ON HAMILTONIAN AVERAGING THEORIES AND RESONANCE

S. FERRAZ-MELLO
Instituto Astronômico e Geofísico, Universidade de São Paulo
São Paulo, Brasil

Abstract. In this article, we review the construction of Hamiltonian perturbation theories with emphasis on Hori's theory and its extension to the case of dynamical systems with several degrees of freedom and one resonant critical angle. The essential modification is the comparison of the series terms according to the degree of homogeneity in both $\sqrt{\varepsilon}$ and a parameter which measures the distance from the exact resonance, instead of just $\sqrt{\varepsilon}$.

1. Introduction. Formal Averaging Theories

The construction of theories aiming at a precise description of the celestial motions is an old problem in astronomy. The complexity of the equations of the N-planets problem ($N \geq 2$) has been overcome by astronomers through very elaborate techniques. Most of these theories were founded on the algebraic skills of some astronomers and classical treatises on perturbations theory assemble a large number of algebraic tricks used to reach the proposed targets. We could quote Sampson's theory of the four Galilean satellites of Jupiter as a milestone in this direction with its mobile frames and "completed time". These theories were generally good enough for the construction of ephemerides of a rather good quality for the needs of that time. However, almost all attempts of extending them by increasing the order of approximation were frustrated by insurmountable difficulties, notwithstanding the possibilities of using computer algebra in order to expedite the calculations and to avoid errors in the algebraic developments.

Among the techniques devised in the past, Delaunay's theory of the motion of the Moon is an exception. Instead of looking for tricks to solve every difficulty, Delaunay proposed a well organized iterative procedure which is a paradigm of what is done nowadays (see Brouwer and Clemence,

Celestial Mechanics and Dynamical Astronomy **66**: 39–50, 1997.

1961, chap. XVII). He first noted that the equations of variation of the elements used by Lagrange and Laplace could be written in a very simple form by using a special set of variables, usually called Delaunay's variables:

$$\begin{array}{ll} \ell\text{=mean anomaly} & L=\sqrt{\mu a} \\ g\text{=argument of the pericenter} & G=L\sqrt{1-e^2} \\ h\text{=longitude of the node} & H=G\cos i \end{array} \tag{1}$$

where μ is the product of the gravitational constant and the mass of the central body, a the semi-major axis, e the orbital eccentricity and i the inclination of the orbit over the reference plane.

With these variables, the equations of variation of the elements are the Delaunay equations

$$\begin{array}{ll} \dfrac{d\ell}{dt}=\dfrac{\partial F}{\partial L} & \dfrac{dL}{dt}=-\dfrac{\partial F}{\partial \ell} \\ \dfrac{dg}{dt}=\dfrac{\partial F}{\partial G} & \dfrac{dG}{dt}=-\dfrac{\partial F}{\partial g} \\ \dfrac{dh}{dt}=\dfrac{\partial F}{\partial H} & \dfrac{dH}{dt}=-\dfrac{\partial F}{\partial h} \end{array} \tag{2}$$

where

$$F=-\frac{\mu^2}{2L^2}+\varepsilon R(L,G,H,\ell,g,h). \tag{3}$$

R is the potential of the disturbing forces expressed in terms of the Delaunay variables, written here as a time independent function only for simplicity, and ε is a small parameter of the order of the relative value of the disturbing masses. The variational equations are in canonical form. Delaunay then introduced his *'operation'* and performed it successively many hundreds of times. Delaunay's operation starts with the choice of one trigonometric term in the Fourier expansion of R, say

$$W_1=A_1(L,G,H)\cos(k_1'\ell+k_1''g+k_1'''h)$$

and the consideration of the dynamical system defined by the abridged Hamiltonian

$$F^{(1)}=-\frac{\mu^2}{2L^2}+\varepsilon W_1.$$

This system is integrable and the main step of Delaunay's operation is to obtain one solution of this abridged system and to use it to derive one canonical transformation leading to the elimination of the term W_1 from the given system [in fact, the transformation achieves the substitution of this term by another one with a coefficient of the order of $\mathcal{O}(\varepsilon^2)$]. After eliminating W_1, one may choose another term, W_2, and repeat the operation. This operation is repeated as many times as possible.

