the orbit of i (respectively μ) under the action of G by \bar{i} (respectively $\bar{\mu}$) and view \bar{i} (respectively $\bar{\mu}$) as both an edge (respectively a vertex) in $\bar{\Gamma}$ and as an orbit set in Γ . For each edge \bar{i} in $\bar{\Gamma}$, we fix an edge $i_* \in \Gamma$ in the orbit \bar{i} . Next, suppose that $\bar{\mu}$ and $\bar{\nu}$ are the endpoints of \bar{i} in $\bar{\Gamma}$. Choose μ_* (respectively ν_*) in the orbit of $\bar{\mu}$ (respectively $\bar{\nu}$) so that i_* is incident with μ_* and ν_* . Note that there is a unique choice for μ_* and ν_* unless \bar{i} is a loop and i_* is not a loop. Now let \bar{j} be the successor of \bar{i} at vertex $\bar{\mu}$ in $\bar{\Gamma}$. Then there is a unique edge $l \in \bar{j}$ such that l is the successor of i_* at vertex μ_* in Γ . Hence $l = (j_*)^g$, for some $g \in G$. We define $W(\bar{i}, \bar{j}) = g$. We note that W is dependent on the choices of i_* and μ_* . We call W the *successor weighting associated with the action of G on $(\Gamma, \mathfrak{o}, m)$* .

LEMMA 6.1. *Let $W: \mathcal{Z}_{\bar{\Gamma}} \rightarrow G$ be the successor weighting associated with the free Brauer action of G on $(\Gamma, \mathfrak{o}, m)$. Suppose that i is an edge of Γ incident with the vertex μ and that $i = i_1, i_2, \dots, i_k, i_1^g, \dots, i_k^g, \dots, i_1^{g^s}, \dots, i_k^{g^s}$ is the successor sequence of i at μ , where $g \in G$. Then*

- (1) $g = \omega_{\bar{\mu}} = W(\bar{i}_1, \bar{i}_2)W(\bar{i}_2, \bar{i}_3) \cdots W(\bar{i}_k, \bar{i}_{k+1})$, where $\bar{i}_{k+1} = \bar{i}_1$.
- (2) The order of $\omega_{\bar{\mu}} = \text{ord}(\bar{\mu}) = s + 1$.
- (3) $\text{ord}(\bar{\mu}) \mid \bar{m}(\bar{\mu})$.
- (4) If $h, h' \in G$ then $\mu^h = \mu^{h'}$ if and only if $h(h')^{-1} \in H_{\bar{\mu}}$, where $H_{\bar{\mu}}$ is the subgroup of G generated by $\omega_{\bar{\mu}}$.
- (5) The index of $H_{\bar{\mu}}$ in G equals the number of vertices in the orbit of μ .

Proof. From our hypothesis, we see that $\bar{i}_1, \bar{i}_2, \dots, \bar{i}_k$ is the successor sequence of \bar{i} at vertex $\bar{\mu}$. From the definitions of the successor weight function W and $\omega(\bar{i}_1, \bar{i}_j)$, we conclude that the successor sequence of $i_* = (i_1)_*$ at μ_* is

$$(i_1)_*, (i_2)_*^{\omega(\bar{i}_1, \bar{i}_2)}, \dots, (i_k)_*^{\omega(\bar{i}_1, \bar{i}_k)}, (i_1)_*^{\omega_{\bar{\mu}}}, \dots, (i_k)_*^{\omega_{\bar{\mu}}}, \dots, (i_1)_*^{\omega_{\bar{\mu}}^s}, \dots, (i_k)_*^{\omega_{\bar{\mu}}^s}.$$

Since the action of G is free on the edges of Γ , and since the smallest positive integer l such that $((i_1)_*)^l = (i_1)_*$ is $\omega_{\bar{\mu}}^{s+1}$, we conclude that the order of $\omega_{\bar{\mu}}$ is $s + 1$. Similarly, since $((i_1)_*)^g = ((i_1)_*)^{\omega_{\bar{\mu}}}$, we see that $g = \omega_{\bar{\mu}}$ and we have shown that (1) and (2) hold.

From Proposition 3.5, we have that $\bar{m}(\bar{\mu}) = m(\mu) \text{val}(\mu) / \text{val}(\bar{\mu})$, and from Lemma 3.3, we have that $\text{val}(\mu) = \text{val}(\bar{\mu})(s + 1)$. So $\bar{m}(\bar{\mu}) = m(\mu)(s + 1)$, and (3) now follows from (2).

Next, consider $(\mu_*)^h$, for $h \in G$. From the successor sequence of $(i_1)_*$ at μ_* , we have that $(\mu_*)^h = \mu_*$ if and only if $h = \omega_{\bar{\mu}}^\theta$ for some $0 \leq \theta < \text{ord}(\bar{\mu})$. Hence, if $h, h' \in G$, then $\mu_*^h = \mu_*^{h'}$ if and only if $h(h')^{-1} \in H_{\bar{\mu}}$ and (4) now follows after noting that $\mu = (\mu_*)^{h''}$ for some $h'' \in G$.

That the index of $H_{\bar{\mu}}$ equals the number of vertices in the orbit of μ follows from (4), and the proof is complete. □

As an immediate consequence of 6.1(3), we have the following result.

COROLLARY 6.2. *The successor weighting associated with the free Brauer action of G on $(\Gamma, \mathfrak{o}, m)$ is a Brauer weighting.*

We call the successor weighting associated with a free Brauer action of G on $(\Gamma, \mathfrak{o}, m)$, the *Brauer weighting associated with the action of G on $(\Gamma, \mathfrak{o}, m)$.*

THEOREM 6.3. *Suppose that G is a finite abelian group with a free Brauer action on the Brauer graph $(\Gamma, \mathfrak{o}, m)$. Let $(\bar{\Gamma}, \bar{\mathfrak{o}}, \bar{m})$ be the Brauer orbit graph and $(\bar{\Gamma}_W, \bar{\mathfrak{o}}_W, \bar{m}_W)$ be the Brauer covering graph obtained from the Brauer weighting W associated with the action of G on $(\Gamma, \mathfrak{o}, m)$. Then $(\Gamma, \mathfrak{o}, m)$ and $(\bar{\Gamma}_W, \bar{\mathfrak{o}}_W, \bar{m}_W)$ are isomorphic as Brauer graphs. Moreover, if $(\Gamma, \mathfrak{o}, m, q)$ is a quantized Brauer graph with free Brauer action by G , then $(\Gamma, \mathfrak{o}, m, q)$ and $(\bar{\Gamma}_W, \bar{\mathfrak{o}}_W, \bar{m}_W, \bar{q}_W)$ are isomorphic as quantized Brauer graphs. In particular, the associated Brauer graph algebras \mathcal{A}_Γ and $\mathcal{A}_{\bar{\Gamma}_W}$ are isomorphic.*

