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On Gâteaux Differentiability of Convex Functions in WCG Spaces

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Abstract. It is shown, using the Borwein–Preiss variational principle that for every continuous convex function f on a weakly compactly generated space X, every $x_0 \in X$ and every weakly compact convex symmetric set K such that $\overline{\operatorname{span}}\, K = X$, there is a point of Gâteaux differentiability of f in $x_0 + K$. This extends a Klee's result for separable spaces.

The well-known Mazur's theorem says that a continuous convex function f on a separable Banach space X is Gâteaux differentiable on a dense G_δ set, [4, Theorem 8.14]. A function f on X is said to be *Gâteaux differentiable* at $x \in X$ if there is $F \in X^*$ such that

$$\lim_{t \to 0} \frac{f(x + th) - f(x)}{t} = F(h),$$

for all $h \in X$. A Banach space is called a *weak Asplund space* if every continuous convex function f on it is Gâteaux differentiable at the points of a dense G_δ set. It is known that weakly compactly generated spaces are weak Asplund spaces, [3, Theorem 1.3.4]. Recall that a Banach space X is called *weakly compactly generated* (WCG) if there is a weakly compact set $K \subset X$ such that $\overline{\operatorname{span}}\, K = X$.

It is proved in [5] that, for a separable Banach space X, the set of points of Gâteaux differentiability of a convex continuous function f is even bigger than dense in the following sense. If $K \subset X$ is a norm compact convex symmetric set such that $\overline{\operatorname{span}}\, K = X$ and $x_0 \in X$, then there is $x \in x_0 + K$, a point of Gâteaux differentiability of f. A set $C \subset X$ is called *symmetric* if $-C = C$. We will extend the above result to weakly compact set in WCG spaces.

Theorem 1 *Let X be a WCG space and K be a weakly compact convex symmetric set such that $\overline{\operatorname{span}}\, K = X$. Let f be a continuous convex function on X and $x_0 \in X$. Then there is $x \in x_0 + K$ such that f is Gâteaux differentiable at x.*

Let us define terms used in the proof. For a closed convex symmetric set C let μ_C denote a *Minkowski functional* of C defined by

$$\mu_C(x) = \inf\{\lambda > 0\,;\, x \in \lambda C\}.$$

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It is known that $\mu_C\colon X \to \mathbb{R} \cup \{\infty\}$ is a convex lower semicontinuous function. A function $f\colon X \to \mathbb{R} \cup \{\infty\}$ is said to be *lower semicontinuous* if its level sets $\{x \in X\,;\, f(x) \le r\}$ are closed for every $r \in \mathbb{R}$. This is equivalent to saying that the *epigraph* of f,

$$\operatorname{epi}(f) = \{(x, r) \in X \times \mathbb{R}\,;\, f(x) \le r\},$$

is closed in $X \times \mathbb{R}$. Thus the epigraph of a convex lower semicontinuous function is a closed convex set. The *subdifferential*, $\partial f(x)$, of f at $x \in X$ is the set of all $\varphi \in X^*$ such that

$$\varphi(y - x) \le f(y) - f(x),$$

for all $y \in X$. A functional $\varphi \in X^*$ is called a *supporting functional* for a set K at a point $k_0 \in K$ if

$$\varphi(k_0) = \sup\{\varphi(k); k \in K\}.$$

A function $f\colon X \to \mathbb{R}$ is called a *Gâteaux smooth bump* if it is a Gâteaux differentiable function with a bounded support. A system $\{x_\gamma, x_\gamma^*\}_{\gamma\in\Gamma} \subset X \times X^*$ is called a *Markushevich basis* for X if $x_\beta^*(x_\gamma) = \delta_{\beta\gamma}$ (the Kronecker delta) for all $\beta, \gamma \in \Gamma$, $\overline{\operatorname{span}}\{x_\gamma; \gamma \in \Gamma\} = X$, and if for every $0 \ne x \in X$ there is $\gamma \in \Gamma$ such that $x_\gamma^*(x) \ne 0$. A norm $\|\cdot\|$ on X is called *strictly convex*, if $x = y$ whenever

$$2\|x\| = 2\|y\| = \|x + y\|.$$

Proof of Theorem 1 The proof will be divided into three steps. First we will show that there is a "smooth" weakly compact set $L \subset K$.