The techniques of Delaunay were improved by Poincaré (1892) who, instead of making the successive elimination of the trigonometric terms, used the Jacobian generating functions of canonical transformations and was able to eliminate all terms of a given kind (say, those with $k_1' \neq 0$) with one only canonical transformation. This is the form used by Brouwer (1959) to solve the problem of the motion of an artificial satellite around an oblate Earth and is generally known as the Von Zeipel or Poincaré-Von Zeipel, or even the Brouwer-Von Zeipel method, since Brouwer used some ideas of Von Zeipel (1916) to deal with secular terms.

2. Lie Mappings, Hori Theory

Hori (1966) considered, instead of Jacobian generating functions, canonical transformations defined by Lie mappings. The use of Lie series was already current amongst physicists and, before 1960, several authors (Sérsic, 1956; Gröbner, 1960) had already suggested to use them to represent the canonical transformations used in the perturbation methods of celestial mechanics. A modified approach to the question was later introduced by Deprit (1969), shown to be equivalent to Hori's formulation by several authors (e.g. Campbell and Jefferys, 1970).

In order to explain Hori's theory, let us start by considering one canonical system of equations:

$$\frac{d\ell_i}{dt} = \frac{\partial F}{\partial L_i} \qquad \frac{dL_i}{dt} = -\frac{\partial F}{\partial \ell_i} \qquad (i = 1, 2, \cdots, N) \tag{4}$$

where $F = F(L_i, \ell_i, \varepsilon)$ is a time-independent Hamiltonian, and consider the transformation $(L_i, \ell_i) \Rightarrow (L_i^*, \ell_i^*)$ defined by the generic equation

$$\begin{aligned}\phi(L_i, \ell_i) &= E_{T^*}\phi(L_i^*, \ell_i^*) = \\ &\phi + \{\phi, T^*\} + \tfrac{1}{2}\{\{\phi, T^*\}, T^*\} + \tfrac{1}{3!}\{\{\{\phi, T^*\}, T^*\}, T^*\} + \cdots\end{aligned} \tag{5}$$

where $T^* = T^*(L_i^*, \ell_i^*, \varepsilon)$. $\{.,.\}$ are Poisson brackets in the variables (L_i^*, ℓ_i^*). The method of Hori is founded on the set of equations obtained by identification of all terms of the same order with respect to ε in the equation of energy conservation:

$$F^*(L_i^*, \ell_i^*, \varepsilon) = E_{T^*}F(L_i^*, \ell_i^*, \varepsilon) \tag{6}$$

(F^* is the Hamiltonian resulting from the transformation). The resulting equations are partial differential equations in the coefficients of the power expansion of an unknown Lie generator:

$$T^*(L_i^*, \ell_i^*, \varepsilon) = \sum_k \varepsilon^k T_k^*(L_i^*, \ell_i^*).$$

The main points of the theory proposed by Hori are:

https://doi.org/10.1017/S0252921100046352 Published online by Cambridge University Press

- All partial differential equations may be written in the homological form
$$F_k^* = \Psi_k + \{F_0, T_k^*\} \tag{7}$$
where F_0 is the undisturbed part of F, F_k^* is the coefficient of ε^k in the expansion of F^* in the powers of ε and Ψ_k is a function which is known if the equations of the previous orders have been solved.
- The Cauchy's characteristics of the homological partial differential are curves whose parametric equations are solutions of the equations
$$\frac{dL_i^*}{du} = -\frac{\partial F_0}{\partial \ell_i^*} \qquad \frac{d\ell_i^*}{du} = \frac{\partial F_0}{\partial L_i^*} \tag{8}$$
$$\frac{dT_k^*}{du} = \Psi_k - F_k^*. \tag{9}$$
- The averaging rule
$$F_k^* = <\Psi_k>$$
expresses the intent of having an averaged F^* and avoids the occurrence of secular terms in T^*. The introduction of this arbitrary condition is allowed by the indeterminacy of eq. 9 where both T_k^* and F_k^* are unknown.
- L_i, ℓ_i are unspecified canonical variables (Hori's theory is not restricted to action-angle variables).