Proof. We begin by defining an isomorphism $\varphi: \bar{\Gamma}_W \rightarrow \Gamma$ of graphs with cyclic ordering on the edges. If \bar{t}_g is an edge in $\bar{\Gamma}_W$, we let $\varphi(\bar{t}_g) = (i_*)^g$. The action of G is free on the edges of Γ so φ is a bijection when restricted to the edge sets. For $d = [\bar{i}, H_{\bar{\mu},s}] \in \mathcal{D}_{\bar{\mu}}$, we define $\varphi(\bar{\mu}_d) = (\mu_*)^h$, where h is an element of the coset $H_{\bar{\mu},s}$. By Lemma 6.1(3) and (4), $\varphi(\bar{\mu}_d)$ is independent of the choice of h , and so φ is a bijection when restricted to the vertex sets.

Let \bar{t} be an edge in $\bar{\Gamma}$ incident with $\bar{\mu}$ and let $\bar{t} = \bar{t}_1, \bar{t}_2, \dots, \bar{t}_k$ be the successor sequence of \bar{t} at vertex $\bar{\mu}$. If $i \in \Gamma_1$ is in the orbit \bar{t} and is incident with vertex μ where $\mu \in \bar{\mu}$, and i has successor sequence $i = i_1, \dots, i_k, i_1^g, \dots, i_k^g, \dots, i_1^{g^s}, \dots, i_k^{g^s}$ at vertex μ , then, since $g = \omega_{\bar{\mu}}$ from Lemma 6.1(1), if \bar{t}_h is in $\bar{\Gamma}_W$ then \bar{t}_h has successor sequence

$$\begin{aligned} \bar{t}_h = & (\bar{t}_1)_h, (\bar{t}_2)_h^{\omega(\bar{t}_1, \bar{t}_2)}, \dots, (\bar{t}_k)_h^{\omega(\bar{t}_1, \bar{t}_k)}, (\bar{t}_1)_h^{\omega_{\bar{\mu}}}, (\bar{t}_2)_h^{\omega_{\bar{\mu}}\omega(\bar{t}_1, \bar{t}_2)}, \dots, (\bar{t}_k)_h^{\omega_{\bar{\mu}}\omega(\bar{t}_1, \bar{t}_k)}, \dots, \\ & (\bar{t}_1)_h^{\omega_{\bar{\mu}}^s}, \dots, (\bar{t}_k)_h^{\omega_{\bar{\mu}}^s\omega(\bar{t}_1, \bar{t}_k)} \end{aligned}$$

at vertex $\bar{\mu}_d$, where $d = [\bar{i}, H_{\bar{\mu},s}]$ and $h \in H_{\bar{\mu},s}$. Using Lemma 6.1(3), the action of G on the edges of Γ , and the definition of φ , we see that the successor sequence of \bar{t}_h at vertex $\bar{\mu}_d$ in $\bar{\Gamma}_W$ is sent, under φ , to the successor sequence of i^h at vertex μ^h in Γ . From this we see that φ is a graph isomorphism that preserves the cyclic ordering on the edges at each vertex.

It remains to show, identifying Γ and $\bar{\Gamma}_W$ via φ , that $m = \bar{m}_W$ and that q and \bar{q}_W generate the same ideal of relations in the associated Brauer graph algebras. From Definition 4.5, we see that if μ is a vertex in Γ and $\bar{\mu}_d$ is a vertex in $\bar{\Gamma}_W$, then $\bar{m}_W(\bar{\mu}_d) = \bar{m}(\bar{\mu}) / \text{ord}(\bar{\mu})$. By Proposition 3.4, $\bar{m}(\bar{\mu}) = \text{val}(\mu)m(\mu) / \text{val}(\bar{\mu})$. However, from Lemmas 3.3 and 6.1(2), we have that $\frac{\text{val}(\mu)}{\text{val}(\bar{\mu})\text{ord}(\bar{\mu})} = 1$. Hence, $\bar{m}_W(\bar{\mu}_d) = m(\mu)$.

If $\mu \in \Gamma_0$ and $i \in \Gamma_1$ is incident with μ , then i is truncated at μ if and only if \bar{t} is truncated at $\bar{\mu}$ if and only if \bar{t}_g is truncated at $\bar{\mu}_d$, for all $g \in G$ and $d \in \mathcal{D}_{\bar{\mu}}$, since $\text{val}(\mu)m(\mu) = \text{val}(\bar{\mu})\bar{m}(\bar{\mu}) = \text{val}(\bar{\mu}_d)\bar{m}_W(\bar{\mu}_d)$. Hence, $(i, \mu) \in \mathcal{X}_\Gamma$ if and only if $(\bar{t}, \bar{\mu}) \in \mathcal{X}_{\bar{\Gamma}}$ if and only if $(\bar{t}_g, \bar{\mu}_d) \in \mathcal{X}_{\bar{\Gamma}_W}$, for all $g \in G$ and $d \in \mathcal{D}_{\bar{\mu}}$. Recall from Section 3 that $\mathcal{Y} = \{i \in \Gamma_1 \mid i \text{ is not truncated at either of its endpoints}\}$, and by Lemma 3.8, we can choose a set function $\mathcal{E}_1: \mathcal{Y} \rightarrow \Gamma_0$ so that $\mathcal{E}_1(i)$ is an endpoint of $i \in \Gamma_1$ and $(\mathcal{E}_1(i))^g = \mathcal{E}_1(i^g)$, for all $g \in G$. Let $i \in \Gamma_1$ be incident with vertex $\mu \in \Gamma_0$, $g \in G$, and $d \in \mathcal{D}_{\bar{\mu}}$. From Section 5, $\bar{q}_W(\bar{t}_g, \bar{\mu}_d) = \bar{q}(\bar{t}, \bar{\mu})$ and from Section 3, $\bar{q}(\bar{t}, \bar{\mu}) = \frac{q_{i,\mu}}{q_{i,\mathcal{E}_1(i)}}$. It is now immediate that there is a correspondence between relations of type one in $K\mathcal{Q}_\Gamma$ and relations of type one in $K\mathcal{Q}_{\bar{\Gamma}_W}$. The reader may check the correspondences for

relations of types two and three. Thus, the associated Brauer graph algebras \mathcal{A}_Γ and $\mathcal{A}_{\overline{\Gamma}_W}$ are isomorphic. \square

