On a theorem in Determinants.

By Sir William Thomson.

The Ancient Methods for the Duplication of the Cube.

By J. S. Mackay, M.A., F.R.S.E.

The object of the present paper is merely to exhibit the methods employed by the ancient Greek geometers in their solution of this celebrated problem. A critical discussion of these methods, of the origin of the problem, and perhaps also an account of more recent researches and a notice of the literature connected therewith may form the subject of a subsequent paper.

The mythical origin of the problem is told by Eratosthenes in his letter to King Ptolemy III. (Euergetes), and mention is there made of the form into which Hippocrates of Chios (about 444 B.C.) recast the problem. Hippocrates's contribution may be set out thus:—

If AB, CD, EF, GH be four straight lines,

then $$AB : GH = \left\{ \begin{matrix} AB : CD \\ CD : EF \\ EF : GH \end{matrix} \right\}.$$

Now if $$AB : CD = CD : EF = EF : GH,$$

then $$AB : GH = \text{triplicate of } AB : CD,$$
$$= AB^3 : CD^3.$$

Hence if $$GH = 2\,AB, \quad CD^3 = 2\,AB^3.$$

If, therefore, AB^3 is a given cube, and it is required to double it, take a straight line GH equal to 2AB, and between AB and GH insert two mean proportionals CD, EF. CD^3 will be the cube required.

The solutions which follow are translated from the commentary of Eutocius of Ascalon (about 555 A.D.) on Archimedes's treatise *Of the Sphere and Cylinder.* See Torelli's *Archimedis Quae Supersunt Omnia* (Oxonii, 1792), pp. 135-149, or Heiberg's *Archimedis Opera Omnia* (Lipsiae, 1881), pp. 67-127. I have retained the order or disorder in which Eutocius gives the solutions, but have affixed approximate dates to their authors. The solution of Pappus, which is also given by Eutocius, I have translated from Pappus himself. See Commandine's *Pappi Alexandrini Mathematicae Collectiones*, or

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Hultsch's *Pappi Alexandrini Collectionis Quae Supersunt* (Berolini, 1876), Book III., proposition 5. The same solution, in almost the same words, occurs again in the Eighth Book of Pappus, and it is from this passage that Eutocius makes his quotation. I have added the remarks which Eutocius makes on Pappus's method. Besides his own solution, Pappus gives those of Eratosthenes, Nicomedes, and Heron, as well as an incorrect solution by a geometer whom he does not name, and his own disproof of the same.

Method of Plato (429-348 b.c.).

Figures 1, 2.

Two straight lines being given to find two mean proportionals in continued proportion.

Let AB, BC at right angles to each other be the two given straight lines between which it is required to find two mean proportionals.

Produce them to D, E.

Construct a rectangular frame FGH, and let a bar KL slide in a groove in one of the legs FG in such a manner as always to be parallel to GH. This will happen if another bar HM be understood to be connected with HG, and to be parallel to FG. For the surfaces FG, HM being grooved from above in dovetail fashion, and pins being fixed in KL to fit these grooves, the movement of KL will always be parallel to GH.

With this construction let any side GH of the angle FGH touch C, and let this angle and the bar KL be shifted until the point G be on the straight line BE, while the side GH touches C; let the bar KL touch the straight line BD at K, and at the other part the point A, so that, as is the case in the figure, the right angle may have a position such as CED, and the bar KL a position such as DA. If this be performed the proposed solution will be effected.

For, since the angles at D and E are right,

$$CB : BE = EB : BD = DB : BA.$$

Method of Heron (about 111 b.c.), As given in his Treatises on Mechanics and Catapults.

Figure 3.

Let AB, BC be the two given straight lines between which it is required to find two mean proportionals.

Place them so as to contain a right angle at B; complete the parallelogram BD; and join AC, BD, which are obviously equal and bisect one another. For the circle described round one of them [as diameter] will pass also through the extremities of the other, since the parallelogram is rectangular.

Let DC, DA be produced to F, G; and about a pin fixed at B suppose a bar FBG to be revolved till it cuts off equal straight lines measured from E, namely EG, EF. Suppose the lines cut off, and the position of the bar to be FBG, EG, EF being as already said equal to each other. From E draw EH perpendicular to CD; EH evidently bisects CD.