Eqs. 8 are known as Hori's auxiliary system or Hori's kernel and their remarkable property is that they are the same at all orders of approximation. The parameter u has the same physical dimension as t and was called "proper-time" in Hori's paper. Interpretations founded on this analogy led to conceptual difficulties which impaired, in the past, the correct understanding of Hori's theory. However, the meaning of u is immaterial, since this parameter disappears as can be seen from the sequence of calculations to be done: a) introduction of the general solution of eq. 8: $L_j^*, \ell_j^* = L_j^*, \ell_j^*(C_1 + u, C_2, \cdots, C_{2N})$ into Ψ_k; b) averaging; c) integration of eq. 9 in u; and d) transformation of the results back to the variables L_j^*, ℓ_j^* and elimination of $C_1 + u, C_2, \cdots, C_{2N}$ by means of the inverses of the general solutions of eq. 8 (see Carathéodory, 1965).

3. Degeneracies, Resonance

The classical difficulty in an averaging theory, let it be the theory of Delaunay, Poincaré, Hori or any other, is the degeneracy of the Hamiltonian system defined by F_0. A Hamiltonian is said to be degenerate when there exists one linear combination with integer coefficients of its proper frequencies
$$\nu_i = \frac{\partial F_0}{\partial L_i}$$

which is equal to zero. This is the case, for instance, when F_0 is the Hamiltonian of the Keplerian motion: $F_0 = -\mu^2/2L^2$, in which case $\frac{\partial F_0}{\partial G} = \frac{\partial F_0}{\partial H} = 0$ since F_0 depends only on the first Delaunay variable. Degeneracies of this kind are said to be essential because they do not depend on the initial conditions: all Keplerian motions have both a fixed plane and a fixed perihelion. Degeneracies may also be accidental and occur only for some particular values of the initial conditions. One example is the motion of an asteroid whose orbital period is commensurable with Jupiter's period. This degeneracy depends on the initial conditions and disappears if we move the asteroid to an orbit with a different period. The main consequence of an accidental degeneracy, or resonance, is the occurrence of small divisors during the integration of the equations of perturbation.

When degeneracies do not occur, the system may be averaged over all Delaunay angles, that is, we may find a canonical transformation such that the transformed Hamiltonian F^* is independent of all new angles ℓ_i^* and is, therefore, easily integrable. The transformation back to the original variables gives the formal solution of the problem at the order of the approximation of the calculated F^*. When degeneracies occur, the averaging may be done only over the non-degenerate angles, that is, we may only find a canonical transformation which makes the transformed Hamiltonian F^* independent of the non-degenerate angles and, as a consequence, the method leads only to a simplification of the given problem When the degeneracy is accidental, that is, in the so-called resonant systems, one more averaging is, however, possible.

Let us consider a resonant system where, after some suitable change of variables, the Hamiltonian is

$$F = F_0(L_1) + \sum_{k=1}^{\infty} \varepsilon^k R_{2k}(L_i, \ell_i) \qquad (i = 1, 2, \cdots, N), \tag{10}$$

and

$$\nu_1 = \frac{\partial F_0}{\partial L_1} = \mathcal{O}(\sqrt{\varepsilon}). \tag{11}$$

that is, one fundamental frequency is close to zero and all other undisturbed fundamental frequencies $\nu_\rho = \partial F_0/\partial L_\rho$ $(\rho \neq 1)$ are equal to zero.