7. Applications. In this section, we provide a number of applications of the theory. These applications lead to two of the theorems announced in Section 1, namely the classification of the coverings of Brauer graph algebras that are again Brauer graph algebras (Theorem 7.1), and the fact that any Brauer graph can be covered by a tower of coverings, the topmost of which is a Brauer graph with no loops, no multiple edges and whose multiplicity function is identically one, that is, there are no exceptional vertices (Theorem 7.7).

7.1. Coverings of Brauer graph algebras that are Brauer graph algebras. Our first application classifies the coverings of Brauer graph algebras that are again Brauer graph algebras.

Let (Δ, σ, m, q) be a quantized Brauer graph and $W: \mathcal{Z}_\Delta \rightarrow G$ be a Brauer weighting for some finite abelian group G . As in Section 5, we let $W^*: (\mathcal{Q}_\Delta)_1 \rightarrow G$ be the weight function induced by W . Theorem 5.3 shows that the covering algebra obtained from W^* is a Brauer graph algebra. We now show the converse, that is, if the covering algebra of \mathcal{A}_Δ , obtained from a weight function on the arrows of \mathcal{Q}_Δ , is a Brauer graph algebra, then it is isomorphic to a covering algebra obtained from a Brauer weighting of (Δ, σ, m, q) .

Let G be a finite abelian group and $W^*: (\mathcal{Q}_\Delta)_1 \rightarrow G$ a weight function such that I_Δ is generated by weight homogeneous elements. Let i be an edge in Δ which is not truncated at either of its endpoints, μ and ν . Let $C_{i,\mu}$ and $C_{i,\nu}$ be the corresponding cycles associated with the edge i . Then $x = q_{i,\mu} C_{i,\mu}^{m(\mu)} - q_{i,\nu} C_{i,\nu}^{m(\nu)}$ is a relation of type one in I_Δ . We note, by the nature of the generating relations of I_Δ , that x must be homogeneous, that is, $W^*(C_{i,\mu}^{m(\mu)}) = W^*(C_{i,\nu}^{m(\nu)})$. We also note that $W^*(C_{i,\mu}) = \omega_\mu$. If $\omega_\mu^{m(\mu)}$ is not id_G , then, in the covering algebra, the liftings of x to $K\mathcal{Q}_{W^*}$ induce relations which are differences of paths that are not cycles. These differences are minimal relations for I_{W^*} . It follows that the covering algebra associated with W^* has minimal generating relations that are not of types one, two, or three, and hence the covering algebra associated with W^* is not a Brauer graph algebra.

On the other hand, if $\omega_\mu^{m(\mu)} = \text{id}_G$, we define the successor weighting $W: \mathcal{Z}_\Delta \rightarrow G$ as follows. If $(i, j) \in \mathcal{Z}_\mu$, that is, if j is the successor of i at vertex μ , then we define $W((i, j)) = W^*(a)$, where a is the arrow in \mathcal{Q}_Δ associated with (i, j) . Since $\omega_\mu^{m(\mu)} = \text{id}_G$, we see that $\text{ord}(\mu) \mid m(\mu)$, and hence W is the desired Brauer weighting.

Summarizing, we have the following result.

THEOREM 7.1. *Let (Δ, σ, m, q) be a quantized Brauer graph with associated Brauer graph algebra $\mathcal{A}_\Delta = K\mathcal{Q}_\Delta/I_\Delta$ and let G be a finite abelian group.*

- (1) *If $W: \mathcal{Z}_\Delta \rightarrow G$ is a Brauer weighting of (Δ, σ, m, q) then there is an associated weight function $W^*: (\mathcal{Q}_\Delta)_1 \rightarrow G$ such that I_Δ is generated by weight homogeneous elements and the covering algebra $(\mathcal{A}_\Delta)_{W^*}$ is isomorphic to the Brauer covering algebra \mathcal{A}_{Δ_W} associated with W . Moreover, for each edge i in Δ , which is not truncated at either of its endpoints μ and ν , we have that $W^*(C_{i,\mu}^{m(\mu)}) = \text{id}_G = W^*(C_{i,\nu}^{m(\nu)})$.*

- (2) Suppose that $W^* : (\mathcal{Q}_\Delta)_1 \rightarrow G$ is a weight function such that I_Δ is generated by weight homogeneous elements, and, for each edge i in Δ which is not truncated at either of its endpoints μ and ν , we have that $W^*(C_{i,\mu}^{m(\mu)}) = \text{id}_G = W^*(C_{i,\nu}^{m(\nu)})$. Then there is a Brauer weighting W of $(\Delta, \mathfrak{o}, m, q)$ such that the covering algebra $(\mathcal{A}_\Delta)_{W^*}$ is isomorphic to the Brauer covering algebra \mathcal{A}_{Δ_W} associated with W .
- (3) If $W^* : (\mathcal{Q}_\Delta)_1 \rightarrow G$ is a weight function such that I_Δ is generated by weight homogeneous elements and there is an edge i in Δ which is not truncated at both endpoints μ and ν , such that $W^*(C_{i,\mu}^{m(\mu)}) \neq \text{id}_G$, then the covering algebra $(\mathcal{A}_\Delta)_{W^*}$ is not isomorphic to a Brauer graph algebra.

7.2. Module categories and coverings. Our next application deals with module categories.

Suppose that $(\Delta, \mathfrak{o}, m, q)$ is a quantized Brauer graph, \mathcal{A}_Δ is the associated Brauer graph algebra, and $W : \mathcal{Z}_\Delta \rightarrow G$ is a Brauer weighting for some finite abelian group G . Let \mathcal{A}_{Δ_W} be the covering algebra associated with W and let $W^* : (\mathcal{Q}_\Delta)_1 \rightarrow G$ be the weight function induced by W . By covering theory, W^* induces a G -grading on \mathcal{A}_Δ . Then the following result holds; see the Appendix, Theorem A.4.

