11

Position and momentum

Field theory is ripe with objects referred to colloquially as coordinates and momenta. These conjugate pairs play a special role in the dynamical formulation but do not necessarily imply any dimensional relationship to actual positions or momenta.

11.1 Position, energy and momentum

In classical particle mechanics, point particles have a definite position in space at a particular time described by a dynamical trajectory $\mathbf{x}(t)$. The momentum $\mathbf{p}(\mathbf{t}) = m\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}$. In addition, one has the energy of the particle, $\frac{p^2}{2m} + V$, as a book-keeping parameter for the history of the particle's momentum transactions.

In the theory of fields, there is no *a priori* notion of particles: no variable in the theory represents discrete objects with deterministic trajectories; instead there is a continuous field covering the whole of space and changing in time. The position x is a coordinate parameter, not a dynamical variable. As Schwinger puts it, the coordinates in field theory play the role of an abstract measurement apparatus [119], a ruler or measuring rod which labels the stage on which the field evolves. Table 11.1 summarizes the correspondence.

The quantum theory is constructed by replacing the classical measures of position, momentum and energy with operators satisfying certain commutation relations:

$$[\mathbf{x}, \mathbf{p}] = \mathrm{i}\hbar \tag{11.1}$$

and

$$[t, E] = -\mathrm{i}\hbar. \tag{11.2}$$

These operators have to act on something, and indeed they act on the fields, but the momentum and energy are represented by the operators themselves

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Table 11.1. Dynamical variables.

Canonical position	Particle mechanics	Field theory
Parameter space	t	$\mathbf{x}, t$
Dynamical variable	$\mathbf{x}(t)$	$\phi(\mathbf{x}, t)$

independently of the nature of the fields. Let us see why this must be so. The obvious solution to the commutators above is to represent t and $\mathbf{x}$ by algebraic variables and E and $\mathbf{p}$ as differential operators:

$$\begin{aligned} p_i &= -\mathrm{i}\hbar\partial_i \\ E &= \mathrm{i}\hbar\partial_t. \end{aligned} \tag{11.3}$$

If we check the dimensions of these operator expressions, we find that $\hbar\partial_i$ has the dimensions of momentum and that $\hbar\partial_t$ has the dimensions of energy. In other words, even though these operators have no meaning until they act on some field, like this

$$\begin{aligned} p_i\psi &= -\mathrm{i}\hbar\partial_i\psi \\ E\psi &= \mathrm{i}\hbar\partial_t\psi, \end{aligned} \tag{11.4}$$

it is the operator, or its eigenvalues, which represent the momentum and energy. The field itself is merely a carrier of the information, which the operator extracts. In this way, it is possible for the classical analogues of energy and momentum, by assumption, to be represented by the same operators for all the fields. Thus the dimensions of these quantities are correct regardless of the dimensions of the field.

The expectation values of these operators are related to the components of the energy–momentum tensor (see section 11.3),

$$\begin{aligned} \overline{p}_i c &= -\int \mathrm{d}\sigma^0\theta_{0i} = \langle p_i c\rangle \\ E_p &= \int \mathrm{d}\sigma^0\theta_{00} = \langle H_D\rangle\,. \end{aligned} \tag{11.5}$$

H_D is the differential Hamiltonian operator, which through the equations of motion is related to $\mathrm{i}\hbar\partial_t$. The relationship does not work for the Klein–Gordon field, because it is quadratic in time derivatives. Because of their relationship with classical concepts of energy and momentum, E_p and P_i may also be considered as mechanical energy and momenta.

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Table 11.2. Canonical pairs for the fields.

Field	'X'	'P'
Klein–Gordon	ϕ	$\hbar^2 c^2 \partial_0 \phi$
Dirac	ψ	$\psi^\dagger$
Schrödinger	ψ	$\mathrm{i}\hbar\psi^*$
Maxwell	A_μ	D_{0i}

Separate from these manifestations of mechanical transport are a number of other conjugate pairs. The field q itself is a basic variable in field theory, whose canonical conjugate $\partial_0 q$ is often referred to as a conjugate momentum; see table 11.2. That these quantities do not have the dimensions of position and momentum should be obvious from these expressions; thus, it should be clear that they are in no way connected with the mechanical quantities known from the classical theory. In classical electrodynamics there is also a notion of 'hidden' momentum which results from self-interactions [71] in the field.

11.2 Particles and position

The word particle is dogged by semantic confusion in the quantum theory of matter. The classical meaning of a particle, namely a localized pointlike object with mass and definite position, no longer has a primary significance in many problems. The quantum theory of fields is often credited with re-discovering the particle concept, since it identifies countable, discrete objects with a number operator in Fock space. The objects which are counted by this operator are really *quanta*, not particles in the classical sense. They are free, delocalized, plane wave objects with infinite extent. This is no problem for physics. In fact, it is possible to speak of momentum and energy transfer, without discussing the nature of the objects which carry these labels. However, it is sometimes important to discuss localizability.

In spite of their conceptual demotion, it is clear that pointlike particle events are measured by detectors on a regular basis and thus have a practical significance. Accordingly, one is interested in determining how sharply it is possible to localize a particle in space, i.e. how sharp a peak can the wavefunction, and hence the probability, develop? Does this depend on the nature of the field, for instance, the other quantum numbers, such as mass and spin? This question was asked originally by Wigner and collaborators in the 1940s and answered for general mass and spin [6, 101].

The localizability of different types of particle depends on the existence of a Hermitian position operator which can measure it. This is related to the issue

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of physical derivatives in section 10.3. Finding such an operator is simple in the case of the non-relativistic Schrödinger field, but is less trivial for relativistic fields. In particular, massless fields, such as the photon, which travel at the speed of light, seem unlikely candidates for localization since they can never be halted in one place.

11.2.1 Schrödinger field

The Schrödinger field has a scalar product

$$
\begin{aligned}
(\psi, \psi) &= \int \mathrm{d}^n\mathbf{x}\; \psi^*(x)\psi(x) \\
&= \int \frac{\mathrm{d}^n\mathbf{k}}{(2\pi)^n}\; \psi^*(k)\psi(k).
\end{aligned} \tag{11.6}
$$

Its wavefunctions automatically have positive energy, and thus the position operator may be written

$$
\begin{aligned}
(\psi, \hat{\mathbf{x}}\psi) &= \int \mathrm{d}^n\mathbf{x}\; \psi^*(x)\hat{\mathbf{x}}\psi(x) \\
&= \int \frac{\mathrm{d}^n\mathbf{k}}{(2\pi)^n}\; \psi^*(k)\left(\mathrm{i}\frac{\partial}{\partial\mathbf{k}}\right)\psi(k).
\end{aligned} \tag{11.7}
$$

This is manifestly Hermitian. If one translates one of these wavefunctions a distance **a** from the other, then, using

$$
\psi(a) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{a}}\psi(0), \tag{11.8}
$$

one has

$$
\begin{aligned}
(\psi(a), \psi(0)) &= \int \mathrm{d}^n\mathbf{x}\; \psi^*(0)\psi(0) \equiv \delta(a) \\
&= \int \mathrm{d}^n\mathbf{x}\; \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{a}}.
\end{aligned} \tag{11.9}
$$

This is an identity. It shows that the Schrödinger wavefunction can be localized with delta-function precision. Point particles exist.

11.2.2 Klein–Gordon field

The Klein–Gordon field does not automatically have only positive energy solutions, so we must restrict the discussion to the set of solutions which have

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strictly positive energy. The scalar product on this positive energy manifold is

$$
\begin{aligned}
(\phi^{(+)}, \phi^{(+)}) &= \int \mathrm{d}^n\mathbf{x}\, (\phi^{(+)*} \overleftrightarrow{\partial_0}\, \phi^{(+)}), \\
&= \int (\mathrm{d}k)\, \phi^{(+)*}(k)\phi^{(+)}(k)\, \theta(-k_0)\delta(p^2c^2 + m^2c^4) \\
&= \int \frac{(\mathrm{d}\mathbf{k})}{2|p_0|} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{a}} |\phi_0^{(+)}|^2.
\end{aligned}
\tag{11.10}
$$

A translation by a such that $\phi^{(+)}(a) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{a}}\phi_0(k)$ makes the states orthogonal;

$$
\begin{aligned}
(\phi^{(+)}(a), \phi^{(+)}(0)) &= \delta^n(\mathbf{a}) \\
&= \int (\mathrm{d}\mathbf{k}) \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{a}} \\
&= \int \frac{(\mathrm{d}\mathbf{k})}{2|p_0|} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{a}} |\phi_0^{(+)}|^2.
\end{aligned}
\tag{11.11}
$$

For the last two lines to agree, we must have

$$
\phi_0^{(+)}(k) = \sqrt{2|p_0|},
\tag{11.12}
$$

and thus the extent of the field about the point $\mathbf{a}$ is given by

$$
\phi^{(+)}(\mathbf{x} - \mathbf{a}) = \int \frac{(\mathrm{d}\mathbf{k})}{\sqrt{2|p_0|}} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{a})},
\tag{11.13}
$$

which is not a delta function, and thus the Klein–Gordon particles do not exist in the same sense that Schrödinger particles do. There exist only approximately localizable concentrations of the field. The result of this integral in n dimensions can be expressed in terms of Bessel functions. For instance, in $n = 3$,

$$
\phi^{(+)}(a) \sim \left(\frac{m}{r}\right)^{\frac{5}{4}} H^{(1)}_{\frac{5}{4}}(\mathrm{i}mr)
\tag{11.14}
$$

where $r = |\mathbf{x}-\mathbf{a}|$. This lack of sharpness is reflected in the nature of the position operator $\hat{\mathbf{x}}$ acting on these states:

$$
(\phi^{(+)}(a), \hat{\mathbf{x}}\phi^{(+)}(a)) = \int \frac{(\mathrm{d}\mathbf{k})}{2|p_0|} \phi^*(k)\, \hat{\mathbf{x}}\, \phi(k).
\tag{11.15}
$$

Clearly, the partial derivative $\frac{\partial}{\partial \mathbf{k}}$ is not a Hermitian operator owing to the factors of p_0 in the measure. It is easy to show (see section 10.3) that the addition of the connection term,

$$
\hat{\mathbf{x}} = \mathrm{i}\frac{\partial}{\partial \mathbf{k}} + \frac{\mathrm{i}}{2}\frac{\mathbf{k}}{p_0^2},
\tag{11.16}
$$

is what is required to make this operator Hermitian.

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11.2.3 Dirac field

The Dirac field also has both positive and negative energy states, and particle wavefunctions must be restricted to positive energies. It shares with the Klein–Gordon field the inability to produce sharp delta-function-like configurations of the field. The expression for the position operator is extremely complicated for the spin-$\frac{1}{2}$ particles, owing to the constraints imposed by the γ-matrices. Although the procedure is the same, in principle, as for the Klein–Gordon field, the details are aggravated by the complexity of the field equations for the Dirac field.

The scalar product for localizable solutions is now, by analogy with eqn. (11.11),

$$(\psi^{(+)}, \psi^{(+)}) = \int \frac{(\mathrm{d}\mathbf{k})}{(2p_0)^2} |\psi|^2, \tag{11.17}$$

since there is no time derivative in the scalar product. Restricting to positive energies is also more complex, owing to the matrix nature of the equation. The normalized positive energy solutions include factors of

$$N = \sqrt{\frac{E}{E + mc^2}} = \sqrt{\frac{-p_0}{(-p_0 + mc)}}, \tag{11.18}$$

giving

$$(\psi^{(+)}, \hat{\mathbf{x}}\psi^{(+)}) = \int \frac{(\mathrm{d}\mathbf{k})}{(2p_0)^2} u^\dagger N \, \hat{\mathbf{x}} \, N u. \tag{11.19}$$

A suitable Hermitian operator for the position

$$\hat{\mathbf{x}} = N \left(-\mathrm{i} \frac{\partial}{\partial \mathbf{k}} + \Gamma \right) N \tag{11.20}$$

must now take into account all of these factors of the momentum.