Since CD is bisected at H, and CF is added,

$$DF \cdot FC + CH^2 = HF^2.$$

Add to each EH^2;

then $$DF \cdot FC + CH^2 + HE^2 = FH^2 + HE^2.$$

But $$CH^2 + HE^2 = CE^2, \text{ and } FH^2 + HE^2 = EF^2;$$

therefore $$DF \cdot FC + CE^2 = EF^2.$$

Similarly it may be shown that

$$DG \cdot GA + AE^2 = EG^2.$$

Now $$AE = EC, \text{ and } GE = EF;$$

therefore $$DF \cdot FC = DG \cdot GA.$$

And if the rectangle under the extremes is equal to the rectangle under the means, the four straight lines are proportional;

therefore $$FD : DG = AG : CF.$$

But $$FD : DG = FC : CB, \quad = BA : AG;$$

for in the triangle FDG, CB has been drawn parallel to DG one of its sides, and AB has been drawn parallel to DF.

Therefore $$BA : AG = AG : CF, \quad = CF : CB.$$

Hence between AB, BC the two mean proportionals are AG, CF, which were to be found.

Method of Philon of Byzantium (about 150 B.C.).

Figure 4.

Let AB, BC be the two given straight lines between which it is required to find two mean proportionals.

Place them so as to contain a right angle at B; join AC, and about it [as diameter] describe a semicircle ABEC. Draw AD per-

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pendicular to BA, and CF to BC. Let a bar be placed at B, moveable round it and cutting AD, CF; and let it be moved round B till the line from B to D may be equal to the line from E to F, that is, to the line between the circumference of the circle and CF.

Suppose then the bar to have the position DBEF, so that, as already said, DB = EF. I say that the lines AD, CF are the mean proportionals between AB, BC.

For suppose the lines DA, FC to be produced and meet at H. It is evident, since BA, FH are parallel, that the angle at H is right, and that the circle AEC if it were completed would pass through H.

Since $$DB = EF,$$
therefore $$ED \cdot DB = BF \cdot FE.$$
But $$ED \cdot DB = HD \cdot DA,$$
since each is equal to the square of the tangent from D;
and $$BF \cdot FE = HF \cdot FC,$$
for in the same way each is equal to the square of the tangent from F;
therefore $$HD \cdot DA = HF \cdot FC;$$
therefore $$DH : HF = CF : DA.$$
Now $$HD : HF = BC : CF = DA : AB;$$
for in the triangle DHF, BC is parallel to DH, and BA to HF;
therefore $$BC : CF = CF : DA = DA : AB,$$
as was to be shown.

It is to be noticed that this construction is almost the same as Heron's. For the parallelogram BH is the same as that taken in Heron's construction, as well as the produced sides HA, HC, and the bar moveable about B. The only point of difference is that in the latter we moved the bar about B till it cut HD, HF in points which were equally distant from K the middle point of AC, so that KD = KF, while in the latter we moved it till DB = EF. With both constructions the same result follows, but the one we have now explained is handier; for it is possible to see that DB, EF are equal, if the bar be divided into consecutively equal parts, and this is much easier than to ascertain by a pair of compasses whether the lines drawn from K to D and F are equal.

Method of Apollonius (about 222 b.c.).

Figure 5.

Let the two given straight lines BA, AC between which two mean proportionals are to be found contain a right angle at A.

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With centre B and radius AC describe the circle HKL, and again with centre C and radius AB describe the circle MHN, cutting HKL at H; and join HA, HB, HC.

Then BC is a parallelogram, and HA a diagonal of it. Bisect HA at X, and with centre X describe a circle cutting AB, AC produced at D, E, in such a way however that D and E may be collinear with H. This would happen if a bar cutting AD, AE were moved about H till the lines from X to D, E were equal. If this be done the solution will be obtained.

For the construction is the same as that of Heron and Philon; and it is obvious that the same demonstration will be applicable.

Method of Diocles (about 180 B.C.?), as given in his treatise on Fire-instruments.

Figures 6, 7.

In a circle let two diameters AB, CD be drawn at right angles, and on opposite sides of B let two equal arcs EB, BF be cut off; and through F let FG be drawn parallel to AB, and let DE be joined. I say that FG, GD are two mean proportionals between CG, GH.