THEOREM 7.2. *Keeping the above notation, the category of G -graded \mathcal{A}_Δ -modules is equivalent to the category of \mathcal{A}_{Δ_W} -modules.*

Furthermore, if S is a simple \mathcal{A}_Δ -module then S is gradable. If \tilde{S} is a graded simple \mathcal{A}_Δ -module which forgets to S , then a minimal graded projective \mathcal{A}_Δ -resolution of \tilde{S} forgets to a minimal projective \mathcal{A}_Δ -resolution of S . By the above theorem, the minimal graded projective \mathcal{A}_Δ -resolutions are precisely the minimal projective \mathcal{A}_{Δ_W} -resolutions. Consequently, some cohomological questions related to \mathcal{A}_Δ -modules can be translated to cohomological questions concerning \mathcal{A}_{Δ_W} -modules, for example, see [9, Theorem 3.2].

7.3. Towers of coverings. In this section, we consider a ‘tower’ of coverings. More precisely, suppose that $(\Delta, \mathfrak{o}, m, q)$ is a quantized Brauer graph and \mathcal{A}_Δ is the associated Brauer graph algebra. Let $W_1 : \mathcal{Z}_\Delta \rightarrow G_1$ be a Brauer weighting for some finite abelian group G_1 and set $(\Delta_1, \mathfrak{o}_1, m_1, q_1)$ to be the associated quantized Brauer covering graph associated with W_1 . Let n be some positive integer. We say that $(\Delta, \mathfrak{o}, m, q), (\Delta_1, \mathfrak{o}_1, m_1, q_1), \dots, (\Delta_n, \mathfrak{o}_n, m_n, q_n)$ is a *tower of quantized Brauer covering graphs* if, for $1 < i \leq n$, there are Brauer weightings $W_i : \mathcal{Z}_{\Delta_{i-1}} \rightarrow G_i$ for finite abelian groups G_i , where $(\Delta_i, \mathfrak{o}_i, m_i, q_i)$ is the quantized Brauer covering graph associated with W_i . Applying Theorem 7.2 we have the following result.

THEOREM 7.3. *Let n be a positive integer and $(\Delta_0, \mathfrak{o}_0, m_0, q_0), (\Delta_1, \mathfrak{o}_1, m_1, q_1), \dots, (\Delta_n, \mathfrak{o}_n, m_n, q_n)$ be a tower of quantized Brauer covering graphs associated with Brauer weightings $W_i : \mathcal{Z}_{\Delta_{i-1}} \rightarrow G_i$, for some finite abelian groups G_i . For $i = 0, \dots, n$, let \mathcal{A}_{Δ_i} be the Brauer graph algebra associated with $(\Delta_i, \mathfrak{o}_i, m_i, q_i)$ and let $\text{Mod}(\mathcal{A}_{\Delta_i})$ denote the category of \mathcal{A}_{Δ_i} -modules. For $i = 1, \dots, n$, let $\mathcal{F}_i : \text{Mod}(\mathcal{A}_{\Delta_i}) \rightarrow \text{Mod}(\mathcal{A}_{\Delta_{i-1}})$ be the forgetful functor and set $\mathcal{G} = \mathcal{F}_1 \mathcal{F}_2 \cdots \mathcal{F}_n : \text{Mod}(\mathcal{A}_{\Delta_n}) \rightarrow \text{Mod}(\mathcal{A}_{\Delta_0})$. Then the following properties hold.*

(i) If S is a simple \mathcal{A}_{Δ_0} -module then there is a simple \mathcal{A}_{Δ_n} -module T such that $\mathcal{G}(T) \cong S$. Moreover, if

$$\mathcal{P} : \dots \rightarrow P^2 \rightarrow P^1 \rightarrow P^0 \rightarrow T \rightarrow 0$$

is a minimal projective \mathcal{A}_{Δ_n} -resolution of T then, applying the exact functor \mathcal{G} to \mathcal{P} , gives a minimal projective \mathcal{A}_{Δ_0} -resolution of S .

(ii) If S' is a simple \mathcal{A}_{Δ_n} -module, then $\mathcal{G}(S')$ is a simple \mathcal{A}_{Δ_0} -module.

As an immediate consequence of Theorem 7.3, we have that if $(\Delta_0, \sigma_0, m_0, q_0), (\Delta_1, \sigma_1, m_1, q_1), \dots, (\Delta_n, \sigma_n, m_n, q_n)$ is a tower of quantized Brauer covering graphs associated with Brauer weightings $W_i: \mathcal{Z}_{\Delta_{i-1}} \rightarrow G_i$ with associated quantized Brauer graph algebras \mathcal{A}_{Δ_i} , then some homological questions related to \mathcal{A}_{Δ_0} -modules can be translated to homological questions concerning \mathcal{A}_{Δ_n} -modules. In particular, this is applied in [10] to determine the Ext algebra of a Brauer graph algebra. The next three applications show how to construct coverings of Brauer graphs with specific properties. We begin by considering multiplicities.

7.4. Coverings and multiplicities. Suppose that (Δ, σ, m) is a Brauer graph for which at least one vertex μ has $m(\mu) > 1$. Let $n = \text{lcm}\{m(\mu) \mid \mu \in \Delta_0\}$ and $\mathbb{Z}_n = \langle g \mid g^n = \text{id} \rangle$ be the cyclic group of order n . We define $W: \mathcal{Z}_{\Delta} \rightarrow \mathbb{Z}_n$ as follows. For each $\mu \in \Delta_0$, choose one edge, i_{μ} , incident with μ . Let $(i, j) \in \mathcal{Z}_{\mu}$, and define

$$W((i, j)) = \begin{cases} g^{\frac{n}{m(\mu)}} & \text{if } i = i_{\mu} \\ \text{id} & \text{otherwise.} \end{cases}$$

We see that W is a Brauer weighting since $\omega_{\mu} = g^{\frac{n}{m(\mu)}}$, and hence $\text{ord}(\mu) = m(\mu)$. If $(\Delta_W, \sigma_W, m_W)$ is the Brauer covering graph associated with W , then, for each vertex μ_d in Δ_W , $m_W(\mu_d) = m(\mu) / \text{ord}(\mu) = 1$. It follows that $(\Delta_W, \sigma_W, m_W)$ has the desired property.