11.2.4 Spin s fields in $3 + 1$ dimensions

The generalization to any half-integral and integral massive spin fields can be accomplished using Dirac's construction for spin $\frac{1}{2}$. It is only sketched here. A spin-s field may be written as a direct product of $2s$ spin-$\frac{1}{2}$ blocks. Following Wigner *et al.* [6, 101], the wavefunction may be written in momentum space as

$$\psi(k)_\alpha \tag{11.21}$$

where $\alpha = 1, \ldots, 2s$ represents the components of $2s$ four-component spin blocks (in total $2s \times 4$ components). The sub-spinors satisfy block-diagonal equations of motion:

$$(\gamma^\mu_\alpha p_\mu + mc)\psi_\alpha = 0. \tag{11.22}$$

The γ-matrices all satisfy the Clifford algebra relation (see chapter 20),

$$\left\{\gamma_\alpha^\mu, \gamma_\alpha^\nu\right\} = -2g^{\mu\nu}. \tag{11.23}$$

The scalar product for localizable positive energy solutions may thus be found by analogy with eqn. (11.17):

$$\begin{aligned}(\psi_1, \psi_2) &= \int (\mathrm{d}p)\overline{\psi}_1\, \gamma_1^0 \dots \gamma_{2s}^0\, \psi_2 \\ &= \int (\mathrm{d}\mathbf{p}) \left(\frac{|mc|}{p_0}\right)^{2s+1} \gamma_1^\dagger \gamma_2,\end{aligned} \tag{11.24}$$

since, in the product over blocks, each normalization factor is multiplied in turn. Wigner *et al.* drop the factors of the mass arbitrarily in their definitions, since these contribute only dimensional factors. It is the factors of p_0 which affect the localizability of the fields. The localizable wavefunction is thus of the form

$$|\psi|^2 \sim p_0^{2s+1}. \tag{11.25}$$

The normalization of the positive energy spinors is

$$\sum_\xi |u|^2 = \left(\frac{p_0 + mc}{2p_0}\right)^{2s}. \tag{11.26}$$

Combining the factors of momentum, one arrives at a normalization factor of

$$N = \left(\frac{p_0}{p_0 + mc}\right)^s \times \sqrt{p_0^{2s+1}} \tag{11.27}$$

and a Hermitian position operator of the form

$$(\psi, \hat{\mathbf{x}}\psi) = \int \frac{(\mathrm{d}\mathbf{p})}{2p_0^{2s+1}} \left(u\, N \left(-\mathrm{i}\frac{\partial}{\partial \mathbf{k}} + \Gamma\right) N\, u\right). \tag{11.28}$$

Notice that the extra factors of the momentum lead to a greater de-localization. This expression contains the expressions for spin 0 and spin $\frac{1}{2}$ as special cases. For massless fields, the above expressions hold for spin 0 and spin $\frac{1}{2}$, but break down for spin 1, i.e. the photon.

11.3 The energy–momentum tensor $\theta_{\mu\nu}$

Translational invariance of the action implies the conservation of momentum. Time-translation invariance implies the conservation of energy. Generally, invariance of one variable implies the conservation of its conjugate variable. In this section, we see how symmetry under translations of coordinates leads to

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the definition of energy, momentum and shear stress in a mechanical system of fields.

In looking at dynamical variations of the action, we have been considering changes in the function $\phi(x)$. Now consider variations in the field which occur because we choose to translate or transform the coordinates x^μ, i.e.

$$\delta_x \phi(x) = (\partial_\mu \phi(x))\delta x^\mu, \tag{11.29}$$

where we use δ_x to distinguish a coordinate variation and

$$\delta x^\mu = x'^\mu - x^\mu. \tag{11.30}$$

The variation of the action under such a change is given by

$$\delta S = \int (\mathrm{d}x')\mathcal{L}(x') - \int (\mathrm{d}x)\mathcal{L}(x), \tag{11.31}$$

which is manifestly zero, in the absence of boundaries, since the first term is simply a re-labelling of the second. We shall consider the action of an infinitesimal change δx^μ and investigate what this tells us about the system. Since we are not making a dynamical variation, we can expect to find quantities which are constant with respect to dynamics.

To calculate eqn. (11.31), we expand the first term formally about x:

$$\begin{aligned}\mathcal{L}(x') &= \mathcal{L}(x) + \delta\mathcal{L}^{(1)} + \cdots \\ &= \mathcal{L}(x) + (\partial_\mu \mathcal{L})\delta x^\mu + \mathrm{O}((\delta x)^2).\end{aligned} \tag{11.32}$$

The volume element transforms with the Jacobian

$$(\mathrm{d}x') = \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)(\mathrm{d}x), \tag{11.33}$$

thus, we require the determinant of

$$\overset{x}{\partial_\nu}\, x'^\mu = \delta^\mu_\nu + (\partial_\nu \delta x^\nu). \tag{11.34}$$

This would be quite difficult to compute generally, but fortunately we only require the result to first order in δx^μ. Writing out the infinite-dimensional matrix explicitly, it is easy to see that all the terms which can contribute to first order lie on the diagonal:

$$\begin{pmatrix} 1+\partial_1\delta x^1 & \partial_1\delta x^2 & \ldots \\ \partial_2\delta x^1 & 1+\partial_2\delta x^2 & \ldots \\ \vdots & \vdots & \end{pmatrix}. \tag{11.35}$$

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Now, the determinant is the product of all the terms along the diagonal, plus some other terms involving off-diagonal elements which do not contribute to first order; thus, it is easy to see that we must have

$$\det(\overset{x}{\partial}_\nu\, x'^\mu) = 1 + \partial_\mu \delta x^\mu + \mathrm{O}((\delta x)^2). \tag{11.36}$$

Using this result in eqn. (11.34), we obtain, to first order,

$$\delta S = \int (\mathrm{d}x) \left\{ \delta \mathcal{L}^{(1)} + (\partial_\mu \delta x^\mu)\mathcal{L} \right\}. \tag{11.37}$$

Let us now use this result to consider the total variation of the action under a combined dynamical and coordinate variation. In principle, we should proceed from here for each Lagrangian we encounter. To make things more concrete, let us make the canonical assumption that we have a Lagrangian density which depends on some generic field $q(x)$ and its derivative $\partial_\mu q(x)$. This assumption leads to correct results in nearly all cases of interest – it fails for gauge theories, because the definition of the velocity is not gauge-covariant, but we can return to that problem later. We take

$$\mathcal{L} = \mathcal{L}\left(q(x), (\partial_\mu q(x)), x^\mu\right). \tag{11.38}$$

Normally, in a conservative system, x^μ does not appear explicitly, but we can include this for generality. Let us denote a functional variation by δq as previously, and the total variation of $q(x)$ by

$$\delta_{\mathrm{T}} q = \delta q + (\partial_\mu q)\delta x^\mu. \tag{11.39}$$

The total variation of the action is now

$$\delta_{\mathrm{T}} S = \int (\mathrm{d}x) \left\{ \frac{\delta \mathcal{L}}{\delta q}\delta q + \frac{\delta \mathcal{L}}{\delta(\partial_\mu q)}\delta(\partial_\mu q) + (\partial_\mu \mathcal{L})\delta x^\mu + (\partial_\mu \delta x^\mu)\mathcal{L} \right\}, \tag{11.40}$$

where the first two terms originate from the functional variation in eqn. (4.21) and the second two arise from the coordinate change in eqn. (11.32). We now make the usual observation that the δ variation commutes with the partial derivative (see eqn. (4.19)), and thus we may integrate by parts in the second and fourth terms of this expression to give

$$\begin{aligned}\delta_{\mathrm{T}} S = &\int (\mathrm{d}x) \left\{ \left(\frac{\delta \mathcal{L}}{\delta q} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu q)} \right) \delta q \right\} \\ &+ \int (\mathrm{d}x) \left\{ \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu q)}\delta q + \mathcal{L}\delta x^\mu \right] \right\}. \end{aligned}\tag{11.41}$$

One identifies the first line as being that which gives rise to the Euler–Lagrange field equations. This term vanishes by virtue of the field equations, for any

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classically acceptable path. The remaining surface term can be compared with eqn. (4.62) and represents a generator for the combined transformation. We recognize the canonical momentum Π_μ from eqn. (4.66). To display this term in its full glory, let us add and subtract

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu q)}(\partial_\nu q)\delta x^\nu \tag{11.42}$$

to the surface term, giving

$$\begin{aligned}\delta_{\mathrm{T}} S &= \frac{1}{c}\int \mathrm{d}\sigma^\mu \left\{\Pi_\mu(\delta q + (\partial_\nu q)\delta x^\nu) - \theta_{\mu\nu}\delta x^\nu\right\} \\ &= \frac{1}{c}\int \mathrm{d}\sigma^\mu \left\{\Pi_\mu \delta_{\mathrm{T}} q - \theta_{\mu\nu}\delta x^\nu\right\},\end{aligned} \tag{11.43}$$

where we have defined

$$\theta_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial^\mu q)}(\partial_\nu q) - \mathcal{L} g_{\mu\nu}. \tag{11.44}$$

This quantity is called the *energy–momentum tensor*. Its $\mu, \nu = 0, 0$ component is the total energy density or Hamiltonian density of the system. Its $\mu, \nu = 0, i$ components are the momentum components. In fact, if we expand out the surface term in eqn. (11.43) we have terms of the form

$$\Pi \delta q - H\delta t + \mathbf{p}\delta \mathbf{x} + \cdots. \tag{11.45}$$

This shows how elegantly the action principle generates all of the dynamical entities of our covariant system and their respective conjugates (the delta objects can be thought of as the conjugates to each of the dynamical generators). Another way of expressing this is to say

- Π is the generator of q translations,
- H is the generator of t translations,
- $\mathbf{p}$ is the generator of $\mathbf{x}$ translations,

and so on. That these differential operators are the generators of causal changes can be understood from method 2 of the example in section 7.1. A single partial derivative has a complementary Green function which satisfies

$$\partial_x G(x, x') = \delta(x, x'). \tag{11.46}$$

This Green function is simply the Heaviside step function $\theta(t - t')$ from Appendix A, eqn. (A.2). What this is saying is that a derivative picks out a direction for causal change in the system. In other words, the response of the system to a source is channelled into a change in the coordinates and vice versa.

11.3.1 Example: classical particle mechanics

To illustrate the energy–momentum tensor in the simplest of cases, we return to the classical system, with the Lagrangian given by eqn. (4.5). This Lagrangian has no $\mu\nu$ indices, so our dogged Lorentz-covariant formalism is strictly wasted, but we may take μ to stand for the time t or position i and use the general expression. Recognizing that the metric for classical particles is $\delta_{\mu\nu}$ rather than $g_{\mu\nu}$, we have

$$\begin{aligned}\theta_{tt} &= \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\delta_{tt} \\ &= p_i\dot{q}^i - L \\ &= \frac{1}{2}m\dot{q}^2 + V(q) \\ &= H.\end{aligned} \tag{11.47}$$

The off-diagonal spacetime components give the momentum,

$$\theta_{ti} = \frac{\partial L}{\partial \dot{q}_j}\frac{\partial q_j}{\partial q_i} = p_j\delta^j{}_i = p_i = m\dot{q}_i, \tag{11.48}$$

and

$$\theta_{ii} = -L, \tag{11.49}$$

which has no special interpretation. The off-diagonal ij components vanish in this case.