Lemma 2 *There is a weakly compact convex symmetric set $L \subset 2^{-1}K$ such that if $\varphi, \psi \in X^*$ are supporting functionals of L at a point $l \in L$ such that $\varphi(l) = \psi(l)$, then $\varphi = \psi$.*

Second, we will use a variational principle to touch the graph of f by a "smooth" function. We may assume that $f(x_0) = -1$. By the continuity of f, we may assume that $|f(x) - f(x_0)| < 1$, for $\|x - x_0\| \le 1$. Let g be a function on X defined by

$$g(x) = \begin{cases} -f(x) & \text{for } \|x - x_0\| \le 1, \\ \infty & \text{for } \|x - x_0\| > 1. \end{cases}$$

Then g is lower semicontinuous and $g > 0$. Set $u_L(x) = \mu_L(x - x_0)$.

Lemma 3 *There is a Gâteaux smooth function $v\colon X \to \mathbb{R}$ and a point $x \in X$ such that $x \in x_0 + 2L \subset x_0 + K$, $0 < \|x - x_0\| < 1$ and $g + u_L - v$ attains its minimum at x.*

Finally, we will show that f is Gâteaux differentiable at x.

Lemma 4 *Let V denote a Gâteaux derivative of v at x. Then there is $\alpha \in \mathbb{R} \setminus \{0\}$ such that $\varphi + V$ is a supporting functional for $x_0 + \alpha L$, for all $\varphi \in \partial f(x)$. Consequently, f is Gâteaux differentiable at x.*

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Proof of Lemma 2 Let $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma} \subset K \times X^*$ be a Markushevich basis of X, see [4, Theorem 11.12]. Then there is a one-to-one operator $T\colon X^* \to c_0(\Gamma)$ defined by

$$T(x^*) = (x^*(x_\gamma))_{\gamma\in\Gamma}.$$

Let $\{e_\gamma\}_{\gamma\in\Gamma}$ denote the standard unit vector basis of $\ell_1(\Gamma)$. The dual operator $T^*\colon \ell_1(\Gamma) \to X^{**}$ satisfies

$$T^*(e_\gamma)(x^*) = e_\gamma(Tx^*) = x^*(x_\gamma),$$

for all $\gamma \in \Gamma$. Thus $T^*(e_\gamma) = x_\gamma$ and $T^*(B_{\ell_1(\Gamma)}) \subset K$. Moreover T^* is a weak*-weak continuous operator from $c_0(\Gamma)^*$ to X.

Let a norm $\|\cdot\|$ on $c_0(\Gamma)$ be (a strictly convex) Day's norm (see [2, Theorem II.7.3]) and let $B \subset \ell_1(\Gamma)$ be its dual unit ball. Put $L = T^*(B)$. We may assume that $\|\cdot\|$ is small enough to have $2L \subset K$. Clearly L is a symmetric convex set. As T^* is weak*-weak continuous, L is weakly compact. Now assume that $\varphi, \psi \in X^*$ are supporting functionals of L at $l \in L$ such that $\varphi(l) = \psi(l)$. We claim that $\varphi = \psi$. Pick $b_0 \in B$ such that $T^*(b_0) = l$ and put $x = T(\varphi)$ and $y = T(\psi)$. Then for all $b \in B$

$$b(x) = b(T(\varphi)) = \varphi(T^*(b)) \le \varphi(l) = b_0(x).$$

Thus $x, y \in c_0(\Gamma)$ are supporting functionals of B at b_0. Moreover

$$\|x\| = \sup\{b(x)\ ;\ b \in B\} = b_0(x) = b_0(y) = \|y\|,$$

and

$$2\|x\| = \|x\| + \|y\| = b_0(x + y) \le \|x + y\| \le \|x\| + \|y\|.$$

Thus $x = y$, as the norm $\|\cdot\|$ is strictly convex. Hence, as T is one-to-one, $\varphi = \psi$. ∎

Proof of Lemma 3 We will use the Deville–Godefroy–Zizler version of the Borwein–Preiss smooth variational principle, see [1] and [2, Theorem 2.3].

Theorem 5 *Let X be a Banach space that admits a Lipschitzian bump function which is Gâteaux differentiable. Then for every lower semicontinuous bounded bellow function F on X and every $\varepsilon > 0$, there exist $x \in X$ and a function $G\colon X \to \mathbb{R}$, which is Lipschitzian and Gâteaux differentiable on X and such that $\|G\| = \sup\{|G(x)|\ ;\ x \in X\} < \varepsilon$, $\|G'\| < \varepsilon$ and $F + G$ attains its minimum on X.*