The natural extension of the classical Jacobian averaging methods to the case where the angle to be eliminated is resonant is Bohlin's method. In Bohlin's method, as it was exposed by Poincaré (1892) and used by many authors, $\nu_1 = \partial F_0/\partial L_1$ is considered as a quantity of the order $\mathcal{O}(\sqrt{\varepsilon})$ and the functions giving the canonical transformation and the averaged Hamiltonian are also expanded in the powers of $\sqrt{\varepsilon}$. However, as first noted by Poincaré (*op.cit.*, p.365), the equations of perturbation of Bohlin's method, at the second approximation, are singular. This singularity cannot be removed if the given system has more than one degree of freedom and the equations of perturbation in Delaunay variables cannot be solved. Another difficulty of Bohlin's method, as first noted by Jupp (1970), is that the results obtained when the equations are solved up to a given order are not complete. Due to the fact that $\nu_1^* = \mathcal{O}(\sqrt{\varepsilon})$ while its derivative with respect to L_1^* is $\mathcal{O}(1)$, differentiation with respect to L_1^* reduces the order of every function having ν_1^* as a factor. Thus, if the generating function is determined up to an order $\mathcal{O}(\varepsilon^{k/2})$, the derivative of the next order term, with respect to L_1^*, will also contribute terms of the order $\mathcal{O}(\varepsilon^{k/2})$ and shall be included in the transformation in order to have it complete to the nominal order $\mathcal{O}(\varepsilon^{k/2})$. This is a less crucial difficulty and it is easily dealt by means of a modification in the definition of the orders of approximation in the theory, as shown below.

The method of Hori is well suited to deal with the first of these difficulties. It leads naturally to a non-Keplerian Hori's kernel whose solution incorporates the main resonant perturbations. It is interesting to remember that some successful attempts in dealing with the singularities of the equations, in the case of systems with one degree of freedom, also introduce a preliminary transformation incorporating the main resonant perturbations in the variables (see Jupp, 1992). Similar transformations are also implicitly done in the Fourier expansions introduced by Bohlin (1888) in his solution and in the method proposed by Sessin (1986) to solve the equations of the Delaunay's theory for resonant systems with several degrees of freedom.

Because of the mixture of orders, a mere generalization, to this case, of the equations of the non-resonant case is not possible. Indeed, in the homological eq. 7, the term Ψ_k is a function of $T_1^*, \cdots, T_{k-1}^*$ through chains of Poisson brackets of these functions with others, including F_0. Therefore, Ψ_k depends on multiple derivatives of F_0 with respect to the variables and, thus, will contain terms of different orders. These terms can be easily taken into account and moved to the equations corresponding to their orders. However, we cannot exclude that the resolution of the equations do not introduce, themselves, terms functionally equal to F_0 which, again, will change in order when differentiated with respect to L_1^*. In order to overcome this difficulty, we will substitute L_1 by a new action variable

$$x = L_1 - L_1^*, \tag{12}$$

where L_1^* is defined by

$$\nu_1^* = \left(\frac{\partial F_0}{\partial L_1}\right)_{L_1 = L_1^*} = 0$$

and assume $x = \mathcal{O}(\sqrt{\varepsilon})$. Then we will take into account that x and $\sqrt{\varepsilon}$ are both small parameters of the same magnitude and they will be considered

https://doi.org/10.1017/S0252921100046352 Published online by Cambridge University Press

on the same footing. Thus, while in the general theories all identifications are done following the powers of the small parameter ε, in this extension of the method of Hori, we will consider both $\sqrt{\varepsilon}$ and x and we will perform all identifications according to the degree of homogeneity in $\sqrt{\varepsilon}$ and x.