The analogous analysis can be carried out for relativistic point particles. Using the action in eqn. (4.32), one finds that

$$\begin{aligned}\theta_{\tau\tau} &= \frac{\partial L}{\partial^t \mathbf{x}}(\partial_t \mathbf{x}) + L \\ &= \frac{\partial L}{\partial^\tau \mathbf{x}}(\partial_\tau \mathbf{x}) + L \\ &= m\mathbf{u}^2 - \frac{1}{2}m\mathbf{u}^2 + V' \\ &= \frac{1}{2}m\mathbf{u}^2 + V,\end{aligned} \tag{11.50}$$

where $\mathbf{u} = \mathrm{d}\mathbf{x}/\mathrm{d}\tau$ is the velocity, or

$$\theta_{tt} = \frac{1}{2}m\mathbf{v}^2 + V. \tag{11.51}$$

11.3.2 Example: the complex scalar field

The application of eqn. (11.44) for the action

$$S = \int (\mathrm{d}x)\left\{\hbar^2c^2(\partial^\mu\phi_A)^*(\partial_\mu\phi_A) + m^2c^4\phi_A^*\phi_A + V(\phi)\right\}, \tag{11.52}$$

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gives us the following components for the energy–momentum tensor:

$$\begin{aligned}\theta_{00} &= \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_A)}(\partial_0 \phi_A) + \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_A^*)}(\partial_0 \phi_A^*) - \mathcal{L} g_{00} \\ &= \hbar^2 c^2 \left[(\partial_0 \phi_A^*)(\partial_0 \phi_A) + (\partial_i \phi_A^*)(\partial_i \phi_A) \right] + m^2 c^4 + V(\phi).\end{aligned} \tag{11.53}$$

Thus, the last line defines the Hamiltonian density $\mathcal{H}$, and the Hamiltonian is given by

$$H = \int \mathrm{d}\sigma \mathcal{H}. \tag{11.54}$$

The off-diagonal spacetime components define a momentum:

$$\begin{aligned}\theta_{0i} = \theta_{i0} &= \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi)_A}(\partial_i \phi)_A + \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_A^*)}(\partial_i \phi_A^*) \\ &= \hbar^2 c^2 \left\{ (\partial_0 \phi_A^*)(\partial_i \phi_A) + (\partial_0 \phi_A)(\partial_i \phi_A^*) \right\}.\end{aligned} \tag{11.55}$$

Taking the integral over all space enables us to integrate by parts and write this in a form which turns out to have the interpretation of the expectation value (inner product) of the field momentum (see chapter 9):

$$\begin{aligned}\int \mathrm{d}\sigma \theta_{0i} &= -\hbar^2 c^2 \int \mathrm{d}\sigma \left(\phi^* \partial_{\mathrm{i}} \partial_0 \phi - (\partial_0 \phi^*) \partial_i \phi \right) \\ &= -(\phi, p_i c \phi),\end{aligned} \tag{11.56}$$

where $p = -\mathrm{i}\hbar \partial_i$. The diagonal space components are given by

$$\begin{aligned}\theta_{ii} &= \frac{\partial \mathcal{L}}{\partial(\partial^i \phi_A)}(\partial_i \phi_A) + \frac{\partial \mathcal{L}}{\partial(\partial^i \phi_A^*)}(\partial_i \phi_A^*) - \mathcal{L} \\ &= 2\hbar^2 c(\partial_i \phi^*)(\partial_i \phi) - \mathcal{L},\end{aligned} \tag{11.57}$$

where i is not summed. Similarly, the off-diagonal 'stress' components are given by

$$\begin{aligned}\theta_{ij} &= \frac{\partial \mathcal{L}}{\partial(\partial^i \phi_A)}(\partial_j \phi_A) + \frac{\partial \mathcal{L}}{\partial(\partial^i \phi_A)}(\partial_j \phi_A) \\ &= \hbar^2 c^2 \left\{ (\partial_i \phi_A^*)(\partial_j \phi_A) + (\partial_j \phi_A^*)(\partial_i \phi_A) \right\} \\ &= \hbar^{-1} c(\phi_A, p_i p_j \phi_A).\end{aligned} \tag{11.58}$$

From eqn. (11.57), we see that the trace over spatial components in $n+1$ dimensions is

$$\sum_i \theta_{ii} = \mathcal{H} - 2m^2 c^4 \phi_A^2 - 2V(\phi) + (n-1)\mathcal{L}, \tag{11.59}$$

so that the full trace gives

$$\theta^{\mu}_{\ \mu} = g^{\mu\nu}\theta_{\nu\mu} = -2m^2c^4\phi_A^2 - 2V(\phi) + (n-1)\mathcal{L}. \tag{11.60}$$

Note that this vanishes in $1+1$ dimensions for zero mass and potential.

11.3.3 Example: conservation

We can also verify the energy–momentum conservation law, when the fields satisfy the equations of motion. We return to this issue in section 11.8.1. For the simplest example of a scalar field with action,

$$S = \int (\mathrm{d}x) \left\{ \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) + \frac{1}{2}m^2\phi^2 \right\}. \tag{11.61}$$

Using eqn. (11.44), we obtain the energy–momentum tensor

$$\theta_{\mu\nu} = \frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m\phi^2. \tag{11.62}$$

The spacetime divergence of this is

$$\partial^\mu\theta_{\mu\nu} = -(-\Box\,\phi + m^2\phi)(\partial_\nu\phi) = 0. \tag{11.63}$$

The right hand side vanishes as a result of the equations of motion, and thus the conservation law is upheld.

It is interesting to consider what happens if we add a potential $V(x)$ to the action. This procedure is standard practice in quantum mechanics, for instance. This can be done by shifting the mass in the action by $m^2 \to m^2 + V(x)$. The result of this is the following expression:

$$\begin{aligned} \partial^\mu\theta_{\mu\nu} &= (\Box\,\phi - (m^2 + V(x))\phi)(\partial_\nu\phi) + (\partial_\nu V)\phi^2 \\ &= (\partial_\nu V(x))\phi^2. \end{aligned} \tag{11.64}$$

The first term vanishes again by virtue of the equations of motion. The spacetime-dependent potential does not vanish, however. Conservation of energy is only assured if there are no spacetime-dependent potentials. This illustrates an important point, which is discussed more generally in section 11.8.1.

The reason that the conservation of energy is violated here is that a static potential of this kind is not physical. All real potentials change in response to an interaction with another field. By making a potential static, we are claiming that the form of $V(x)$ remains unchanged no matter what we scatter off it. It is an immovable barrier. Conservation is violated because, in a physical system, we would take into account terms in the action which allow $V(x)$ to change in response to the momentum imparted by ϕ. See also exercise 1, at the end of this chapter.

11.4 Spacetime invariance and symmetry on indices

For reasons which should become apparent in section 11.6.1, the energy–momentum tensor, properly defined under maximal symmetry, is symmetrical under interchange of its indices. This reflects the symmetry of the metric tensor under interchange of indices. If the Lorentz symmetry is broken, however (for instance, in the non-relativistic limit), then this property ceases to apply. In a relativistic field theory, a non-symmetrical tensor may be considered simply incorrect; in the non-relativistic limit, only the spatial part of the tensor is symmetrical.

11.5 $\theta_{\mu\nu}$ for gauge theories

Consider the Maxwell action

$$S = \int (\mathrm{d}x) \left\{ \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu \right\}. \tag{11.65}$$

A direct application of the formula in eqn. (11.44) gives an energy–momentum tensor which is not gauge-invariant:

$$\theta_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\alpha)} (\partial_\nu A^\alpha) - \frac{1}{4\mu_0} F^{\rho\sigma} F_{\rho\sigma} g_{\mu\nu}. \tag{11.66}$$

The explicit appearance of A_μ in this result shows that this definition cannot be physical for the Maxwell field. The reason for this lack of gauge invariance can be traced to an inaccurate assumption about the nature of a translation, or conformal transformation of the gauge field [44, 76]; it is related to the gauge invariance of the theory. The expression for $\theta_{\mu\nu}$ in eqn. (11.44) relies on the assumption in eqn. (11.29) that the expression for the variation in the field by change of coordinates is given by

$$\delta_x A_\mu = (\partial_\alpha A_\mu) \delta x^\alpha. \tag{11.67}$$

It is clear that this translation is not invariant with respect to gauge transformations, but this seems to be wrong. After all, potential differences are observable as electric and magnetic fields between two points, and observable quantities should be gauge-invariant. In terms of this quantity, the energy–momentum tensor can be written as

$$\theta_{\mu\nu} \delta x^\nu = \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\alpha)} (\delta_x A^\alpha) - \frac{1}{4\mu_0} F^{\rho\sigma} F_{\rho\sigma} g_{\mu\nu} \delta x^\nu. \tag{11.68}$$

Suppose now that we use this as a more fundamental definition of $\theta_{\mu\nu}$. Our problem is then to find a more appropriate definition of $\delta_x A_\mu$, which leads to a gauge-invariant answer. The source of the problem is the implicit assumption

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that the field at one point in spacetime should have the same phase as the field at another point. In other words, under a translation of coordinates, we should expect the field to transform like a vector only up to a gauge transformation. Generalizing the transformation rule for the vector potential to account for this simple observation cures the problem entirely. The correct definition of this variation was derived in section 4.5.2.

The correct (gauge-invariant) transformation is now found by noting that we may write

$$\begin{aligned}\delta_x A_\mu &= (\partial_\nu A'_\mu(x))\epsilon^\nu + (\overset{x'}{\partial_\mu}\,\epsilon^\nu)A_\nu\\ &= \epsilon_\nu F^\nu{}_\mu + \partial_\mu(\epsilon_\nu A^\nu).\end{aligned} \tag{11.69}$$

This last term has the form of a gauge-invariant translation plus a term which can be interpreted as a gauge transformation $\partial^\mu s$ (where $s = \epsilon_\nu A^\nu$). Thus we may now re-define the variation $\delta_x A^\mu$ to include a simultaneous gauge transformation, leading to the gauge-invariant expression

$$\delta_x A^\mu(x) \equiv \delta_x A^\mu - \partial^\mu s = \epsilon_\nu F^{\nu\mu}, \tag{11.70}$$

where $\epsilon^\mu = \delta x^\mu$. The most general description of the translation ϵ^μ, in $3+1$ dimensions is a 15-parameter solution to Killing's equation for the conformal symmetry [76],

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu - \frac{1}{2}g_{\mu\nu}\partial_\gamma\epsilon^\gamma = 0, \tag{11.71}$$

with solution

$$\epsilon^\mu(x) = a^\mu + bx^\mu + \omega^{\mu\nu}x_\nu + 2x^\mu c^\nu x_\nu - c^\mu x^2, \tag{11.72}$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$. This explains why the conformal variation in the tensor $T_{\mu\nu}$ gives the correct result for gauge theories: the extra freedom can accommodate x-dependent scalings of the fields, or gauge transformations.

The anti-symmetry of $F_{\mu\nu}$ will now guarantee the gauge invariance of $\theta_{\mu\nu}$. Using this expression in eqn. (11.43) for the energy–momentum tensor (recalling $\epsilon^\mu = \delta x^\mu$) gives

$$\begin{aligned}\theta'_{\mu\nu} &= \frac{\delta\mathcal{L}}{\delta(\partial^\mu A^\alpha)}F_\nu{}^\alpha - \mathcal{L}g_{\mu\nu}\\ &= 2\frac{\delta\mathcal{L}}{\delta F^{\mu\alpha}}F_\nu{}^\alpha - \mathcal{L}g_{\mu\nu}\\ &= \mu_0^{-1}F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma}g_{\mu\nu}.\end{aligned} \tag{11.73}$$

This result is manifestly gauge-invariant and can be checked against the traditional expressions obtained from Maxwell's equations for the energy density and

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the momentum flux. It also agrees with the Einstein energy–momentum tensor $T_{\mu\nu}$.

The components in 3 + 1 dimensions evaluate to:

$$\begin{aligned}\theta_{00} &= \mu_0^{-1}\left(F_{0i}F_0^{\ i} - \mathcal{L}g_{00}\right)\\ &= \frac{E_iE_i}{c^2\mu_0} + \frac{1}{2\mu_0}\left(B_iB_i - \frac{E_iE_i}{c^2}\right)\\ &= \frac{1}{2\mu_0}\left(\frac{\mathbf{E}^2}{c^2} + \mathbf{B}^2\right)\\ &= \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}),\end{aligned} \tag{11.74}$$

which has the interpretation of an energy or Hamiltonian density. The spacetime off-diagonal components are given by

$$\begin{aligned}\theta_{0j} = \theta_{j0} &= \mu_0^{-1}F_{0i}F_j^{\ i}\\ &= \mu_0^{-1}\epsilon_{ijk}E_iB_k/c\\ &= -\frac{(\mathbf{E}\times\mathbf{H})_k}{c},\end{aligned} \tag{11.75}$$

which has the interpretation of a ‘momentum’ density for the field. This quantity is also known as Poynting’s vector divided by the speed of light. The conservation law is

$$\partial^\mu\theta_{\mu 0} = -\frac{1}{c}\partial_t\mathcal{H} + \partial_i(\mathbf{H}\times\mathbf{E})^i = \frac{1}{c}\partial_\mu S^\mu = 0, \tag{11.76}$$

which may be compared with eqns. (2.70) and (2.73). Notice finally that

$$\frac{\delta S}{\delta x^0} = -\int \mathrm{d}\sigma\,\theta_{00}, \tag{11.77}$$

and thus that

$$\frac{\delta S}{\delta t} = -H, \tag{11.78}$$

which is the energy density or Hamiltonian. We shall have use for this relation in chapter 14.