Through E draw EK parallel to AB.

Then EK = FG, and KC = GD.

This will be evident if LE, LF be joined.

For angle CLE = angle FLD, and the angles at K, G are right; therefore, since LE = LF, all the parts of the one are equal to all the [corresponding] parts of the other;

therefore the remainder CK = GD.

Hence since DK : KE = DG : GH,

and DK : KE = EK : KC,

for EK is a mean proportional between DK, KC;

therefore DK : KE = EK : KC = DG : GH.

Now DK = CG, KE = FG, KC = GD;

therefore CG : GF = FG : GD = DG : GH.

If therefore on both sides of B there be taken equal arcs MB, BN, and through N there be drawn NX parallel to AB, and DM be joined, NX, XD will be mean proportionals between CX, XO.

If therefore we draw several consecutive parallels between B and D, and from B towards C set off arcs equal to those cut off by the parallels towards B, and draw straight lines from D to the points thus obtained, like DE, DM, the parallels between B and D will be cut at certain points (O and H in the accompanying figure 6). If these points by the application of a ruler be joined by straight lines we shall have in the circle a certain line described in which if any point be assumed and through it a parallel be drawn to LB, the parallel so drawn and the segment of the diameter towards D will be mean proportionals between the other segment of the diameter towards C, and that part of the parallel between the assumed point and the diameter.

These preliminaries being arranged, let A, B be the two given straight lines between which two mean proportionals are to be found. In the circle whose two diameters CD, EF are at right angles, let there be described by continuous points, as already shown, the curve DHF. As A is to B, so let CG be to GK; join CK and produce it to meet the curve at H; through H draw LM parallel to EF. Then, by what has been already said, ML, LD are the mean proportionals between CL, LH.

Now since $CL : LH = CG : GK,$

and $CG : GK = A : B;$

if between A and B we insert means N, X in the same ratio as CL, LM, LD, LH, then N, X will be the mean proportionals between A and B, as was to be found.

Method of Pappus (about 300 A.D. ?).

Figure 8.

A cube is found not only double of a cube by means of the instrument described below and invented by us, but also generally having any prescribed ratio to it.

Let a semicircle ABC be constructed, and from the centre D let BD be drawn at right angles. Let a rule move round the point A in such a manner that the one end of it may be fastened with a nail at A, and the remaining part revolve between B and C round the nail as a centre. This construction being made, let it be enjoined to find two cubes having to each other a given ratio.

Let the ratio of BD to DE be made the same as the given ratio; join CE and produce it to F. Let the rule be moved between B and C till the part of it intercepted between the straight lines FE, EB be equal to the part between the straight line BE and the circumference BKC; for this by continued trials and transference of the rule we

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shall easily effect. Let it be done, and let the rule have the position AGHK, so that GH, HK are equal:
I say that the cube described on BD has to the cube described on DH the prescribed ratio, namely that of DB to DE.

Let the circle be understood to be completed;
and having joined KD, produce it to L.
Join LG, which is parallel to BD, because KH = HG and KD = DL;
join also AL and LC.

Since the angle GAL in a semicircle is right, and AM is a perpendicular

$$LM : MA = MA : MG ;$$

therefore $$LM^2 : MA^2 = AM^2 : MG^2 ;$$

and therefore $$CM : MA = AM^2 : MG^2 .$$

With each of these ratios let the ratio of AM to MG be compounded;

then $$\left\{\begin{matrix} CM : MA \\ AM : MG \end{matrix}\right\} = \left\{\begin{matrix} AM^2 : MG^2 \\ AM : MG \end{matrix}\right\} ;$$

that is $$CM : MG = AM^3 : MG^3 .$$

But $$CM : MG = CD : DE = BD : DE ,$$

and $$AM : MG = AD : DH = DB : DH ;$$

therefore $$BD : DE \text{ (the given ratio) } = BD^3 : DH^3 .$$

Accordingly if, as BD is to DH, we make DH to another straight line such as DN, then DH, DN will be two mean proportionals between BD, DE.