For instance, the Poisson bracket of two functions $\psi_1(x, L_\rho, \ell_i)$ and $\psi_2(x, L_\rho, \ell_i)$ $(\rho = 2, 3, \cdots, N; i = 1, 2, \cdots, N)$ homogeneous with respect to the quantities $\sqrt{\varepsilon}$ and x will be split in two parts:

$$\{\psi_1, \psi_2\} = \left(\frac{\partial\psi_1}{\partial\ell_1}\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial\ell_1}\frac{\partial\psi_1}{\partial x}\right) + \sum_{\rho=2}^{N}\left(\frac{\partial\psi_1}{\partial\ell_\rho}\frac{\partial\psi_2}{\partial L_\rho} - \frac{\partial\psi_1}{\partial\ell_\rho}\frac{\partial\psi_2}{\partial L_\rho}\right). \quad (13)$$

The second part of this Poisson bracket is an ordinary operation and the degree of homogeneity of the result is equal to the sum of the degrees of homogeneity of ψ_1 and ψ_2. However, in the first part of it, the operation $\partial/\partial x$ reduces by one the degree of homogeneity and the resulting degree of homogeneity of the first parenthesis with respect to $\sqrt{\varepsilon}$ and x is one unit less than that of the terms of the second bracket.

It is convenient to introduce the notations

$$\{\psi_1, \psi_2\}_1 = \frac{\partial\psi_1}{\partial\ell_1}\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial\ell_1}\frac{\partial\psi_1}{\partial x}, \quad (14)$$

$$\{\psi_1, \psi_2\}_\rho = \sum_{\rho=2}^{N}\left(\frac{\partial\psi_1}{\partial\ell_\rho}\frac{\partial\psi_2}{\partial L_\rho} - \frac{\partial\psi_2}{\partial\ell_\rho}\frac{\partial\psi_1}{\partial L_\rho}\right) \quad (15)$$

and write the Poisson bracket as

$$\{\psi_1, \psi_2\} = \{\psi_1, \psi_2\}_1 + \{\psi_1, \psi_2\}_\rho; \quad (16)$$

For the case of three functions we have

$$\{\{\psi_1, \psi_2\}, \psi_3\} = \{\{\psi_1, \psi_2\}_1, \psi_3\}_1 + \{\{\psi_1, \psi_2\}_1, \psi_3\}_\rho + \quad (17)$$
$$\{\{\psi_1, \psi_2\}_\rho, \psi_3\}_1 + \{\{\psi_1, \psi_2\}_\rho, \psi_3\}_\rho.$$

This notation is helpful in keeping trace of terms of different degrees of homogeneity in $\sqrt{\varepsilon}$ and x. For instance in eq. 17, the degree of homogeneity of the last term is the sum of the degrees of homogeneity of the three functions, the two brackets showing the subscripts 1 and ρ have a degree of homogeneity one unit less, and the bracket showing twice the subscript 1 has a degree of homogeneity two units less. This notation helps in avoiding the mixture terms of different degrees.

This new approach changes the form of the Lie-series expansion of one function. Let f be an homogeneous function of degree D in the variables

$\sqrt{\varepsilon}, x$ and let us consider the canonical transformation $\phi_p : (x, L_\rho, \ell_i) \Rightarrow (x^*, L_\rho^*, \ell_i^*)$ determined by the Lie generator

$$T^* = \sum_{k=2}^{p} T_k^*(x^*, L_\rho^*, \ell_i^*) \tag{18}$$

where the generic $T_k^*(x^*, L_\rho^*, \ell_i^*)$ are homogeneous functions of degree k in the variables $\sqrt{\varepsilon}, x^*$. The *Lie series expansion of* f may be written as in eq. 5, or, making the substitution of the Lie generator T^* by its expansion,