11.6 Another energy–momentum tensor $T_{\mu\nu}$

11.6.1 Variational definition

Using the action principle and the Lorentz invariance of the action, we have viewed the energy–momentum tensor $\theta_{\mu\nu}$ as a generator for translations in space

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and time. There is another quantity which we can construct which behaves as an energy–momentum tensor: it arises naturally in Einstein's field equations of general relativity as a source term for matter. This tensor is defined by the variation of the action with respect to the metric tensor:

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}. \tag{11.79}$$

Clearly, this definition assumes that the action is covariant with respect to the metric $g_{\mu\nu}$, so we should not expect this to work infallibly for non-relativistic actions.

The connection between $T_{\mu\nu}$ and $\theta_{\mu\nu}$ is rather subtle and has to do with conformal transformations. Conformal transformations (see section 9.6) are related to re-scalings of the metric tensor, and they form a super-group, which contains and extends the Lorentz transformation group; thus $T_{\mu\nu}$ admits more freedom than $\theta_{\mu\nu}$. As it turns out, this extra freedom enables it to be covariant even for local gauge theories, where fields are re-defined by spacetime-dependent functions. The naive application of Lorentz invariance for scalar fields in section 11.3 does not automatically lead to invariance in this way; but it can be fixed, as we shall see in the next section. The upshot of this is that, with the exception of the Maxwell field and the Yang–Mills field, these two tensors are the same.

To evaluate eqn. (11.79), we write the action with the metric made explicit, and write the variation:

$$\delta S = \int \mathrm{d}^{n+1}x \sqrt{g} \left(\frac{1}{\sqrt{g}} \frac{\delta g}{\delta g^{\mu\nu}} \mathcal{L} + \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} \right), \tag{11.80}$$

where we recall that $g = -\det g_{\mu\nu}$. To evaluate the first term, we note that

$$\frac{\delta g}{\delta g^{\mu\nu}} = -\frac{\delta \det g_{\mu\nu}}{\delta g^{\mu\nu}}, \tag{11.81}$$

and use the identity

$$\ln \det g_{\mu\nu} = \mathrm{Tr} \ln g_{\mu\nu}. \tag{11.82}$$

Varying this latter result gives

$$\delta \ln(\det g_{\mu\nu}) = \mathrm{Tr}\delta \ln g_{\mu\nu}, \tag{11.83}$$

or

$$\frac{\delta(\det g_{\mu\nu})}{\det g_{\mu\nu}} = \frac{\delta g_{\mu\nu}}{g^{\mu\nu}}. \tag{11.84}$$

Using this result, together with eqn. (11.81), in eqn. (11.80), we obtain

$$T_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}. \tag{11.85}$$

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This definition is tantalizingly close to that for the Lorentz symmetry variation, except for the replacement of the first term. In many cases, the two definitions give the same result, but this is not the case for the gauge field, where $T_{\mu\nu}$ gives the correct answer, but a naive application of $\theta_{\mu\nu}$ does not. The clue as to their relationship is to consider how the metric transforms under a change of coordinates (see chapter 25). Relating a general action $g_{\mu\nu}$ to a locally inertial frame $\eta_{\mu\nu}$, one has

$$g_{\mu\nu} = V_\mu{}^\alpha V_\nu{}^\beta \, \eta_{\alpha\beta}, \tag{11.86}$$

where the *vielbein* $V_\mu^\alpha = \partial'_\mu x^\alpha$, so that

$$g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) = \eta^{\alpha\beta} V^\mu_\alpha V^\nu_\beta (\partial_\mu\phi)(\partial_\nu\phi). \tag{11.87}$$

In terms of these quantities, one has

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{V_{\alpha\mu}}{\det V} \frac{\delta S}{\delta V_\alpha{}^\mu}. \tag{11.88}$$

Thus, one sees that variation with respect to a vector, as in the case of $\theta_{\mu\nu}$ will only work if the vector transforms fully covariantly under every symmetry. Given that the maximal required symmetry is the conformal symmetry, one may regard $T_{\mu\nu}$ as the correct definition of the energy–momentum tensor.

11.6.2 The trace of the energy–momentum tensor $T_{\mu\nu}$

The conformal invariance of the field equations is reflected in the trace of the energy–momentum tensor $T_{\mu\nu}$, which we shall meet in the next chapter. Its trace vanishes for actions which are conformally invariant. To see this, we note that, in a conformally invariant theory,

$$\frac{\delta S}{\delta \Omega} = 0. \tag{11.89}$$

If we express this in terms of the individual partial transformations, we have

$$\frac{\delta S}{\delta \Omega} = \frac{\delta S}{\delta g^{\mu\nu}} \frac{\delta g^{\mu\nu}}{\delta \Omega} + \frac{\delta S}{\delta \phi} \frac{\delta \phi}{\delta \Omega} = 0. \tag{11.90}$$

Assuming that the transformation is invertible, and that the field equations are satisfied,

$$\frac{\delta S}{\delta \phi} = 0, \tag{11.91}$$

we have

$$\frac{1}{2}\sqrt{g}\, T_{\mu\nu} \frac{\delta g^{\mu\nu}}{\delta \Omega} = 0. \tag{11.92}$$

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Since $\frac{\delta g^{\mu\nu}}{\delta\Omega}$ must be proportional to $g^{\mu\nu}$, we have simply that

$$T_{\mu\nu} g^{\mu\nu} = \mathrm{Tr}\; T_{\mu\nu} = 0. \tag{11.93}$$

A similar argument applies to the tensor $\theta_{\mu\nu}$, since the two tensors (when defined correctly) agree. In the absence of conformal invariance, one may expand the trace in the following way:

$$T^{\mu}_{\ \mu} = \beta_i \mathcal{L}^i, \tag{11.94}$$

where $\mathcal{L}^i$ are terms in the Lagrangian of ith order in the fields. β^i is then called the beta function for this term. It occurs in renormalization group and scaling theory.

11.6.3 The conformally improved $T_{\mu\nu}$

The uncertainty in the definition of the energy–momentum tensors $\theta_{\mu\nu}$ and $T_{\mu\nu}$ is usually understood as the freedom to change boundary conditions by adding total derivatives, i.e. surface terms, to the action. However, another explanation is forthcoming: such boundary terms are generators of symmetries, and one would therefore be justified in suspecting that symmetry covariance plays a role in the correctness of the definition. It has emerged that covariance, with respect to the conformal symmetry, frequently plays a role in elucidating a sensible definition of this tensor. While this symmetry might seem excessive in many physical systems, where one would not expect to see such a symmetry, its structure encompasses a generality which ensures that all possible terms are generated, before any limit is taken.

In the case of the energy–momentum tensor, the conformal symmetry motivates improvements not only for gauge theories, but also with regard to scaling anomalies. The tracelessness of the energy–momentum tensor for a massless field is only guaranteed in the presence of conformal symmetry, but such a symmetry usually demands a specific spacetime dimensionality. What is interesting is that a fully covariant, curved spacetime formulation of $T_{\mu\nu}$ leads to an invariant definition, which ensures a vanishing $T^{\mu}_{\ \mu}$ in the massless limit [23, 26, 119].

The freedom to add total derivatives means that one may write

$$T_{\mu\nu} \to T_{\mu\nu} + \nabla^\rho \nabla^\sigma m_{\mu\nu\rho\sigma}, \tag{11.95}$$

where $m_{\mu\nu\rho\sigma}$ is a function of the metric tensor, and is symmetrical on μ, ν and ρ, σ indices; additionally it satisfies:

$$m_{\mu\nu\rho\sigma} + m_{\rho\nu\sigma\mu} + m_{\sigma\nu\mu\rho} = 0. \tag{11.96}$$

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These are also the symmetry properties of the Riemann tensor (see eqn. (25.24)). This combination ensures that the additional terms are conserved:

$$\nabla^\mu \Delta T_{\mu\nu} = \nabla^\mu \nabla^\rho \nabla^\sigma m_{\mu\nu\rho\sigma} = 0. \tag{11.97}$$

The properties of the Riemann tensor imply that the following additional invariant term may be added to the action:

$$\Delta S = \int (\mathrm{d}x)\; \xi\, m^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}. \tag{11.98}$$

For spin-0 fields, the only invariant combination of correct dimension is

$$m^{\mu\nu\rho\sigma} = \left(g^{\mu\nu} g^{\rho\sigma} - \frac{1}{2} g^{\rho\nu} g^{\mu\sigma} - \frac{1}{2} g^{\rho\mu} g^{\nu\sigma} \right) \phi^2, \tag{11.99}$$

which gives the term

$$\Delta S = \int \frac{1}{2} \xi R \phi^2, \tag{11.100}$$

where R is the scalar curvature (see chapter 25). Thus, the modified action, which must be temporarily interpreted in curved spacetime, is

$$S = \int (\mathrm{d}x) \left\{ \frac{1}{2} (\nabla^\mu \phi)(\nabla_\mu \phi) + \frac{1}{2} (m^2 + \xi R) \phi^2 \right\}, \tag{11.101}$$

where $(\mathrm{d}x) = \sqrt{g}\,\mathrm{d}^{n+1}x$. Varying this action with respect to the metric leads to

$$\begin{aligned} T_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{2} g_{\mu\nu} \left[(\nabla^\lambda \phi)(\nabla_\lambda \phi) + m^2 \phi^2 \right] \\ + \xi (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) \phi^2. \end{aligned} \tag{11.102}$$

Notice that the terms proportional to ξ do not vanish, even in the limit $R \to 0$, i.e. $\nabla_\mu \to \partial_\mu$. The resulting additional piece is a classic $(n+1)$ dimensional, transverse (conserved) vector displacement. Indeed, it has the conformally invariant form of the Maxwell action, stripped of its fields. The trace of this tensor may now be computed, giving:

$$T^\mu_{\ \mu} = \left[\frac{1-n}{2} + 2\xi n \right] (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{2} (n+1) m^2 \phi^2. \tag{11.103}$$

One now sees that it is possible to choose ξ such that it vanishes in the massless limit; i.e.

$$T^\mu_{\ \mu} = -\frac{1}{2} (n+1) m^2 \phi^2, \tag{11.104}$$

where

$$\xi = \frac{n-1}{4n}. \tag{11.105}$$

This value of ξ is referred to as conformal coupling. In $3+1$ dimensions, it has the value of $\frac{1}{6}$, which is often assumed explicitly.

11.7 Angular momentum and spin[1]

The topic of angular momentum in quantum mechanics is one of the classic demonstrations of the direct relevance of group theory to the nature of microscopic observables. Whereas linear momentum more closely resembles its Abelian classical limit, the microscopic behaviour of rotation at the level of particles within a field is quite unexpected. The existence of intrinsic, half-integral spin **S**, readily predicted by representation theory of the rotation group in $3+1$ dimensions, has no analogue in a single-valued differential representation of the orbital angular momentum **L**.

