Introduction

This volume splits into three main strands in representation theory corresponding to the topics covered by the three LMS networks BLOC, ARTIN and FCG.

The first four chapters contributed by BLOC introduce different aspects of the representation theory of associative algebras focusing on new developments in the subject, in particular, in connection with geometric surface models, cluster theory and model theory.

Raquel Coelho-Simões surveys Auslander–Reiten theory, one of the main tools underlying modern representation theory. Following the classical definitions and, in particular, the introduction of the Auslander–Reiten translate, her summary of geometric surface models describes recent geometric models for the module categories of algebras and cluster-tilted algebras of type A_n and culminates in geometric surface models for the module category of gentle and skew-gentle algebras. This places this survey at the center of recent new trends and developments in the representation theory of finite dimensional algebras.

Hipolito Treffinger gives a comprehensive account of τ-tilting theory that was introduced as such in the early 2010s by Adachi, Iyama and Reiten with τ, the Auslander–Reiten translate of a finite dimensional algebra, playing a central role. The ideas of τ-tilting theory are a generalisation of classical tilting theory and are closely related to categorifications of Cluster algebras and their mutations. Treffinger's contribution explains how τ-tilting theory has been shown to be the right language for many classical concepts in representation theory such as torsion classes, the Brauer–Thrall conjectures and wide subcategories but that it also gives a representation theoretic approach to subjects such as stability conditions and scattering diagrams.

Matthew Pressland in his survey on frieze patterns, cluster algebras and cluster categories gives another representation theoretic interpretation of cluster theory. Namely, he shows in his survey how cluster algebras and cluster categories can be seen as a conceptual explanation of the combinatorics of Coxeter

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Conway frieze patterns. More precisely, after an introduction of Coxeter Conway frieze patterns and their relation to cluster algebras, he shows how for a quiver of Dynkin type A_n, the Coxeter Conway frieze patterns of height n are in bijection with the cluster-tilting objects of the corresponding cluster category defined by Buan–Marsh–Reineke–Reiten–Todorov as a quotient of the derived category of the module category of a type A_n quiver.

Rosie Laking gives an introduction to infinite dimensional representations. In contrast to the finite dimensional case, the situation is much more complicated and a different approach and different methods are needed. Through carefully chosen examples, Laking not only gives a good overview of important classes of infinite dimensional representations such as indecomposable pure-injectives, she also explains connections to logic and model theory and why such an abstract approach is useful. She introduces the Ziegler spectrum – a topological space formed by the indecomposable pure-injectives up to isomorphism – and gives, for finite dimensional algebras, an overview of the basic properties of this topological space and how these can be used to characterize the notion of finite representation type of a finite dimensional algebra.

The next four chapters contributed by ARTIN introduce the reader to four important developments of Lie theory in the past two decades. What ties these four strands together is that each one unites a purely algebraic study of representation theory with some central topic in algebraic geometry and/or low-dimensional topology.

The lectures by Sam Gunningham give a modern perspective on Springer theory – the study of the geometry and topology of the flag variety and related homogeneous spaces, and its relation to the Weyl group and Hecke algebras. The chapter proceeds first in the more elementary framework of Borel–Moore homology and convolution algebras, and then retells the story in the more sophisticated framework of perverse sheaves. As presented in these lectures, Springer theory is a gateway to the sprawling field of geometric representation theory, in which one seeks to build representations – of finite groups, Coxeter groups, Lie groups, Lie algebras, quantum groups, W-algebras, Hecke algebras, etc. – as sections of geometrically constructed sheaves on moduli spaces arising in algebraic geometry.

Amit Hazi's lectures introduce the reader to the theory of Soergel bimodules, its algebraic realization via the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ and finally its elementary diagrammatic incarnation. Soergel bimodules and their diagrammatic incarnations define monoidal categorifications of Hecke algebras. The remarkable realization that the intricate algebraic structures of BGG category $\mathcal{O}$ and of Hecke algebras can be encoded efficiently using elementary diagrams on the plane with relations defined locally and diagrammatically points to a deep relation between these categories and low-dimensional topology, one that has yielded major advances in the study of categorified knot invariants, mathematical physics, higher categorical representation theory and many others.