Noting that if (Δ, σ, m, q) is a quantized Brauer graph, the above arguments can be extended to the following result.

PROPOSITION 7.4. *Let (Δ, σ, m, q) be a quantized Brauer graph. Then there is a finite abelian group G and a Brauer weighting $W: \mathcal{Z}_{\Delta} \rightarrow G$ of (Δ, σ, m, q) so that the quantized Brauer covering graph $(\Delta_W, \sigma_W, m_W, q_W)$ satisfies $m_W(\mu) = 1$, for all vertices μ in Δ_W .*

7.5. Coverings and the removal of loops. Our next application is the removal of loops.

Let (Δ, σ, m) be a Brauer graph with n loops, labelled ℓ_1, \dots, ℓ_n . Let p be an integer greater than 1 and let $G = \prod_{k=1}^n \mathbb{Z}_p$, where \mathbb{Z}_p is the cyclic group of order p with generator g . For $k = 1, \dots, n$, let $z_k = (\text{id}, \dots, \text{id}, g, \text{id}, \dots, \text{id}) \in G$, where g occurs in the k th component. For each vertex μ in Δ , choose one edge, i_{μ} , incident with μ .

We now define a Brauer weighting $W: \mathcal{Z}_{\Delta} \rightarrow G$ as follows. Suppose that $(i, j) \in \mathcal{Z}_{\mu}$. If i is not a loop, then set $W((i, j)) = \text{id}_G$. If i is the loop ℓ_k and the edge i is the first occurrence of i in the successor sequence of i_{μ} , then set $W((i, j)) = z_k$. If i is the second occurrence of i in the successor sequence of i_{μ} , then set $W((i, j)) = z_k^{-1}$. It is immediate that $\omega_{\mu} = \text{id}_G$, for each vertex μ , and hence W is, in fact, a Brauer weighting.

Let (Δ_W, \circ_W, m_W) be the Brauer covering graph with respect to W . For each vertex μ , since $\omega_\mu = \text{id}_G$ and the order of G is p^n , there are p^n cosets of H_μ in G , each of which contains a single element. Thus, for each vertex μ in Δ , we have p^n vertices $\mu_{[i, H_{\mu,1}]}, \dots, \mu_{[i, H_{\mu,p^n}]}$ in Δ_W , where i is an edge incident with μ . These vertices are independent of the choice of i , see Section 4. To show there are no loops in Δ_W , consider the loop ℓ_k at the vertex v in Δ . For ease of notation we set $\ell = \ell_k$. Let $h \in G$ and consider ℓ_h and $\hat{\ell}_h$, where ℓ and $\hat{\ell}$ are the first and second occurrences of ℓ in the successor sequence of i_v . We show that ℓ_h is not a loop in Δ_W . A similar argument shows that $\hat{\ell}_h$ is also not a loop.

From the remark after Definition 4.3, the edge ℓ_h has endpoints $v_{[\ell, H_{v,s}]}$ and $v_{[\hat{\ell}, H_{v,s}]}$, where $H_{v,s}$ is the unique coset of H_v in G containing h . From Lemma 4.2, $[\ell, H_{v,s}] = [\hat{\ell}, H_{v,s}]$ if and only if $H_{v,s} = H_{v,s}\omega(\ell, \hat{\ell})\omega_v^\theta$, for some $0 \leq \theta < \text{ord}(v)$. But $\omega_v = \text{id}_G$, and, from the definition of W , $\omega(\ell, \hat{\ell})$ is not id_G . Since $H_{v,s} = \{h\}$, we see that ℓ_h is not a loop in Δ_W .

Note that we could have taken G to be any finite product of n non-trivial abelian groups in the above construction and each z_k to be an n -tuple with non-identity element in the k th component and identity elements in all other components.

Summarizing, we have the following result.

PROPOSITION 7.5. *Let (Δ, \circ, m, q) be a quantized Brauer graph. Then there is a finite abelian group G and a Brauer weighting $W: \mathcal{Z}_\Delta \rightarrow G$ of (Δ, \circ, m, q) so that the quantized Brauer covering graph $(\Delta_W, \circ_W, m_W, q_W)$ has the property that Δ_W contains no loops.*

7.6. Coverings and the removal of multiple edges. Our final application is the removal of multiple edges. Let $(\Delta_0, \circ_0, m_0, q_0)$ be a quantized Brauer graph. By Proposition 7.5 there is a finite abelian group G_1 and a Brauer weighting $W_1: \mathcal{Z}_{\Delta_0} \rightarrow G_1$ of $(\Delta_0, \circ_0, m_0, q_0)$ so that the quantized Brauer covering graph $(\Delta_{W_1}, \circ_{W_1}, m_{W_1}, q_{W_1})$ has the property that Δ_{W_1} contains no loops.

For simplicity of notation, let $(\Delta, \circ, m, q) = (\Delta_{W_1}, \circ_{W_1}, m_{W_1}, q_{W_1})$. Thus, Δ contains no loops. We say the pair of vertices $\{\mu, \nu\}$ is α -marked if $\alpha \geq 2$ and there are precisely α edges with endpoints μ and ν . List the α -marked pairs $\{\mu_1, \nu_1\}, \dots, \{\mu_n, \nu_n\}$, where $\{\mu_k, \nu_k\}$ is α_k -marked. Note that a vertex in Δ can occur in more than one α -marked pair. Let G be the product $G = \prod_{k=1}^n \mathbb{Z}_{\alpha_k}$, where \mathbb{Z}_{α_k} is the cyclic group of order α_k with generator g_k . For $k = 1, \dots, n$, let $z_k = (\text{id}, \dots, \text{id}, g_k, \text{id}, \dots, \text{id})$, where g_k occurs in the k th component. For each vertex μ in Δ , choose one edge, i_μ , incident with μ . For each α -marked pair $\{\mu, \nu\}$ choose either μ or ν to be the distinguished vertex of that pair.

We now define a successor weighting $W: \mathcal{Z}_\Delta \rightarrow G$ as follows. Suppose that $(i, j) \in \mathcal{Z}_\mu$. If the endpoints of i are the α -marked pair $\{\mu_k, \nu_k\}$, and μ is the distinguished vertex of this pair, then set $W((i, j)) = z_k$. In all other cases, set $W((i, j)) = \text{id}_G$. The reader may check that $\omega_\mu = z_k^{\alpha_k} = \text{id}_G$, for each vertex μ , and hence W is, in fact, a Brauer weighting.