11.7.1 *Algebra of orbital motion in* $3+1$ *dimensions*

The dynamical commutation relations of quantum mechanics fix the algebra of angular momentum operators. It is perhaps unsurprising, at this stage, that the canonical commutation relations for position and momentum actually correspond to the Lie algebra for the rotation group. The orbital angular momentum of a body is defined by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{11.106}$$

In component notation in n-dimensional Euclidean space, one writes

$$L_i = \epsilon_{ijk} x^j p^k. \tag{11.107}$$

The commutation relations for position and momentum

$$[x^i, p^j] = \mathrm{i}\, \chi_h\, \delta^{ij} \tag{11.108}$$

then imply that (see section 11.9)

$$[L_i, L_j] = \mathrm{i}\, \chi_h\, \epsilon_{ijk}\, L_k. \tag{11.109}$$

This is a Lie algebra. Comparing it with eqn. (8.47) we see the correspondence between the generators and the angular momentum components,

$$\begin{aligned} T^a &\leftrightarrow L^a / \chi_h \\ f_{abc} &= -\epsilon_{abc}, \end{aligned} \tag{11.110}$$

with the group space $a, b, c \leftrightarrow i, j, k$ corresponding to the Euclidean spatial basis vectors. What this shows, however, is that the group theoretical description of rotation translates directly into the operators of the dynamical theory, with a

[1] A full understanding of this section requires a familiarity with Lorentz and Poincaré symmetry from section 9.4.

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dimensionful scale χ_h, which in quantum mechanics is $\chi_h = \hbar$. This happens, as discussed in section 8.1.3, because we are representing the dynamical variables (fields or wavefunctions) as tensors which live on the representation space of the group (spacetime) by a mapping which is adjoint (the group space and representation space are the same).

11.7.2 The nature of angular momentum in $n + 1$ dimensions

In spite of its commonality, the nature of rotation is surprisingly non-intuitive, perhaps because many of its everyday features are taken for granted. The freedom for rotation is intimately linked to the dimension of spacetime. This much is clear from intuition, but, as we have seen, the physics of dynamical systems depends on the group properties of the transformations, which result in rotations. Thus, to gain a true intuition for rotation, one must look to the properties of the rotation group in $n + 1$ dimensions.

In one dimension, there are not enough degrees of freedom to admit rotations. In $2 + 1$ dimensions, there is only room for one axis of rotation. Then we have an Abelian group $U(1)$ with continuous eigenvalues $\exp(\mathrm{i}\theta)$. These 'circular harmonics' or eigenfunctions span this continuum. The topology of this space gives boundary conditions which can lead to any statistics under rotation. i.e. anyons.

In $3 + 1$ dimensions, the rank 2-tensor components of the symmetry group generators behave like two separate 3-vectors, those arising in the timelike components T^{0i} and those arising in the spacelike components $\frac{1}{2}\epsilon^{ijk}T_{ij}$; indeed, the electric and magnetic components of the electromagnetic field are related to the electric and magnetic components of the Lorentz group generators. Physically, we know that rotations and coils are associated with magnetic fields, so this ought not be surprising. The rotation group in $3 + 1$ dimensions is the non-Abelian $SO(3)$, and the maximal Abelian sub-group (the centre) has eigenvalues ± 1. These form a Z_2 sub-group and reflect the topology of the group, giving rise to two possible behaviours under rotation: symmetrical and anti-symmetrical boundary conditions corresponding in turn to Bose–Einstein and Fermi–Dirac statistics.

In higher dimensions, angular momentum has a tensor character and is characterized by n-dimensional spherical harmonics [130].

11.7.3 Covariant description in $3 + 1$ dimensions

The angular momentum of a body at position $\mathbf{r}$, about an origin, with momentum $\mathbf{p}$, is defined by

$$\mathbf{J} = \mathbf{L} + \mathbf{S} = (\mathbf{r} \times \mathbf{p}) + \mathbf{S}. \tag{11.111}$$

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The first term, constructed from the cross-product of the position and linear momentum, is the contribution to the orbital angular momentum. The second term, **S**, is the spin, or intrinsic angular momentum, of the body. The total angular momentum is a conserved quantity and may be derived from the energy–momentum tensor in the following way. Suppose we have a conserved energy–momentum tensor $\theta_{\mu\nu}$, which is symmetrical in its indices (Lorentz-invariant), then

$$\partial_\mu \theta^{\mu\nu} = 0. \tag{11.112}$$

We can construct the new axial tensor,

$$L^{\mu\nu\lambda} = x^\nu \, \theta^{\lambda\mu} - x^\lambda \, \theta^{\nu\mu}, \tag{11.113}$$

which is also conserved, since

$$\partial_\mu L^{\mu\nu\lambda} = \theta^{\lambda\nu} - \theta^{\nu\lambda} = 0. \tag{11.114}$$

Comparing eqn. (11.113) with eqn. (11.111), we see that $L^{\mu\nu\lambda}$ is a generalized vector product, since the components of $\mathbf{r} \times \mathbf{p}$ are of the form $L_1 = r_2 p_3 - r_3 p_2$, or $L_i = \epsilon_{ijk} r_j p_k$. We may then identify the angular momentum 2-tensor as the anti-symmetrical matrix

$$J^{\mu\nu} = \int \mathrm{d}\sigma \; L^{0\mu\nu} = -J^{\nu\mu}, \tag{11.115}$$

which is related to the generators of homogeneous Lorentz transformations (generalized rotations on spacetime) by

$$J^{\mu\nu}\Big|_{p_i=0} = \chi_h \; T^{\mu\nu}_{3+1}; \tag{11.116}$$

see eqn. (9.95). The ij components of $J^{\mu\nu}$ are simply the components of $\mathbf{r} \times \mathbf{p}$. The $i0$ components are related to boosts. Clearly, this matrix is conserved,

$$\partial_\mu J^{\mu\nu} = 0. \tag{11.117}$$

Since the coordinates x^μ appear explicitly in the definition of $J^{\mu\nu}$, it is not invariant under translations of the origin. Under the translation $x^\mu \to x^\mu + a^\mu$, the components transform into

$$J^{\mu\nu} \to J^{\mu\nu} + (a^\mu p^\nu + a^\mu p^\mu)\,. \tag{11.118}$$

(see eqn. (11.5)). This can be compared with the properties of eqn. (9.153). To isolate the part of $T_{\mu\nu}$ which is intrinsic to the field (i.e. is independent of position), we may either evaluate in a rest frame $p_i = 0$ or define, in $3+1$ dimensions, the *dual* tensor

$$S_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda} J^{\lambda\rho} = S^*_{\mu\nu}. \tag{11.119}$$

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The anti-symmetry of the Levi-Cevita tensor ensures that the extra terms in eqn. (11.118) cancel. We may therefore think of this as being the generator of the intrinsic angular momentum of the field or spin. This dual tensor is rather formal though and not very useful in practice. Rather, we consider the Pauli–Lubanski vector as introduced in eqn. (9.161). We define a spin 4-vector by

$$-\frac{1}{2}mc\,S_\mu \equiv \chi_h\; W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}J^{\nu\rho}p^\lambda, \tag{11.120}$$

so that, in a rest frame,

$$\chi_h\; W^\mu_{\text{rest}} = -\frac{1}{2}mc(0, S^i), \tag{11.121}$$

where S^i is the intrinsic spin angular momentum, which is defined by

$$S^i = J^i\Big|_{p^i=0} = \chi_h\, T_{B\,i} = \frac{1}{2}\,\chi_h\,\epsilon_{ijk}T_{\rm R}^{jk}, \tag{11.122}$$

with eigenvalues $s(s+1)\,\chi_h{}^2$ and $m_s\;\chi_h$, where $s = e + f$.

11.7.4 Intrinsic spin of tensor fields in $3 + 1$ dimensions

Tensor fields are classified by their intrinsic spin in $3 + 1$ dimensions. We speak of fields with intrinsic spin $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. These labels usually refer to $3 + 1$ dimensions, and may differ in other number of dimensions since they involve counting the number of independent components in the tensors, which differs since the representation space is spacetime for the Lorentz symmetry. The number depends on the dimension and transformation properties of the matrix representation, which defines a rotation of the field. The homogeneous (translation independent) Lorentz group classifies these properties of the field in $3 + 1$ dimensions,

Field	Spin
$\phi(x)$	0
$\psi_\alpha(x)$	$\frac{1}{2}$
A_μ	1
Ψ^μ_α	$\frac{3}{2}$
$g_{\mu\nu}$	2

where $\mu, \nu = 0, 1, 2, 3$. Although fields are classified by their spin properties, this is not enough to be able to determine the rotational modes of the field. The

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mass also plays a role. This is perhaps most noticeable for the spin-1 field A_μ. In the massless case, it has helicities $\lambda = \pm 1$, whereas in the massive case it can take on the additional value of zero. The reason for the difference follows from a difference in the true spacetime symmetry of the field in the two cases. We shall explore this below.

From section 9.4.3 we recall that the irreducible representations of the Lorentz group determine the highest weight or spin $s \equiv e + f$ of a field. If we set the generators of boosts to zero by taking $\omega_{0i} T^{0i} = 0$ in eqn. (9.95), then we obtain the pure spatial rotations of section 8.5.10. Then the generators of the Lorentz group E_i and F_i become identical, and we may define the *spin* of a representation by the operator

$$S_i = E_i + F_i = \chi_h \, T_{B\,i}. \tag{11.123}$$

The Casimir operator for the defining (vector field) representation is then

$$S^2 = \chi_h^2 \, T_B^2 = \chi_h^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \tag{11.124}$$

This shows that the rotational 3-vector part of the defining representation forms an irreducible module, leaving an empty scalar component in the time direction. One might expect this; after all, spatial rotations ought not to involve timelike components. If we ignore the time component, then we easily identify the spin of the vector field as follows. From section 8.5.10 we know that in representation G_{R}, the Casimir operator is proportional to the identity matrix with value

$$S^2 = S^i S_i = s(s+1)\chi_h^2 \, I_{\mathrm{R}}, \tag{11.125}$$

and $s = e + f$. Comparing this with eqn. (11.124) we have $s(s+1) = 2$, thus $s = 1$ for the vector field. We say that a vector field has spin 1.

Although the vector transformation leads us to a value for the highest weight spin, this does not necessarily tell us about the intermediate values, because there are two ways to put together a spin-1 representation. One of these applies to the massless (transverse) field and the other to the massive Proca field, which was discussed in section 9.4.4. As another example, we take a rank 2-tensor field. This transforms like

$$G_{\mu\nu} \to L_\mu{}^\rho L_\nu{}^\sigma G_{\rho\sigma}. \tag{11.126}$$

In other words, two vector transformations are required to transform this, one for each index. The product of two such matrices has an equivalent vector form with irreducible blocks:

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$$\underbrace{(1,1)}_{\substack{\text{traceless}\\ \text{symmetric}}} \quad \oplus \quad \underbrace{(1,0)\oplus(0,1)}_{\text{anti-symmetric}} \quad \oplus \quad \underbrace{(0,0)}_{\text{trace}}.$$
$$\qquad\qquad + \qquad\qquad\qquad + $$

This is another way of writing the result which was found in section 3.76 using more pedestrian arguments. The first has $(2e+1)(2f+1) = 9$ $(e = f = 1)$ spin $e + f = 2$ components; the second two blocks are six spin-1 parts; and the last term is a single scalar component, giving 16 components in all, which is the number of components in the second-rank tensor.

Another way to look at this is to compare the number of spatial components in fields with $2s + 1$. For scalar fields (spin 0), $2s + 1$ gives one component. A 4-vector field has one scalar component and $2s+1 = 3$ spatial components (spin 1). A spin-2 field has nine spatial components: one scalar (spin-0) component, three vector (spin-1) components and $2s + 1 = 5$ remaining spin-2 components. This is reflected in the way that the representations of the Lorentz transformation matrices reduce into diagonal blocks for spins 0, 1 and 2. See ref. [132] for a discussion of spin-2 fields and covariance.

It is coincidental for $3 + 1$ dimensions that spin-0 particles have no Lorentz indices, spin-1 particles have one Lorentz index and spin-2 particles have two Lorentz indices.

What is the physical meaning of the spin label? The spin is the highest weight of the representation which characterizes rotational invariance of the system. Since the string of values produced by the stepping operators moves in integer steps, it tells us how many distinct ways, $m + m'$, a system can spin in an 'equivalent' fashion. In this case, equivalent means about the same axis.