Now, it is to be observed that this construction is the same as that which Diocles has proposed, and differs only in this that he describes by consecutive points between A and B a certain curve, on which the point G is taken by producing CE to cut the said curve, while here the point G is found by means of a rule movable round A. That the point G is the same whether it is obtained as here, by means of a rule, or as Diocles says, we may learn thus.

Produce MG to N,* and join KN [cutting DB at X].
Since KH = HG, and GN is parallel to HB,
therefore KX = XN.
Now XB is common and at right angles [to KN],

* This addition to Pappus's figure is left to be made by the reader. The point N here is different from the point N spoken of in the last sentence of Pappus's solution.

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for KN is bisected perpendicularly by the line through the centre; therefore the base is equal to the base, and, on that account, the circumference KB is equal to BN.

The point G, therefore, is the point on Diocles's curve, and the demonstration is the same.

For Diocles said that

$$CM : MN = MN : MA = AM : MG;$$

and NM = ML, for the diameter cuts it perpendicularly;

therefore $CM : ML = LM : MA = AM : MG$.

Hence between CM, MG, the mean proportionals are LM, MA.

But $CM : MG = CD : DE$,

and $CM : ML = AM : MG = CD : DH$;

therefore when two mean proportionals between CD, DE [are to be found], the second [line, that is the first mean proportional] is DH, which also Pappus obtained.

Method of Sporus.*

Figure 9.

Let AB, BC be the two given unequal straight lines; it is required to find between AB, BC two mean proportionals in continued proportion.

From B draw DBE at right angles to AB;

with centre B and radius BA describe the semicircle DAE.

Join EC, and produce it to F; and from D let a straight line be drawn so that GH is equal to HK; for this is possible.

From G and K let the perpendiculars GL, KNM be drawn to DE.

Since $KH : HG = MB : BL$,

and KH = HG; therefore MB = BL;

therefore the remainder $ME = LD$;

therefore the whole $DM = LE$;

therefore $MD : DL = KM : GL$.

Again since $DM : MK = KM : ME$;

therefore $DM : ME = DM^2 : MK^2$

$= DB^2 : BH^2$

$= AB^2 : BH^2$, for DB = AB.

Again since $MD : DB = LE : EB$;

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* Eutocius is, as far as I know, the only ancient author who mentions this geometer.

but $MD : DB = KM : HB,$
and $LE : EB = GL : CB;$
therefore $KM : HB = GL : CB.$
Now $KM : GL = MD : DL,$
$= DM : ME,$
$= AB^2 : HB^2;$
therefore $AB^2 : HB^2 = BH : BC.$

Between HB, BC take a mean proportional X.

Since $AB^2 : BH^2 = HB : BC;$
but $AB^2 : BH^2 =$ duplicate of $AB : BH,$
and $HB : BC =$ duplicate of $HB : X;$
therefore $AB : BH = BH : X.$
But $BH : X = X : BC;$
therefore $AB : BH = BH : X = X : BC.$

It is obvious that this method is the same as that of Pappus and Diocles.

Method of Menechmus (about 350 B.C.).

Figure 10.

Let A, E be the two given straight lines; it is required to find two mean proportionals between A, E.

Let it be done, and let them be B, C;
and let the straight line DG, given in position and terminated at D, be drawn.
At D let DF be made equal to C, and let HF be drawn at right angles, and FH made equal to B.

Since then the three straight lines A, B, C are proportional, the rectangle $A \cdot C = B^2$;
therefore the rectangle under the given line A and C, that is DF, $= B^2 = FH^2$;
therefore H lies on a parabola described through D.

Let the parallels HK, DK be drawn.

Since the rectangle $B \cdot C$ is given, for it is equal to the rectangle $A \cdot E$;
therefore the rectangle $KH \cdot HF$ is also given;
therefore the point H lies on a hyperbola with KD, DF for asymptotes.
The point H is therefore given, and therefore also the point F.

The synthesis will be the following:—

Let A, E be the given straight lines, and let DG be given in position and terminated at D. Through D let there be described a parabola whose axis is DG and parameter [literally, right side of the figure] A. Let the squares of the ordinates drawn at right angles to DG be equal to the rectangles [literally, let the straight lines drawn at right angles to DG be equal in power to the spaces] applied to A, and having as breadths the lines cut off by them to the point D. Let DH be the parabola, and let DK be perpendicular.