Let (Δ_W, \circ_W, m_W) be the Brauer covering graph associated with W . An argument similar to the one given in Section 2 shows that (Δ_W, \circ_W, m_W) has the desired properties.

PROPOSITION 7.6. *Let (Δ, \circ, m, q) be a quantized Brauer graph such that Δ contains no loops. Then there is a finite abelian group G and a Brauer weighting $W: \mathcal{Z}_\Delta \rightarrow G$*

of (Δ, σ, m, q) so that the quantized Brauer covering graph $(\Delta_W, \sigma_W, m_W, q_W)$ has the property that Δ_W does not contain multiple edges between any two vertices.

7.7. Main theorem. We combine Propositions 7.4, 7.5 and 7.6 to obtain our final result.

THEOREM 7.7. *Let $(\Delta_0, \sigma_0, m_0, q_0)$ be a quantized Brauer graph. Then there is a tower of quantized Brauer covering graphs $(\Delta_0, \sigma_0, m_0, q_0), (\Delta_1, \sigma_1, m_1, q_1), (\Delta_2, \sigma_2, m_2, q_2), (\Delta_3, \sigma_3, m_3, q_3)$ such that the quantized Brauer covering graph $(\Delta_3, \sigma_3, m_3, q_3)$ has the following properties:*

- (1) the multiplicity function m_3 is identically one,
- (2) the graph Δ_3 has no loops, and
- (3) the graph Δ_3 has no multiple edges.

Proof. By Proposition 7.4, there is a finite abelian group G_1 and a Brauer weighting $W_1: \mathcal{Z}_{\Delta_0} \rightarrow G_1$ such that the associated quantized Brauer covering graph $((\Delta_0)_{W_1}, (\sigma_0)_{W_1}, (m_0)_{W_1}, (q_0)_{W_1})$ has $(m_0)_{W_1}$ identically 1. Set $(\Delta_1, \sigma_1, m_1, q_1) = ((\Delta_0)_{W_1}, (\sigma_0)_{W_1}, (m_0)_{W_1}, (q_0)_{W_1})$.

By Proposition 7.5, there is a finite abelian group G_2 and a Brauer weighting $W_2: \mathcal{Z}_{\Delta_1} \rightarrow G_2$ such that the associated quantized Brauer covering graph $((\Delta_1)_{W_2}, (\sigma_1)_{W_2}, (m_1)_{W_2}, (q_1)_{W_2})$ has the property that $(\Delta_1)_{W_2}$ contains no loops. Set $(\Delta_2, \sigma_2, m_2, q_2) = ((\Delta_1)_{W_2}, (\sigma_1)_{W_2}, (m_1)_{W_2}, (q_1)_{W_2})$. Note that m_2 is also identically 1, for if μ_d is a vertex in Δ_2 with $\mu \in \Delta_1$, then we have $\text{ord}(\mu) = 1$ so that $m_2(\mu_d) = m_1(\mu) = 1$.

Finally, by Proposition 7.6, there is a finite abelian group G_3 and a Brauer weighting $W_3: \mathcal{Z}_{\Delta_2} \rightarrow G_3$ such that the associated quantized Brauer covering graph $((\Delta_2)_{W_3}, (\sigma_2)_{W_3}, (m_2)_{W_3}, (q_2)_{W_3})$ has the property that $(\Delta_2)_{W_3}$ contains no multiple edges. Set $(\Delta_3, \sigma_3, m_3, q_3) = ((\Delta_2)_{W_3}, (\sigma_2)_{W_3}, (m_2)_{W_3}, (q_2)_{W_3})$. We then note that Δ_3 has no loops since Δ_2 has no loops, and that m_3 is identically 1 for reasons similar to m_2 being identically 1. This completes the proof. □

Appendix. In this appendix, we review covering theory for path algebras and their quotients. For further information and proofs see [4, 7–9]. We allow G to be any finite group whereas we assumed that G is always a finite abelian group in the previous sections.

Let K be a field and \mathcal{Q} be a finite quiver. Let G be a finite, not necessarily abelian, group. We begin by showing that if G acts freely on \mathcal{Q} , then there is an orbit quiver $\overline{\mathcal{Q}}$ associated with this action. Let G act freely on \mathcal{Q} . If x is either a vertex or an arrow in \mathcal{Q} , then we denote the action of $g \in G$ on x by x^g , and denote the orbit of x under the G -action by \bar{x} . We now construct $\overline{\mathcal{Q}}$. The vertices of $\overline{\mathcal{Q}}$ are the orbits of vertices of \mathcal{Q} and the arrows of $\overline{\mathcal{Q}}$ are the orbits of arrows of \mathcal{Q} , that is, if $a: v \rightarrow w$ is an arrow in \mathcal{Q}_1 , then $\bar{a}: \bar{v} \rightarrow \bar{w}$ is an arrow in $\overline{\mathcal{Q}}_1$.

Consider the path algebras $K\mathcal{Q}$ and $K\overline{\mathcal{Q}}$. The action of G on \mathcal{Q} extends to an action of G on the paths of \mathcal{Q} and hence to the path algebra $K\mathcal{Q}$. Let I be an ideal in $K\mathcal{Q}$ and set $\Lambda = K\mathcal{Q}/I$. Assume that I satisfies $r \in I$ if and only if $r^g \in I$, for all $g \in G$. Let \bar{I} denote the set of orbits of elements of I under the action of G . It is immediate that \bar{I} is an ideal in $K\overline{\mathcal{Q}}$. Let $\bar{\Lambda} = K\overline{\mathcal{Q}}/\bar{I}$. We call $\bar{\Lambda}$ the orbit algebra associated with the action of G on Λ .