11.7.5 Helicity versus spin

Helicity is defined by

$$\lambda = J_i\, \hat{p}^i. \tag{11.127}$$

Spin s and helicity λ are clearly related quite closely, but they are subtly different. It is not uncommon to refer loosely to helicity as spin in the literature since that is often the relevant quantity to consider. The differences in rotation algebras, as applied to physical states are summarized in table 11.3. Because the value of the helicity is not determined by an upper limit on the total angular momentum, it is conventional to use the component of the spin of the irreducible representation for the Lorentz group which lies along the direction of the direction of travel. Clearly these two definitions are not the same thing. In the massless case, the labels for the helicity are the same as those which would occur for m_j in the rest frame of the massive case.

From eqn. (11.127) we see that the helicity is rotationally invariant for massive fields and generally Lorentz-invariant for massless $p_0 = 0$ fields.

Table 11.3. Spin and helicity.

	Casimir	$\Lambda_c = m_j$
Massive	$j(j+1)$	$0, \pm\frac{1}{2}, \pm 1, \ldots, \pm j$
Massless	0	$0, \pm\frac{1}{2}, \pm 1, \ldots, \infty$

It transforms like a pseudo-scalar, since J_i is a pseudo-vector. Thus, the sign of helicity changes under parity transformations, and a massless particle which takes part in parity conserving interactions must have both helicity states $\pm\lambda$, i.e. we must represent it by a (reducible) symmetrized pair of irreducible representations:

$$\begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix}. \tag{11.128}$$

The former is the case for the massless Dirac field ($\lambda = \pm\frac{1}{2}$), while the latter is true for the photon field $F^{\mu\nu}$ ($\lambda = \pm 1$), where the states correspond to left and right circularly polarized radiation. Note that, whereas a massive particle could have $\lambda = 0, \pm 1$, representing left transverse, right transverse and longitudinal angular momentum, a massless (purely transverse) field cannot have a longitudinal mode, so $\lambda = 0$ is absent. This can be derived more rigorously from representation theory.

In refs. [45, 55], the authors study massless fields with general spin and show that higher spins do not necessarily have to be strictly conserved; only the Dirac-traceless part of the divergence has to vanish.

11.7.6 Fractional spin in $2 + 1$ dimensions

The Poincaré group in $2 + 1$ dimensions shares many features of the group in $3 + 1$ dimensions, but also conceals many subtleties [9, 58, 77]. These have specific implications for angular momentum and spin. In two spatial dimensions, rotations form an Abelian group $SO(2) \sim U(1)$, whose generators can, in principle, take on eigenvalues which are unrestricted by the constraints of spherical harmonics. This leads to continuous phases [89, 138], particle statistics and the concept of fractional spin. It turns out, however, that there is a close relationship between vector (gauge) fields and spin in $2 + 1$ dimensions, and that fractional values of spin can only be realized in the context of a gauge field coupling. This is an involved topic, with a considerable literature, which we shall not delve into here.

11.8 Work, force and transport in open systems

The notion of interaction and force in field theory is unlike the classical picture of particles bumping into one another and transferring momentum. Two fields interact in the manner of two waves passing through one another: by interference, or *amplitude modulation*. Two fields are said to interact if there is a term in the action in which some power of one field multiplies some power of another. For example,

$$S_{\text{int}} = \int (\mathrm{d}x) \left\{ \phi^2 A_\mu A^\mu \right\}. \tag{11.129}$$

Since the fields multiply, they modulate one another's behaviour or perturb one another. There is no explicit notion of a force here, and precisely what momentum is transferred is rather unclear in the classical picture; nevertheless, there is an interaction. This can lead to scattering of one field off another, for instance.

The source terms in the previous section have the form of an interaction, in which the coupling is linear, and thus they exert what is referred to as a *generalized force* on the field concerned. The word generalized is used because J does not have the dimensions of force – what is important is that the source has an *influence* on the behaviour of the field.

Moreover, if we place all such interaction terms on the right hand side of the equations of motion, it is clear that interactions also behave as sources for the fields (or currents, if you prefer that name). In eqn. (11.129), the coupling between ϕ and A_μ will lead to a term in the equations of motion for ϕ and for A_μ, thus it acts as a source for both fields.

We can express this in other words: *an interaction can be thought of as a source which transfers some 'current' from one field to another.* But be wary that what we are calling heuristically 'current' might be different in each case and have different dimensions.

A term in which a field multiplies itself, ϕ^n, is called a self-interaction. In this case the field is its own source. Self-interactions lead to the scattering of a field off itself. The classical notion of a force was described in terms of the energy–momentum tensor in section 11.3.

11.8.1 The generalized force $F_\nu = \partial_\mu T^{\mu\nu}$

There is a simple proof which shows that the tensor $T_{\mu\nu}$ is conserved, provided one has Lorentz invariance and the classical equations of motion are satisfied. Consider the total dynamical variation of the action

$$\delta S = \int \frac{\delta S}{\delta g^{\mu\nu}} \delta g^{\mu\nu} + \int \frac{\delta S}{\delta q} \delta q = 0. \tag{11.130}$$

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Since the equations of motion are satisfied, the second term vanishes identically, leaving

$$\delta S = \frac{1}{2}\sqrt{g}\int (\mathrm{d}x) T_{\mu\nu}\delta g^{\mu\nu}. \tag{11.131}$$

For simplicity, we shall assume that the metric $g_{\mu\nu}$ is independent of x, so that the variation may be written (see eqn. (4.88))

$$\delta S = \int (\mathrm{d}x) T_{\mu\nu}\left[g_{\mu\lambda}(\partial_\nu \epsilon^\lambda) + g_{\lambda\nu}(\partial_\mu \epsilon^\lambda)\right] = 0. \tag{11.132}$$

Integrating by parts, we obtain

$$\delta S = \int (\mathrm{d}x)\left[-2\partial_\mu T^{\mu\nu}\right]\epsilon_\nu = 0. \tag{11.133}$$

Since $\epsilon^\mu(x)$ is arbitrary, this implies that

$$\partial_\mu T^{\mu\nu} = 0, \tag{11.134}$$

and hence $T^{\mu\nu}$ is conserved. From this argument, it would seem that $T^{\mu\nu}$ must always be conserved in every physical system, and yet one could imagine constructing a physical model in which energy was allowed to leak away. The assumption of Lorentz invariance and the use of the equations of motion provide a catch, however. While it is true that the energy–momentum tensor is conserved in any *complete* physical system, it does not follow that energy or momentum is conserved in every part of a system individually. If we imagine taking two partial systems and coupling them together, then those two systems can exchange energy. In fact, energy will only be conserved if the systems are in perfect balance: if, on the other hand, one system does work on the other, then energy flows from one system to the other. No energy escapes the total system, however.

Physical systems which are coupled to other systems, about which we have no knowledge, are called *open systems*. This is a matter of definition. Given any closed system, we can make an open system by isolating a piece of it and ignoring the rest. Clearly a description of a piece of a system is an incomplete description of the total system, so it appears that energy is not conserved in the small piece. In order to see conservation, we need to know about the whole system. This situation has a direct analogue in field theory. Systems are placed in contact with one another by interactions, often through currents or sources. For instance, Dirac matter and radiation couple through a term which looks like $J^\mu A_\mu$. If we look at only the Dirac field, the energy–momentum tensor is not conserved. If we look at only the radiation field, the energy–momentum tensor is not conserved, but the sum of the two parts is. The reason is that we have to be 'on shell' – i.e., we have to satisfy the equations of motion.

Consider the following example. The (incomplete) action for the interaction between the Dirac field and the Maxwell field is

$$S = \int (\mathrm{d}x) \left\{ \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu \right\}, \tag{11.135}$$

where $J^\mu = \overline{\psi} \gamma^\mu \psi$. Now, computing the energy–momentum tensor for this action, we obtain

$$\partial_\mu T^{\mu\nu} = F^{\mu\nu} J_\mu. \tag{11.136}$$

This is not zero because we are assuming that the current J^μ is not zero. But this is not a consistent assumption in the action, because we have not added any dynamics for the Dirac field, only the coupling $J^\mu A_\mu$. Consider the field equation for ψ from eqn. (11.135). Varying with respect to $\overline{\psi}$,

$$\frac{\delta S}{\delta \overline{\psi}} = \mathrm{i} e \gamma^\mu A_\mu \psi = 0. \tag{11.137}$$

This means that either $A_\mu = 0$ or $\psi = 0$, but both of these assumptions make the right hand side of eqn. (11.136) zero! So, in fact, the energy–momentum tensor is conserved, as long as we obey the equations of motion given by the variation of the action.

The 'paradox' here is that we did not include a piece in the action for the Dirac field, but that we were sort of just assuming that it was there. This is a classic example of writing down an incomplete (open) system. The full action,

$$S = \int (\mathrm{d}x) \left\{ \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu + \overline{\psi}(\gamma^\mu \partial_\mu + m)\psi \right\}, \tag{11.138}$$

has a conserved energy–momentum tensor, for more interesting solutions than $\psi = 0$.

From this discussion, we can imagine the imbalance of energy–momentum on a partial system as resulting in an external force on this system, just as in Newton's second law. Suppose we define the generalized external force by

$$F^\nu = \int \mathrm{d}\sigma \; \partial_\mu T^{\mu\nu}. \tag{11.139}$$

The spatial components are

$$F^i = \int \mathrm{d}\sigma \; \partial_0 T^{0i} = \partial_t P^i = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \tag{11.140}$$

which is just Newton's second law. Compare the above discussion with eqn. (2.73) for the Poynting vector.

An important lesson to learn from this is that a source is not only a generator for the field (see section 14.2) but also a model for what we do not know about an external system. This is part of the essence of source theory as proposed by Schwinger. For another manifestation of this, see section 11.3.3.

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11.8.2 Work and power

In chapter 5 we related the imaginary part of the Feynman Green function to the instantaneous rate at which work is done by the field. We now return to this problem and use the energy–momentum tensor to provide a new perspective on the problem.

In section 6.1.4 we assumed that the variation of the action with time, evaluated at the equations of motion, was the energy of the system. It is now possible to justify this; in fact, it should already be clear from eqn. (11.78). We can go one step further, however, and relate the power loss to the notion of an open system. If a system is open (if it is coupled to sources), it does work, w. The rate at which it does work is given by

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \int \mathrm{d}\sigma\ \partial_\mu T^{\mu 0}. \tag{11.141}$$

This has the dimensions of energy per unit time. It is clearly related to the variation of the action itself, evaluated at value of the field which satisfies the field equations, since

$$\Delta w = -\int \mathrm{d}\sigma \mathrm{d}t\ \partial_\mu T^{\mu 0} = -\left.\frac{\delta S}{\delta t}\right|_{\text{field eqns}}. \tag{11.142}$$

The electromagnetic field is the proto-typical example here. If we consider the open part of the action (the source coupling),

$$S_J = \int (\mathrm{d}x)\ J^\mu A_\mu, \tag{11.143}$$

then, using

$$A_\mu = \int (\mathrm{d}x)\ D_{\mu\nu}(x, x') J^\nu(x'), \tag{11.144}$$

we have

$$\begin{aligned}
\delta S[A_J] &= \delta \int (\mathrm{d}x)\ J^\mu \delta A_\mu \\
&= \int (\mathrm{d}x)(\mathrm{d}x') J^\mu(x) D_{\mu\nu}(x, x') \delta J^\nu(x') \\
&= \int (\mathrm{d}x)(\partial_\mu T^{\mu 0}) \delta t \\
&= \Delta w \delta t.
\end{aligned} \tag{11.145}$$

The Green function we choose here plays an important role in the discussion, as noted in section 6.1.4. There are two Green functions which can be used

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in eqn. (11.144) as the inverse of the Maxwell operator: the retarded Green function and the Feynman Green function. The key expression here is

$$W = \frac{1}{2}\int (\mathrm{d}x)(\mathrm{d}x')J^{\mu}(x)D_{\mu\nu}(x,x')J^{\nu}(x'). \tag{11.146}$$

Since the integral is spacetime symmetrical, only the symmetrical part of the Green function contributes to the integral. This immediately excludes the retarded Green function

11.8.3 Hydrodynamic flow and entropy

Hydrodynamics is not usually regarded as field theory, but it is from hydrodynamics (fluid mechanics) that we derive notions of macroscopic transport. All transport phenomena and thermodynamic properties are based on the idea of *flow*. The equations of hydrodynamics are the Navier–Stokes equations. These are non-linear vector equations with highly complex properties, and their complete treatment is outside the scope of this book. In their linearized form, however, they may be solved in the usual way of a classical field theory, using the methods of this book. We study hydrodynamics here in order to forge a link between field theory and thermodynamics. This is an important connection, which is crying out to be a part of the treatment of the energy–momentum tensor. We should be clear, however, that this is a phenomenological addition to the field theory for statistically large systems.