With KD, DF as asymptotes let a hyperbola be described such that the lines drawn parallel to KD, DF shall make a space equal to the rectangle $A \cdot E$; it will cut the parabola. Let it cut it at H, and let the perpendiculars HK, HF be drawn.

Since then $FH^2 = A \cdot DF$;

therefore $A : FH = HF : FD$.

Again since $A \cdot E = HF \cdot FD$;

therefore $A : FH = FD : E$.

But $A : FH = FH : FD$;

therefore $A : FH = FH : FD = FD : E$.

Let B be made $= HF$, and $C = DF$;

then $A : B = B : C = C : E$;

therefore A, B, C, E are in continued proportion, as was to be found.

Otherwise.

Figure 11.

Let the two given straight lines AB, BC be at right angles to each other, and let the means between them be DB, BE, so that

$$CB : BD = BD : BE = BE : BA,$$

and let DF, EF be drawn at right angles.

Since $CB : BD = DB : BE$;

therefore the rectangle $CB \cdot BE$, that is the rectangle under a given line and $BE = BD^2 = EF^2$.

Since then the rectangle under a given line and $BE = EF^2$; therefore F touches a parabola about the axis BE.

Again since $AB : BE = BE : BD$;

therefore the rectangle $AB \cdot BD$, that is the rectangle under a given line and $BD = EB^2 = DF^2$;

therefore F touches a parabola about the axis BD.

But it touches also another given [parabola] about [the axis] BE; therefore F is given.

And FD, FE are perpendiculars;
therefore D, E are given.

The synthesis will be the following:—

Let the two given straight lines AB, BC be at right angles to each other, and let them be produced indefinitely from B. About the axis BE let there be described a parabola whose parameter is BC. Again about the axis DB let there be described a parabola whose parameter is AB. These parabolas will cut each other; let them cut at F, and from F let the perpendiculars FD, FE be drawn.

Since from the parabola FE, that is DB, has been drawn,

therefore $CB \cdot BE = BD^2$;

therefore $CB : BD = DB : BE$.

Again since from the parabola FD, that is EB, has been drawn,

therefore $DB \cdot BA = EB^2$;

therefore $DB : BE = BE : BA$.

But $DB : BE = CB : BD$;

therefore $CB : BD = BD : BE = EB : BA$,

as was to be found.

The parabola is described by means of the compass invented by the engineer Isidorus of Miletus, our master, and described by him in the commentary he wrote on Heron's *Arches*.

Method of Archytas (about 400 B.C.), as Eudemus gives it.

Figure 12.

Let the two given straight lines be AD, C: it is required to find two mean proportionals between AD, C.

About the greater AD describe a circle ABDF, and in it let there be inflected AB equal to C; let AB produced meet at P the tangent to the circle drawn from D, and let BEF be drawn parallel to PD. Let a semicylinder be conceived to be situated uprightly on the semicircle ABD, and on AD an upright semicircle lying in the parallelogram of the cylinder. Now, while A the end of the diameter remains fixed, if this semicircle be revolved from D towards B, it will in its revolution cut the cylindrical surface, and will describe on it a certain line. Again, if, while AD remains fixed, the triangle APD be revolved with a motion opposite to that of the semicircle, it will describe a conical surface with the straight line AP, and this in its revolution will meet the line on the cylinder in some point; at the same time B will describe a semicircle on the surface of the cone. At the place where these lines meet, let the revolving semicircle have the position DKA, and the oppositely revolving triangle the position DLA; and let K be the point of the said meeting. Let also the semicircle described by the point B be BMF, and let the common section of it and the circle BDFA be BF. From K to the plane of the semicircle BDA let a perpendicular be drawn: it will fall on the circumference of the circle, because the cylinder stands uprightly. Let it be KI, and let the line joining I to A meet BF at H, and the line AL meet the semicircle BMF at M. Let KD, MI, MH be joined.

Then since both the semicircles DKA, BMF are perpendicular to the underlying plane, their common section MH is perpendicular to the plane of the circle, so that MH is perpendicular to BF;

therefore $HB \cdot HF$, that is $HA \cdot HI = MH^2$.

Hence triangle AMI is similar to both triangles MIH, MAH, and angle IMA is right.