The lectures by Bart Vlaar introduce the reader to the theory of quantum groups and quantum symmetric pairs. Quantum groups were first studied in the context of statistical mechanics, where they described certain generalized symmetries of a statistical lattice model on the plane. Their representation theory was at the heart of the famous Witten–Reshetikhin–Turaev construction of 3-manifold invariants building on Chern–Simons theory, and the full extent of their role in the theory of 3- and 4-dimensional topological field theories is still a very active area of research. These lectures recall some of this history, and focus on recent developments with so-called quantum symmetric pairs. Roughly quantum symmetric pairs describe (quantizations of) involution-fixed-point subgroups, and so they exhibit analogous algebraic structure to real forms of algebraic groups. Their representation theory has proved to be even richer, yet comparable to study, than that of the quantum group itself, and there is an active community of mathematicians fleshing out this theory in recent years.

Brian Williams' lectures recount the algebro–geometric theory of affine Lie algebras and vertex algebras more generally, as factorization algebras on algebraic curves. This approach to vertex algebras in was first developed by Beilinson–Drinfeld and Frenkel–Ben-Zvi, and in these lectures is recast in the modern and more flexible language of holomorphic factorization algebras, after Costello–Gwilliam, Williams, Faonte–Hennion–Kapranov and others. A remarkable consequence of the reformulation as holomorphic factorization algebras, explored in these lectures, is that the definitions naturally extend to higher-dimensional algebraic varieties, which are replete with new classes of examples. An essential new feature of this extension is the deployment of derived algebraic geometry, and the outcome is a new theory of Kac–Moody algebras in higher dimensions.

The final three chapters contributed by FCG introduce the reader to three strikingly different aspects of group theory.

Gareth Tracey introduces the notion of *crowns* in a finite group G, which, loosely, are subquotients of G that are obtained from the non-Frattini chief factors of G. Crowns were introduced by Gaschütz in 1962 in the context of finite soluble groups. Tracey's article is self-contained, including traditional results on the structure of finite groups necessary to understand the theory of crowns. The chapter starts with the review of chief series, chief factors and an equivalence relation between them up to introducing the notion of a crown in a finite group. The chapter concludes with an application of crowns to find the minimum number of generators of a finite group.

Ilaria Castellano's chapter is an introduction to totally disconnected locally compact (TDLC) groups, their properties and applications, with a focus on the compactly generated TDLC groups. A locally compact group is an extension of its connected component containing the identity by a TDLC group. As a consequence of a result by Gleason and Yamabe, connected locally compact groups can be approximated by Lie groups, in a certain sense. Hence it "remains" to study TDLC groups to understand the structure of locally compact groups. TDLC groups are also the automorphism groups of locally finite connected graphs, and they come with a family of Cayley–Abels graphs unique up to quasi-isometry, a geometric invariant of the group. After discussing some intriguing geometric and topological comparisons of the properties of TDLC groups, the final part of Castellano's chapter introduces homological finiteness conditions and presents some recent work of Willis.

Andreas Bode and Nicolas Dupré survey Schneider and Teitelbaum's theory of admissible locally analytic representations of p-adic Lie groups. The study of such representations is motivated by the conjecture of the p-adic Langlands correspondence. Bode and Dupré introduce locally analytic representations, and they describe the construction of the distributions algebra. Then, they explain the notion of a Fréchet–Stein algebra, and they define admissible representations, providing some context on the recent developments.

Clearly, each of these chapters can only scratch the surface of the topics they introduce. However, our hope is that they provide motivated readers – beginning PhD students and subject-adjacent non-experts alike – with a springboard into the vast and growing literature in each area.