A fluid is represented as a velocity field, $U^{\mu}(x)$, such that each point in a system is moving with a specified velocity. The considerations in this section do not depend on the specific nature of the field, only that the field is composed of matter which is flowing with the velocity vector U^{μ}. Our discussion of flow will be partly inspired by the treatment in ref. [134], and it applies even to relativistic flows. As we shall see, the result differs from the non-relativistic case only by a single term. A stationary field (fluid) with maximal spherical symmetry, in flat spacetime, has an energy–momentum tensor given by

$$\begin{aligned} T_{00} &= \mathcal{H} \\ T_{0i} &= T_{i0} = 0 \\ T_{ij} &= P\delta_{ij}. \end{aligned} \tag{11.147}$$

In order to make this system flow, we may perform a position-dependent boost which places the observer in relative motion with the fluid. Following a boost, the energy–momentum tensor has the form

$$T^{\mu\nu} = Pg^{\mu\nu} + (P+\mathcal{H})U^{\mu}U^{\nu}/c^2. \tag{11.148}$$

The terms have the dimensions of energy density. P is the pressure exerted by the fluid (clearly a thermodynamical average variable, which summarizes the microscopic thermal motion of the field). $\mathcal{H}$ is the internal energy density of the field. Let us consider the generalized thermodynamic force $F^\mu = \partial_\nu T^{\mu\nu}$. In a closed thermodynamic system, we know that the energy–momentum tensor is conserved:

$$F^\mu = \partial_\nu T^{\mu\nu} = 0, \tag{11.149}$$

and that the matter density $N(x)$ in the field is conserved,

$$\partial_\mu N^\mu = 0, \tag{11.150}$$

where $N_\mu = N(x)U_\mu$. If we think of the field as a plasma of particles, then $N(x)$ is the number of particles per unit volume, or number density. Due to its special form, we may write

$$\partial_\mu N^\mu = (\partial_\mu N)U^\mu + (\partial_\mu U^\mu), \tag{11.151}$$

which provides a hint that the velocity boost acts like a local scaling or conformal transformation on space

$$-c^2 \mathrm{d}t^2 + \mathrm{d}x_i \mathrm{d}x^i \to -c^2 \mathrm{d}t^2 + \Omega^2(U)\mathrm{d}x_i \mathrm{d}x^i. \tag{11.152}$$

The average rate of work done by the field is zero in an ideal, closed system:

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= \int \mathrm{d}\sigma \; U_\nu F^\nu \\ &= \int \mathrm{d}\sigma \left[U^\mu \partial_\mu P - \partial_\mu \left((P+\mathcal{H})U^\mu\right)\right] \\ &= 0.\end{aligned} \tag{11.153}$$

Now, noting the identity

$$N\partial_\mu \left(\frac{P+\mathcal{H}}{N}\right) = \partial(P+\mathcal{H}) - \left(\frac{\partial_\mu N}{N}\right)(P+\mathcal{H}), \tag{11.154}$$

we may write

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \int \mathrm{d}\sigma \; U^\mu \left[\partial_\mu P - N\left(\frac{P+\mathcal{H}}{N}\right)\right]. \tag{11.155}$$

Then, integrating by parts, assuming that U^μ is zero on the boundary of the system, and using the identity in eqn. (11.151)

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= -\int \mathrm{d}\sigma \; NU^\mu \left[P\partial_\mu\left(\frac{1}{N}\right) + \partial_\mu\left(\frac{\mathcal{H}}{N}\right)\right] \\ &= -\int \mathrm{d}\sigma \; NU^\mu \left[P\partial_\mu V + \partial_\mu H\right],\end{aligned} \tag{11.156}$$

where V is the volume per particle and H is the internal energy. This expression can be compared with

$$T\mathrm{d}S = P\mathrm{d}V + \mathrm{d}H. \tag{11.157}$$

Eqn. (11.156) may be interpreted as a rate of entropy production due to the hydrodynamic flow of the field, i.e. it is the rate at which energy becomes unavailable to do work, as a result of energy diffusing out over the system uniformly or as a result of internal losses. We are presently assuming this to be zero, in virtue of the conservation law, but this can change if the system contains hidden degrees of freedom (sources/sinks), such as friction or viscosity, which convert mechanical energy into heat in a non-useful form. Combining eqn. (11.156) and eqn. (11.157) we have

$$-\int \mathrm{d}\sigma N U^\mu (\partial_\mu S) T = \int \mathrm{d}\sigma U_\nu \partial_\mu T^{\mu\nu} = 0. \tag{11.158}$$

From this, it is useful to define a covariant entropy density vector $\mathcal{S}^\mu$, which symbolizes the rate of loss of energy in the hydrodynamic flow. In order to express the right hand side of eqn. (11.158) in terms of gradients of the field and the temperature, we integrate by parts and define. Let

$$c(\partial_\mu \mathcal{S}^\mu) = \partial_\mu \left(\frac{U_\nu}{T}\right) T^{\mu\nu}, \tag{11.159}$$

where

$$\mathcal{S}^\mu = N S U^\mu - \frac{U_\nu T^{\mu\nu}}{T}. \tag{11.160}$$

The zeroth component, $c\mathcal{S}^0 = NS$, is the entropy density, so we may interpret $\mathcal{S}^\mu$ as a spacetime entropy vector. Let us now assume that hidden losses can cause the conservation law to be violated. Then we have the rate of entropy generation given by

$$c(\partial_\mu \mathcal{S}^\mu) = \left[-\frac{1}{T}(\partial_\nu U_\mu) + \frac{1}{T^2}(\partial_\nu T) U_\mu\right] T^{\mu\nu}. \tag{11.161}$$

We shall assume that the temperature is independent of time, since the simple arguments used to address statistical issues at the classical level do not take into account time-dependent changes properly: the fluctuation model introduced in section 6.1.5 gives rise only to instantaneous changes or steady state flows. If we return to the co-moving frame in which the fluid is stationary, we have

$$U_i = \partial_\mu U^0 = \partial_t T = 0, \tag{11.162}$$

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and thus

$$(\partial_\mu \mathcal{S}^\mu) = -\left[-\frac{1}{c^2 T}(\partial_t U_i) + \frac{1}{T^2}(\partial_i T)\right] T^{0i} \\ + \frac{1}{2T}(\partial_i U_j + \partial_j U_i) T^{ij}. \quad (11.163)$$

Note the single term which vanishes in the non-relativistic limit $c \to \infty$. This is the only sign of the Lorentz covariance of our formulation. Also, we have used the symmetry of T^{ij} to write $\partial_i U_j$ in an ij-symmetric form.

So far, these equations admit no losses: the conservation law cannot be violated: energy cannot be dissipated. To introduce, phenomenologically, an expression of dissipation, we need so-called *constitutive relations* which represent average 'frictional forces' in the system. These relations provide a linear relationship between gradients of the field and temperature and the rate of entropy generation, or energy stirring. The following well known forms are used in elementary thermodynamics to define the thermal conductivity κ in terms of the heat flux Q_i and the temperature gradient; similarly the viscosity η in terms of the pressure P:

$$Q_i = -\kappa \frac{\mathrm{d}T}{\mathrm{d}x^i} \\ P_{ij} = -\eta \frac{\partial U_i}{\partial x^j}. \quad (11.164)$$

The relations we choose to implement these must make the rate of entropy generation non-negative if they are to make thermodynamical sense. It may be checked that the following definitions fulfil this requirement in n spatial dimensions:

$$T^{0i} = -\kappa\left(\partial_i T + T\partial_t U_i/c^2\right) \\ T^{ij} = -\eta\left(\partial_i U_j + \partial_j U_i - \frac{2}{n}(\partial_k U^k)\delta_{ij}\right) - \zeta(\partial_k U^k)\delta_{ij}, \quad (11.165)$$

where κ is the thermal conductivity, η is the shear viscosity and ζ is the bulk viscosity. The first term in this last equation may be compared with eqn. (9.217). This makes use of the definition of *shear* σ_{ij} for a vector field V_i as a conformal deformation

$$\Delta_{ij} = \partial_i V_j + \partial_j V_i - \frac{2}{n}(\partial_k V^k)\delta_{ij}. \quad (11.166)$$

This is a measure of the non-invariance of the system to conformal, or shearing transformations. Substituting these constitutive equations into eqn. (11.163), one obtains

$$
\begin{aligned}
c\partial_\mu \mathcal{S}^\mu &= \frac{\kappa}{T^2}(\partial_i T + T\partial_0 U_i/c)(\partial^i T + T\partial_0 U^i/c) \\
&= \frac{\eta}{2T}\left(\partial_i U_j + \partial_j U_i\right)\left(\partial^i U^j + \partial^j U^i\right) \\
&= \left(\zeta + \frac{4}{n}\zeta\right)\frac{1}{T}(\partial_k U^k)^2.
\end{aligned}
\tag{11.167}
$$

11.8.4 Thermodynamical energy conservation

The thermodynamical energy equations supplement the conservation laws for mechanical energy, but they are of a different character. These energy equations are average properties for bulk materials. They summarize collective microscopic conservation on a macroscopic scale.

$$\partial_\mu T^{\mu\nu} = H + T\mathrm{d}S + P\mathrm{d}V + \mathrm{d}F \tag{11.168}$$

$$S = k\ln\Omega \tag{11.169}$$

$$T\mathrm{d}S = kT\frac{\mathrm{d}\Omega}{\Omega} = \frac{1}{\beta}\frac{\mathrm{d}\Omega}{\Omega}. \tag{11.170}$$

11.8.5 Kubo formulae for transport coefficients

In section 6.1.6, a general scheme for computing transport coefficients was presented, but only the conductivity tensor was given as an example. Armed with a knowledge of the energy–momentum tensor, entropy and the dissipative processes leading to viscosity, we are now in a position to catalogue the most important expressions for these transport coefficients. The construction of the coefficients is based on the general scheme outlined in section 6.1.6. In order to compute these coefficients, we make use of the assumption of linear dissipation, which means that we consider only first-order gradients of thermodynamic averages. This assumes a slow rate of dissipation, or a linear relation of the form

$$\langle \text{variable} \rangle = k\nabla_\mu \langle \text{source} \rangle, \tag{11.171}$$

where ∇_μ represents some spacetime gradient. This is the so-called constitutive relation. The expectation values of the variables may be derived from the generating functional W in eqn. (6.7) by adding source terms, or variables conjugate to the ones we wish to find correlations between. The precise meaning of the sources is not important in the linear theory we are using, since the source

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Table 11.4. Conductivity tensor.

Component	Response	Measure
σ_{00}/c^2	induced density	charge compressibility
σ_{0i}/c	density current	–
σ_{ii}	induced current	linear conductivity
σ_{ij}	induced current	transverse (Hall) conductivity

cancels out of the transport formulae completely (see eqn. (6.66)). Also, there is a symmetry between the variables and their conjugates. If we add source terms

$$S \to S + \int (\mathrm{d}x)(J \cdot A + J^{\mu} A_{\mu} + J^{\mu\nu} A_{\mu\nu}), \tag{11.172}$$

then the J's are sources for the A's, but conversely the A's are also sources for the J's.