Now angle DKA is right;

therefore KD, MI are parallel, and from the similarity of the triangles there will be proportional

$$DA : AK = KA : AI = IA : AM;$$

therefore the four DA, AK, AI, AM are continually proportional.

Now AM is equal to C, since it is equal to AB;

therefore between the two given lines AD, C two mean proportionals have been found AK, AI.

Method of Eratosthenes (276–194 b.c.).

To King Ptolemy Eratosthenes greeting.

They say that an ancient tragedian introduced Minos constructing a tomb for Glaucus, and that when he learned that it was to be a hundred feet every way, he said: You have indicated a small enclosure for a royal tomb: let it be double; and failing not of the beautiful form, double quickly each side of the tomb. But it appeared that he had made a mistake; for when the sides were doubled, the surface became quadrupled, and the solidity octupled. Now, the inquiry was made by geometers how one could double a given solid and retain it in the same shape, and such a problem was called the duplication of the cube; for having supposed a cube they sought to double it. When all were for a long time at a loss, Hippocrates of

Chios was the first to perceive that if between two straight lines, the greater of which was double the less, there could be found two mean proportionals in continued proportion, the cube would be doubled, so that his difficulty was exchanged for another difficulty equally great. At a later time, it is said that certain Delians, in obedience to an oracle, attempted to double one of the altars, and fell into the same difficulty. They sent accordingly to the geometers, who were with Plato at the Academy, requesting them for a solution of the problem. While they were industriously applying themselves to the matter, and endeavouring to find two means between the two given lines, Archytas of Tarentum is said to have discovered them by means of his semi-cylinders, and Eudoxus by means of the so-called curved lines. Now, all these succeeded in writing demonstratively, but they were not able to make practically useful applications of their methods, except Menechmus to some small extent, and that with difficulty. But an easy instrumental method has been contrived by us whereby between the two given lines we shall find not only two means, but as many as any one shall prescribe. This being found, we shall be able to change into a cube any given solid contained by parallelograms, or to transform from one shape into another, and to make similar, and to increase while preserving the similarity, as, for example, altars and temples. We shall be able also to change into a cube liquid and dry measures, such as a bushel or a medimnus, and by means of its side measure the capacity of these vessels. The contrivance will be useful also to those who wish to enlarge instruments for throwing bolts or stones. For all the parts must be enlarged proportionally, thicknesses, lengths, apertures, naves (?), and inserted cords, if the discharge is to be proportionally increased. Now, these things cannot be performed without finding the means. The demonstration and the construction of the said instrument I have explained to you below.

Figures 13, 14.

Let AE, DH be the two given unequal straight lines between which it is required to find two mean proportionals in continued proportion.

On any straight line EH let AE be placed perpendicularly, and on EH let three consecutive [equal] parallelograms be constructed AF, FI, IH, and in them let there be drawn the diagonals AF, LG, IH which will therefore be parallel. Let the middle parallelogram FI remain fixed, but let the parallelogram AF be pushed above the middle one, and IH below it, as in the second diagram [fig. 14] till the points A, B, C, D are in the same straight line. Through the points A, B, C, D let a straight line be drawn, and let it meet EH produced at K.

Then from the parallels AE, FB,

$$AK : KB = EK : KF;$$

and from the parallels AF, BG,

$$AK : KB = FK : KG;$$

therefore $AK : KB = EK : KF = KF : KG.$

Again from the parallels BF, CG,

$$BK : KC = FK : KG;$$

and from the parallels BG, CH,

$$BK : KC = GK : KH;$$

therefore $BK : KC = FK : KG = GK : KH.$

But $FK : KG = EK : KF;$

therefore $EK : KF = FK : KG = GK : KH.$

Now $EK : KF = AE : BF,$

$FK : KG = BF : CG,$

and $GK : KH = CG : DH;$

therefore $AE : BF = BF : CG = CG : DH.$

Hence between AE, DH two means BF, CG have been found.