We now show that $\bar{\Lambda}$ is G -graded. We start by constructing a set function $\bar{W}: \bar{\mathcal{Q}}_1 \rightarrow G$ which we call the *weight function on $\bar{\mathcal{Q}}$ induced by the action of G* . For each vertex \bar{v} in $\bar{\mathcal{Q}}_0$, choose a vertex v_* in \mathcal{Q}_0 such that $v_* \in \bar{v}$. Let $\bar{a}: \bar{v} \rightarrow \bar{w}$ be an arrow in $\bar{\mathcal{Q}}_1$. Then, by the freeness of the action of G on \mathcal{Q} , there is a unique $g \in G$ and $b \in \bar{a}$ such that $b: v_*^{\text{id}_G} \rightarrow w_*^g$ is an arrow in \mathcal{Q}_1 . We define $\bar{W}(\bar{a}) = g$ and remark that \bar{W} is dependent on the choices of the v_* . We see that \bar{W} extends linearly to $K\bar{\mathcal{Q}}$ by setting $\bar{W}(\bar{v}) = \text{id}_G$, for $\bar{v} \in \bar{\mathcal{Q}}_0$, and, if $p = \bar{a}_1\bar{a}_2 \cdots \bar{a}_n$ is a path in $\bar{\mathcal{Q}}$ with $\bar{a}_i \in \bar{\mathcal{Q}}_1$, then $\bar{W}(p) = \bar{W}(\bar{a}_n) \cdots \bar{W}(\bar{a}_2)\bar{W}(\bar{a}_1)$. Note that in Section 5, the product for $\bar{W}(p)$ is written in reverse order. However, the definitions coincide since G is assumed to be abelian in that section. This choice of \bar{W} induces a G -grading on $K\bar{\mathcal{Q}}$ by setting $(K\bar{\mathcal{Q}})_g$ to be the K -span of paths p such that $\bar{W}(p) = g$. This G -grading on $K\bar{\mathcal{Q}}$ induces a G -grading on $\bar{\Lambda} = K\bar{\mathcal{Q}}/\bar{I}$ if and only if \bar{I} can be generated by weight homogeneous elements, that is, by elements each of which is in $(K\bar{\mathcal{Q}})_g$, for some $g \in G$. It remains to show that \bar{I} can be generated by weight homogeneous elements.

For this, we recall that $r \in K\mathcal{Q}$ is said to be *uniform* if there are vertices v and w in \mathcal{Q}_0 such that $r = vrw$. Since every non-zero element of $K\mathcal{Q}$ is uniquely a sum of uniform elements, the ideal I can be generated by uniform elements. Let $r \in I$ and $v, w \in \mathcal{Q}_0$ so that $r = vrw$. Then, since $r^g \in I$, for all $g \in G$, there exists $h \in G$ such that $r^h = (v_*)^{\text{id}_G} r^h (w_*)^k$, for some $k \in G$. Hence, if p is a path occurring in r , then $W(\bar{p}) = k$. It follows that $\bar{r} \in (K\bar{\mathcal{Q}})_k$ is a homogeneous element, and hence, \bar{I} can be generated by weight homogeneous elements.

We summarize the above discussion in the following result.

PROPOSITION A.1. *Let K be a field and let G be a finite group which acts freely on a finite quiver \mathcal{Q} . Assume that I is an ideal in $K\mathcal{Q}$ such that $r \in I$ if and only if $r^g \in I$, for all $g \in G$. Let $\bar{\Lambda} = K\bar{\mathcal{Q}}/\bar{I}$ be the orbit algebra associated with the action of G on $\Lambda = K\mathcal{Q}/I$. Then there is a weight function $\bar{W}: \bar{\mathcal{Q}}_1 \rightarrow G$ induced by the action of G so that \bar{I} can be generated by weight homogeneous elements. This weight function induces a G -grading on $\bar{\Lambda}$.*

We now start with a weight function and construct an associated covering algebra. Let $W: \mathcal{Q}_1 \rightarrow G$ be a weight function on \mathcal{Q} . We begin by defining the quiver \mathcal{Q}_W . Set $(\mathcal{Q}_W)_0 = \mathcal{Q}_0 \times G$ and $(\mathcal{Q}_W)_1 = \mathcal{Q}_1 \times G$. We denote the vertices of \mathcal{Q}_W by v_g if $v \in \mathcal{Q}_0$ and $g \in G$, and the arrows of \mathcal{Q}_W by a_g if $a \in \mathcal{Q}_1$ and $g \in G$. The arrows in \mathcal{Q}_W are defined as follows: if $a: v \rightarrow w$ is an arrow in \mathcal{Q}_1 and $g \in G$, then $a_g: v_g \rightarrow w_{W(a)g}$. There is a surjection $\pi: K\mathcal{Q}_W \rightarrow K\mathcal{Q}$ induced by $\pi(v_g) = v$ and $\pi(a_g) = a$, for all $g \in G, v \in \mathcal{Q}_0$ and $a \in \mathcal{Q}_1$. If I is generated by a set ρ of weight homogeneous elements, then let I_W be the ideal in $K\mathcal{Q}_W$ generated by $\pi^{-1}(\rho)$. We call $\Lambda_W = K\mathcal{Q}_W/I_W$ the *covering algebra associated with W* .

The group G acts freely on \mathcal{Q}_W in a canonical way; namely, if x is either a vertex or an arrow in \mathcal{Q} and $g, h \in G$, we set $(x_g)^h = x_{gh}$. This action can be extended to an action of G on $K\mathcal{Q}_W$. From the definition of I_W , it is clear that $r \in I_W$ if and only if $r^g \in I_W$, for all $g \in G$. It follows that the action of G on $K\mathcal{Q}_W$ induces an action of G on $\Lambda_W = K\mathcal{Q}_W/I_W$.

The above discussion yields the following result.

PROPOSITION A.2. *Let K be a field, G be a finite group, \mathcal{Q} be a finite quiver, and $W: \mathcal{Q}_1 \rightarrow G$ be a weight function on \mathcal{Q} . Suppose that I is an ideal in $K\mathcal{Q}$ which is generated by weight homogeneous elements. Then G acts freely on \mathcal{Q}_W and there is an*

induced G -action on $\Lambda_W = K\mathcal{Q}_W/I_W$, the covering algebra of $\Lambda = K\mathcal{Q}/I$ associated with W .

The next result shows that the above constructions may be considered as inverse to one another.