We can use it, as X admits a Gâteaux smooth norm [4, Theorem 11.20] and thus it admits a Lipschitzian Gâteaux smooth bump. Let us fix $\varepsilon \in (0, 1/4)$. To assure that a point x we get by the variational principle is different from x_0, we will first modify the function $g + u_L$. Let $x_1 \in X$ be such that

$$(g + u_L)(x_1) < (g + u_L)(x_0) + \varepsilon/4.$$

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Let $v_1\colon X \to \mathbb{R}$ be a continuous Gâteaux smooth bump function such that $\|v_1\| < \varepsilon/2$ and

$$(g + u_L - v_1)(x_1) < (g + u_L - v_1)(x_0) - \varepsilon/4.$$

By applying the variational principle with $\varepsilon' = \varepsilon/8$ on $g+u_L-v_1$, we get a Gâteaux smooth function v_2, $\|v_2\| < \varepsilon/8$ and a point $x \in X$, such that $g+u_L-(v_1+v_2)$ attains its minimum at x. Thus

$$\begin{aligned}(g + u_L - v_1 - v_2)(x) &\le (g + u_l - v_1)(x_1) - v_2(x_1)\\ &< (g + u_L - v_1 - v_2)(x_0) < \infty.\end{aligned}$$

It means that $x \neq x_0$, $g(x) < \infty$, and thus $0 < \|x - x_0\| < 1$. Put $v = v_1 + v_2$. Then $\|v\| < \varepsilon$ and thus $g(x) - v(x) > -\varepsilon$. We claim that $u_L(x) < 1 + 3\varepsilon < 2$. Really, if we assume a contrary, then

$$1 + 2\varepsilon \le u_L(x) - \varepsilon < (g + u_L - v)(x) \le (g + u_L - v)(x_0) \le 1 + \varepsilon,$$

a contradiction. Thus $x \in x_0 + 2L \subset x_0 + K$. ∎

Proof of Lemma 4 As f is a continuous convex function, $\partial f(x) \neq \varnothing$ and we only need to show that there is only one $\varphi \in \partial f(x)$, see [6]. For the rest of the proof we will assume, without loss of generality, that $g + u_L - v = g - (v - u_L)$ attains its minimum at $x = 0$, $g(0) = 0$ and $g(0) - (v - u_L)(0) = 0$. In particular, $0 < \|x_0\| < 1$ and $u_L(0) = v(0)$.

Pick any $\varphi \in \partial f(0)$. Let $\delta > 0$ be small enough to have $g(ty) = -f(ty) < \infty$ for $y \in S_X$, $|t| < \delta$. Then

$$-\varphi(ty) \ge -f(ty) = g(ty) \ge (v - u_L)(ty).$$

Let V be a Gâteaux derivative of v at 0. Then

$$v(ty) = v(0) + V(ty) + o_y(t),\ t \to 0,$$

for all $y \in S_X$, $|t| < \delta$, where $o_y(t)$ is a function (depending on y), such that $o_y(t)/t \to 0$, as $t \to 0$. Thus

$$(\varphi + V)(ty) + o_y(t) \le u_L(ty) - v(0), t \to 0. \tag{1}$$

From that it follows that

$$(\varphi + V)(ty) \le u_L(ty) - v(0) = u_L(ty) - u_L(0), \tag{2}$$

for all $y \in S_X$ and all $t \in \mathbb{R}$. Indeed, if (2) does not hold, then there is $y_0 \in S_X$, $0 \neq t_0 \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that

$$(\varphi + V)(t_0 y_0) - \varepsilon_0 > u_L(t_0 y_0) - v(0).$$

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By convexity of u_L, we may assume that $0 < |t_0| < \delta$. Because

$$(\varphi + V)(0) = 0 = (u_L - v)(0), \tag{3}$$

one has that for all $t \in (0, |t_0|]$

$$u_L(ty_0) - v(0) \leq t\,\frac{u_L(t_0y_0) - v(0)}{t_0} < t\,\frac{(\varphi + V)(t_0y_0) - \varepsilon_0}{t_0},$$

a contradiction with (1).

Notice that (2) says that $(\varphi + V) \in \partial u_L(0)$, and thus $(\varphi + V)(x_0) = u_L(0)$, as u_L is linear on lines going from x_0.

Thus, by (2) and (3), $(\varphi + V)$ is a support functional of $x_0 + v(0)L$ at the point $x = 0$. Indeed, by an assumption $u_L(0) = v(0)$, and thus $0 \in x_0 + v(0)L$. Moreover $(\varphi + V)(0) = 0$, and by (2),

$$(\varphi + V)(z) \leq u_L(z) - v(0) \leq 0,$$

for all $z \in x_0 + v(0)L$. Equivalently, $(\varphi + V)$ is a support functional of $v(0)L$ at $-x_0$ with $(\varphi + V)(-x_0) = -u_L(0)$. Because $x_0 \neq 0$, $v(0) = u_L(0) \neq 0$, by Lemma 2, there is only one support functional ψ of $v(0)L$ at $-x_0$ with $\psi(0) = -u_L(0)$. Thus there is only one $\varphi \in \partial f(0)$. This concludes the proof of Lemma 4 and the proof of Theorem 1. ∎

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