These things then have been demonstrated on geometrical surfaces. But in order that we may be able to find the two means instrumentally, let there be constructed a rectangular frame of wood, ivory, or bronze, having three equal tablets as thin as possible, of which the middle one is fixed, but the two others can be moved along in grooves of such a size and fit as anybody may please; for what relates to the demonstration will be carried out in the same way. But in order that the lines may be found more accurately, care must be taken in moving the tablets that all [corresponding] lines may remain parallel, and the parts fit each other smoothly and without gaps. Now, in the votive offering the instrument is of bronze, and is fastened with lead just under the crown of the slab; under it is the demonstration concisely worded and the diagram, and finally the epigram. Let me therefore transcribe these things for you, that you may have them exactly as they are on the votive offering. Of the two diagrams, only the second has been drawn on the slab.

Between two given straight lines to find two mean proportionals in continued proportion.

Let AE, DH be given.

In the instrument I move the tablets till the points A, B, C, D, are in the same straight line, as in the diagram, suppose.

Then AK : KB = EK : KF, from the parallels AE, BF;

and = FK : KG, from the parallels AF, BG;

therefore EK : KF = KF : KG.

Now these same are to each other as AE : BF and BF : CG.

Similarly we shall show that

FB : CG = CG : DH.

therefore AE, BF, CG, DH are proportional.

Between the two given straight lines therefore two means have been found.

But if the given lines should not be equal to AE, DH; having made AE, DH proportional to them, we shall take the means between these [that is, AE, DH], and shall transfer them [by proportion] to those, and we shall have performed what was prescribed. Should it be enjoined to find several means, we shall place in the instrument one more tablet than the number of means to be found. The demonstration will be the same.

[The epigram is here omitted.]

Method of Nicomedes (about 180 B.C.), as given in his treatise on Conchoidal Lines.

Figure 15.

Let CD, DA perpendicular to each other be the two given straight lines between which it is required to find two means continually proportional.

Complete the parallelogram* ABCD, and bisect the sides AB, BC at tbe points L, E.

Having joined LD produce it and let it meet CB produced at G; draw EF perpendicular to BC, and make CF equal to AL.

Join FG, and make CH parallel to it;

and since an angle is contained by KC, CH, from F let there be drawn FHK, making HK equal to AL or CF (which has been shown to be possible by the conchoid).

Having joined KD produce it and let it meet BA produced at M:

then CD : KC = KC : MA = MA : AD.

* The parallelogram is lettered ABCL in Eutocius, ABCD in Pappus. The Greek letters for D and L are easily confounded.

Since BC is bisected at E and KC is added to it,

therefore $BK \cdot KC + CE^2 = EK^2$.

Add EF^2 to both sides;

then $BK \cdot KC + CE^2 + EF^2 = KE^2 + EF^2$;

therefore $BK \cdot KC + CF^2 = KF^2$.

And since $MA : AB = MD : DK$,

and $MD : DK = BC : CK$;

therefore $MA : AB = BC : CK$.

Now $AL = \frac{1}{2}AB$, and $CG = 2BC$;

therefore $MA : AL = GC : CK$.

But $GC : CK = FH : HK$, on account of the parallels GF, CH;

therefore $ML : LA = FK : KH$, by composition.

Now $AL = HK$, since $AL = CF$;

therefore $ML = FK$, and therefore $ML^2 = FK^2$.

And $BM \cdot MA + LA^2 = ML^2$,

and $BK \cdot KC + CF^2 = FK^2$,

of which $AL^2 = CF^2$, for $AL = CF$;

therefore the remainder $BM \cdot MA$ = the remainder $BK \cdot KC$;

and therefore $MB : BK = KC : AM$.

But $BM : BK = CD : CK$;

therefore $CD : CK = CK : AM$.

And $DC : CK = MA : AD$;

therefore $DC : CK = CK : AM = AM : AD$.

Notes on Euclid I., 47.

By William Harvey, B. A.

ABC is a triangle, right-angled at A, and X, Y, Z are the centres of the squares described on the sides opposite the angles A, B, C; XD, XM, XN are respectively perpendicular to BC, CA, AB; MY, DY are joined, and DY meets AC at E; NZ, DZ are joined, and DZ meets AB at F.

In fig. 16 the squares are drawn outwardly, and the points X, Y, Z are on the sides of BC, CA, AB remote from the opposite angles A, B, C.

In fig. 17 the squares are drawn inwardly, and the points X, Y, Z are on the same sides of BC, CA, AB as the opposite angles A, B, C.