THEOREM A.3. *Let K be a field, G a finite group, \mathcal{Q} a finite quiver, and I an ideal in $K\mathcal{Q}$.*

- (1) *Suppose G acts freely on \mathcal{Q} and $r \in I$ if and only if $r^g \in I$, for all $g \in G$. Then, by Propositions A.1 and A.2, G acts freely on \mathcal{Q}_W . There is an isomorphism of quivers from \mathcal{Q} to $\overline{\mathcal{Q}}_W$ which induces a K -algebra isomorphism from $\Lambda = K\mathcal{Q}/I$ to $\overline{\Lambda}_W = K\overline{\mathcal{Q}}_W/\overline{I}_W$.*
- (2) *Suppose $\Lambda = K\mathcal{Q}/I$ and $W: \mathcal{Q}_1 \rightarrow G$ is a weight function on \mathcal{Q} such that I can be generated by weight homogeneous elements. Then, by Propositions A.1 and A.2, there is an isomorphism of quivers from \mathcal{Q} to $\overline{\mathcal{Q}}_W$ which induces a K -algebra isomorphism from $\Lambda = K\mathcal{Q}/I$ to $\overline{\Lambda}_W = K\overline{\mathcal{Q}}_W/\overline{I}_W$.*

We also have the following theorem, which we apply in Theorem 7.2. A proof may be found in [9, Theorem 2.5].

THEOREM A.4. *Let K be a field, G be a finite group, \mathcal{Q} be a finite quiver and I be an ideal in $K\mathcal{Q}$. Suppose $\Lambda = K\mathcal{Q}/I$ and $W: \mathcal{Q}_1 \rightarrow G$ is a weight function on \mathcal{Q} such that I can be generated by weight homogeneous elements. Give Λ the G -grading induced from W . Then the category of G -graded Λ -modules is equivalent to the category of $\Lambda_W = K\mathcal{Q}_W/I_W$ -modules.*

Finally, let $\Lambda = K\mathcal{Q}/I$ be a G -graded algebra and let $P: K\mathcal{Q} \rightarrow \Lambda$ be the canonical surjection. Suppose that the G -grading of Λ is induced from a weight function. It follows that, if x is either a vertex or an arrow of \mathcal{Q} , then $P(x)$ is in some $(K\mathcal{Q})_g$, for some $g \in G$; that is, $P(x)$ is a homogeneous element of degree g . Our final proposition shows that the converse also holds. The proof is left to the reader.

PROPOSITION A.5. *Let $\Lambda = K\mathcal{Q}/I$ be a G -graded algebra and let $P: K\mathcal{Q} \rightarrow \Lambda$ be the canonical surjection. Suppose, for all $x \in \mathcal{Q}_0 \cup \mathcal{Q}_1$, there exists $g \in G$ such that $P(x)$ is a homogeneous element of degree g . Then the G -grading of Λ is induced by a weight function on \mathcal{Q} .*

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REFERENCES

1. J. L. Alperin, *Local representation theory: Modular representations as an introduction to the local representation theory of finite groups*, Cambridge Studies in Advanced Mathematics, vol. 11 (Cambridge University Press, Cambridge, UK, 1986).
2. H. Asashiba, A generalization of Gabriel's galois covering functors and derived equivalences, *J. Algebra* **334** (2011), 109–149.
3. D. J. Benson, *Representations and cohomology. I. Basic representation theory of finite groups and associative algebras*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 30 (Cambridge University Press, Cambridge, UK, 1998).
4. K. Bongartz and P. Gabriel, Covering spaces in representation-theory, *Invent. Math.* **65** (1981–82), 331–378.

5. R. Brauer, Investigations on group characters, *Ann. Math.* **42** (1941), 936–958.
6. C. Cibils and E. N. Marcos, Skew category, Galois covering and smash product of a k -category, *Proc. Amer. Math. Soc.* **134** (2006), 39–50.
7. R. Gordon and E. L. Green, Graded Artin algebras, *J. Algebra* **76** (1982), 111–137.
8. E. L. Green, Graphs with relations, coverings and group-graded algebras, *Trans. Amer. Math. Soc.* **279** (1983), 297–310.
9. E. L. Green, J. R. Hunton and N. Snashall, Coverings, the graded center and Hochschild cohomology, *J. Pure Appl. Algebra* **212** (2008), 2691–2706.
10. E. L. Green, S. Schroll, N. Snashall and R. Taillefer, The Ext algebra of a Brauer graph algebra, preprint (arXiv:1302.6413).
11. J. A. Green, Walking around the Brauer tree, *J. Austral. Math. Soc.* **17** (1974), 197–213 (collection of articles dedicated to the memory of Hanna Neumann, VI).
12. G. Janusz, Indecomposable modules for finite groups, *Ann. Math.* **89** (1969), 209–241.
13. M. Kauer, Derived equivalences of graph orders, PhD Thesis (Universität Stuttgart, 1996).
14. M. Kauer, Derived equivalence of graph algebras, in *Trends in the representation theory of finite-dimensional algebras*, Contemporary Mathematics, vol. 229 (Green E. L., Editor) (American Mathematical Society, Providence, RI, 1998), 201–213.
15. M. Kauer and K. W. Roggenkamp, Higher-dimensional orders, graph-orders, and derived equivalences, *J. Pure Appl. Algebra* **155** (2001), 181–202.
16. F. H. Membrillo-Hernández, Brauer tree algebras and derived equivalence, *J. Pure Appl. Algebra* **114** (1997), 231–258.
17. I. Reiten, Almost split sequences for group algebras of finite representation type, *Trans. Amer. Math. Soc.* **233** (1977), 125–136.
18. J. Rickard, Derived categories and stable equivalence, *J. Pure Appl. Algebra* **61** (1989), 303–317.
19. Ch. Riedtmann, Algebren, Darstellungsköcher, Überlagerungen und zurück, *Comment. Math. Helv.* **55** (1980), 199–224.
20. K. W. Roggenkamp, Biserial algebras and graphs, in *Algebras and modules*, II, CMS Conf. Proc., vol. 24 (Reiten I., Smalø S. O. and Solberg Ø., Editors) (American Mathematical Society, Providence, RI, 1998), 481–496.
21. A. Skowroński, Selfinjective algebras: Finite and tame type, in *Trends in representation theory of algebras and related topics*, Contemporary Mathematics, vol. 406 (de la Peña J. A. and Bautista R., Editors) (American Mathematical Society, Providence, RI, 2006), 169–238.
22. A. Skowroński, K. Yamagata, Positive Galois coverings of self-injective algebras, *Adv. Math.* **194** (2005), 398–436.