We begin therefore by looking at the constitutive relations for the transport coefficients, in turn. The generalization of the conductivity derived in eqn. (6.75) for the spacetime current is

$$J_{\mu} = \sigma^{\mu\nu} \partial_t A^{\nu}. \tag{11.173}$$

Although derivable directly from Ohm's law, this expresses a general dissipative relationship between any current J^{μ} and source A^{μ}, so we would expect this construction to work equally well for any kind of current, be it charged or not. From eqn. (11.171) and eqn. (6.66) we have the Fourier space expression for the spacetime conductivity tensor in terms of the Feynman correlation functions

$$\sigma_{\mu\nu}(\omega) = \lim_{\mathbf{k}\to 0} \frac{\mathrm{i}}{\hbar\omega} \int (\mathrm{d}x) \mathrm{e}^{-\mathrm{i}k(x-x')} \langle J_{\mu}(x) J_{\nu}(x') \rangle, \tag{11.174}$$

or in terms of the retarded functions. In general the products with Feynman boundary conditions are often easier to calculate, since there are theorems for their factorization.

$$\sigma_{\mu\nu}(\omega)\Big|_{\beta} \equiv \lim_{\mathbf{k}\to 0} \frac{(1 - \mathrm{e}^{-\hbar\beta\omega})}{\hbar\omega} \int (\mathrm{d}x) \mathrm{e}^{-\mathrm{i}k(x-x')} \langle J_{\mu}(x) J_{\nu}(x') \rangle. \tag{11.175}$$

The D.C. conductivity is given by the $\omega \to 0$ limit of this expression. The components of this tensor are shown in table 11.4: The constitutive relations for the viscosities are given in eqn. (11.165). From eqn. (6.67) we have

$$\langle T_{\mu\nu}(x) \rangle = \frac{\delta W}{\delta J^{\mu\nu}(x)} \tag{11.176}$$

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and

$$\frac{\delta\langle T_{\mu\nu}(x)\rangle}{\delta J^{\rho\sigma}(x')} = \frac{\mathrm{i}}{\hbar}\langle T_{\mu\nu}(x)T_{\rho\sigma}(x')\rangle$$
$$= \frac{\mathrm{i}}{\hbar}\langle T_{\mu\rho}(x)T_{\nu\sigma}(x')\rangle, \tag{11.177}$$

where the last line is a consequence of the connectivity of Feynman averaging. Note that this relation does not depend on our ability to express $W[J^{\mu\nu}]$ in a quadratic form analogous to eqn. (6.35). The product on the right hand side can be evaluated by expressing $T_{\mu\nu}$ in terms of the field. The symmetry of the energy–momentum tensor implies that

$$J^{\mu\nu} = J^{\nu\mu}, \tag{11.178}$$

and, if the source coupling is to have dimensions of action, $J_{\mu\nu}$ must be dimensionless. The only object one can construct is therefore

$$J^{\mu\nu} = g^{\mu\nu}. \tag{11.179}$$

Thus, the source term is the trace of the energy–momentum tensor, which vanishes when the action is conformally invariant. To express eqn. (11.165) in momentum space, we note that Fourier transform of the velocity is the phase velocity of the waves,

$$U^i(x) = \gamma\frac{\mathrm{d}x^i}{\mathrm{d}t} = \gamma\int\frac{\mathrm{d}^n k}{(2\pi)^n}\mathrm{e}^{\mathrm{i}k^\mu x_\mu}\frac{\omega}{k_i}$$
$$= \gamma\int\frac{\mathrm{d}^n k}{(2\pi)^n}\mathrm{e}^{\mathrm{i}k^\mu x_\mu}\frac{\omega k^i}{\mathbf{k}^2}. \tag{11.180}$$

The derivative is given by

$$\partial_j U^i = \mathrm{i}\gamma\int\frac{\mathrm{d}^n k}{(2\pi)^n}\mathrm{e}^{\mathrm{i}k^\mu x_\mu}\frac{\omega k^i k_j}{\mathbf{k}^2}. \tag{11.181}$$

Thus, eqn. (11.165) becomes

$$\langle T_{ij}\rangle = -\left(\zeta + \frac{4}{n}\eta\right)\mathrm{i}\gamma\omega g_{ij} - \eta\,\mathrm{i}\gamma\omega\frac{k_i k_j}{\mathbf{k}^2}. \tag{11.182}$$

Comparing this with eqn. (11.176), we have, for the spatial components,

$$-\left(\zeta + \frac{4}{n}\eta\right)g_{ij}g_{lm} - \eta\frac{k_i k_j}{\mathbf{k}^2}g_{lm} =$$
$$\frac{1}{\hbar\omega}\int(\mathrm{d}x)\ \mathrm{e}^{-\mathrm{i}k(x-x')}\langle T_{il}(x)T^i_{\ m}(x')\rangle. \tag{11.183}$$

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Contracting both sides with $g^{ij} g^{lm}$ leaves

$$\left(\zeta(\omega) + \frac{4-n}{n}\,\eta(\omega)\right) = \lim_{\mathbf{k}\to 0} -\frac{1}{n^2\hbar\omega}\int (\mathrm{d}x)\mathrm{e}^{-\mathrm{i}k(x-x')}\langle T_{ij}(x)T^{ij}(x')\rangle. \tag{11.184}$$

The two viscosities cannot be separated in this relation, but η can be related to the diffusion coefficient, which can be calculated separately. Assuming causal (retarded relation between field and source), at finite temperature we may use eqn. (6.74) to write

$$\left(\zeta(\omega) + \frac{4-n}{n}\,\eta(\omega)\right)\Big|_{\beta} \equiv \lim_{\mathbf{k}\to 0} -\frac{(1-\mathrm{e}^{-\hbar\omega\beta})}{n^2\hbar\omega}\int (\mathrm{d}x)\mathrm{e}^{-\mathrm{i}k(x-x')}\langle T_{ij}(x)T^{ij}(x')\rangle. \tag{11.185}$$

The temperature conduction coefficient κ is obtained from eqn. (11.165). Following the same procedure as before, we obtain

$$\begin{aligned}\frac{\mathrm{i}}{\hbar}\int (\mathrm{d}x)\mathrm{e}^{-\mathrm{i}k^{\mu}(x-x')_{\mu}}&\langle T^{0i}(x)T^{0j}(x')\\ &= -g^{0j}\kappa(\partial^i T + T\partial_t U^i/c^2)\rangle\\ &= -\mathrm{i}g^{0j}\kappa(k^i T - T\gamma\omega^2 k^i/\mathbf{k}^2).\end{aligned} \tag{11.186}$$

Rearranging, we get

$$\kappa(\omega) = \lim_{\mathbf{k}\to 0} -\frac{g_{0j}k_i(1-\mathrm{e}^{-\hbar\omega\beta})}{\hbar\left(\mathbf{k}^2 - \gamma\omega^2/c^2\right)}\int (\mathrm{d}x)\mathrm{e}^{-\mathrm{i}k^{\mu}(x-x')_{\mu}}\langle T^{0i}(x)T^{0j}(x'). \tag{11.187}$$

To summarize, we note a list of properties with their relevant fluctuations and conjugate sources. See table 11.5.

11.9 Example: Radiation pressure

The fact that the radiation field carries momentum means that light striking a material surface will exert a pressure equal to the change in momentum of the light there. For a perfectly absorbative surface, the pressure will simply be equal to the momentum striking the surface. At a perfectly reflective (elastic) surface, the change in momentum is twice the momentum of the incident radiation in that the light undergoes a complete change of direction. Standard expressions for the radiation pressure are for reflective surfaces.

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Table 11.5. Fluctuation generators.

Property	Fluctuation	Source
Electromagnetic radiation	A_μ	J^μ
Electric current	J_i	A^i
Compressibility	N_0	A^0
Temperature current	T	T^{0i} (heat Q)

The pressure (kinetic energy density) in a relativistic field is thus

$$P_i = -2T_{0i} = p_i c/\sigma \tag{11.188}$$

with the factor of two coming from a total reversal in momentum, and σ being the volume of the uniform system outside the surface. Using the arguments of kinetic theory, where the kinetic energy density of a gas with average velocity $\langle v \rangle$ is isotropic in all directions,

$$\frac{1}{2}m\langle v^2 \rangle = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) \sim \frac{3}{2}mv_x^2, \tag{11.189}$$

we write

$$P_i \sim \frac{1}{3}\langle P \rangle. \tag{11.190}$$

Thus, the pressure of diffuse radiation on a planar reflective surface is

$$P_i = -\frac{2}{3}T_{0i}. \tag{11.191}$$

Using eqn. (7.88), we may evaluate this, giving:

$$P_i = -\frac{2}{3}\frac{(\mathbf{E} \times \mathbf{H})_i}{c} = \frac{2}{3}\epsilon_0 E^2. \tag{11.192}$$

Exercises

Although this is not primarily a study book, it is helpful to phrase a few outstanding points as problems, to be demonstrated by the reader.

(1) In the action in eqn. (11.61), add a kinetic term for the potential $V(x)$

$$\Delta S = \int (\mathrm{d}x)\ \frac{1}{2}(\partial^\mu V)(\partial_\mu V). \tag{11.193}$$

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Vary the total action with respect to $\phi(x)$ to obtain the equation of motion for the field. Then vary the action with respect to $V(x)$ and show that this leads to the equations

$$-\Box\,\phi + (m^2 + V)\phi = 0$$
$$-\Box\,V + \frac{1}{2}\phi^2 = 0.$$

Next show that the addition of this extra field leads to an extra term in the energy–momentum tensor, so that

$$\theta_{\mu\nu} = \frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi) + \frac{1}{2}(\partial_\mu V)(\partial_\nu V) - \frac{1}{2}(m^2 + V)\phi^2. \quad (11.194)$$

Using the two equations of motion derived above, show that

$$\partial^\mu\theta_{\mu\nu} = 0 \quad (11.195)$$

so that energy conservation is now restored. This problem demonstrates that energy conservation can always be restored if one considers all of the dynamical pieces in a physical system. It also serves as a reminder that fixed potentials such as $V(x)$ are only a convenient approximation to real physics.

(2) Using the explicit form of a Lorentz boost transformation, show that a fluid velocity field has an energy–momentum tensor of the form,

$$T^{\mu\nu} = Pg^{\mu\nu} + (P + \mathcal{H})U^\mu U^\nu/c^2. \quad (11.196)$$

Start with the following expressions for a spherically symmetrical fluid at rest:

$$\begin{aligned} T_{00} &= \mathcal{H} \\ T_{0i} &= T_{i0} = 0 \\ T_{ij} &= P\delta_{ij}. \end{aligned} \quad (11.197)$$

(3) Consider a matter current $N^\mu = (N, N\mathbf{v}) = N(x)U^\mu(x)$. Show that the conservation equation $\partial_\mu N^\mu = 0$ may be written

$$\partial_\mu N^\mu = [\partial_t + \pounds_v]\,, \quad (11.198)$$

where $\pounds_D = U^i\partial_i + (\partial_i U^i)$. This is called the Lie derivative. Compare this with the derivatives found in section 10.3 and the discussion found in section 9.6. See also ref. [111] for more details of this interpretation.

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(4) By writing the orbital angular momentum operator in the form $L_i = \epsilon_{ijk}x^j p^k$ and the quantum mechanical commutation relations $[x^i, p^j] = \mathrm{i}\hbar\delta^{ij}$ in the form $\epsilon_{ijk}x^j p^k = \mathrm{i}\hbar$, show that

$$L_\mathrm{i}\epsilon_{ilm} = [x_l, p_m] = \mathrm{i}\hbar\, \delta_{lm}, \tag{11.199}$$

and thence

$$\epsilon_{ilm}L_i L_l = \mathrm{i}\hbar L_m. \tag{11.200}$$

Hence show that the angular momentum components satisfy the algebra relation

$$[L_i, L_j] = \mathrm{i}\hbar\, \epsilon_{ijk}\, L_k. \tag{11.201}$$

Show that this is the Lie algebra for $so(3)$ and determine the dimensionless generators T^a and structure constants f_{abc} in terms of L_i and $\hbar$.

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