$$\begin{aligned} E_{T^*} f = \; & f + \\ & \{f, T_2^*\}_1 + \\ & \{f, T_3^*\}_1 + \{f, T_2^*\}_\rho + \tfrac{1}{2}\{\{f, T_2^*\}_1, T_2^*\}_1 + \\ & \{f, T_4^*\}_1 + \{f, T_3^*\}_\rho + \tfrac{1}{2}\{\{f, T_2^*\}_1, T_3^*\}_1 + \tfrac{1}{2}\{\{f, T_3^*\}_1, T_2^*\}_1 \\ & \qquad + \tfrac{1}{2}\{\{f, T_2^*\}_\rho, T_2^*\}_1 + \tfrac{1}{2}\{\{f, T_2^*\}_1, T_2^*\}_\rho \\ & \qquad + \tfrac{1}{6}\{\{\{f, T_2^*\}_1, T_2^*\}_1, T_2^*\}_1 + \\ & \{f, T_5^*\}_1 + \{f, T_4^*\}_\rho + \cdots \end{aligned} \tag{19}$$

where we have taken full account of the fact that the Poisson brackets carry terms of different orders and have grouped them so as to have terms of degree D in the first row, terms of degree $D+1$ in the second row, etc. The mere inspection of this result allows us to justify the assumption $T_1^* = 0$, which serves to avoid an unlimited number of terms at every order.

4. Resonant Averaging Theory

In order to construct perturbation equations in this problem, all functions shall be expanded in series of terms with the same degree of homogeneity with respect to $\sqrt{\varepsilon}$ and x. Initially, the functions F_0 and R_{2k} are expanded to become

$$F_0 = F_0(L_1^*) + X_2(x) + X_3(x) + \cdots \tag{20}$$

and

$$R_{2k}(L_i, \ell_i) = R_{2k}(L^*, L_\rho, \ell_i) + R_{2k}^{(1)}(x, L_\rho, \ell_i) + R_{2k}^{(2)}(x, L_\rho, \ell_i) + \cdots \tag{21}$$

where $X_{k'}$ and $R_{2k}^{(k')}$ are homogeneous functions of degree k' in x. The term $X_1(x)$ is absent from the series for F_0 since, again, by Taylor's theorem and the definition of L_1^*,

$$X_1(x) = \frac{dF_0(L_1^*)}{dL_1^*} x = \nu_1^* x = 0.$$

When these expansions are introduced into eq. 6, we obtain

$$F = F_0(L_1^*) + \sum_{k \geq 2} F_k(x, L_\rho, \ell_i, \varepsilon), \tag{22}$$

https://doi.org/10.1017/S0252921100046352 Published online by Cambridge University Press

where F_k is homogeneous of degree k with respect to x and $\sqrt{\varepsilon}$. In particular, we have $F_1 = 0$ and

$$\begin{array}{lll} F_2 & = X_2(x) & +\varepsilon R_2(L_1^*, L_\rho, \ell_i) \\ F_3 & = X_3(x) & +\varepsilon R_2^{(1)}(x, L_\rho, \ell_i) \\ F_4 & = X_4(x) & +\varepsilon R_2^{(2)}(x, L_\rho, \ell_i) + \varepsilon^2 R_4(L_1^*, L_\rho, \ell_i) \\ F_5 & = X_5(x) & +\varepsilon R_2^{(3)}(x, L_\rho, \ell_i) + \varepsilon^2 R_4^{(1)}(x, L_\rho, \ell_i) \\ \cdots & & \cdots \end{array} \tag{23}$$

We then consider the conservation equation

$$F^*(x^*, L_\rho^*, \ell_i^*) = E_{T^*} F(x^*, L_\rho^*, \ell_i^*) \tag{24}$$

and introduce the Lie generator of the canonical transformation ϕ_p, given by eq. 18, and the transformed Hamiltonian F^* in the expanded form

$$F^* = \sum_{k=1}^{p} F_k^*. \tag{25}$$

where, again, the subscripts indicate the degree of homogeneity in $\sqrt{\varepsilon}$, x. The comparison of the terms of the same degree of homogeneity in $\sqrt{\varepsilon}$, x yields a system of equations similar to the equations of Hori's general theory, but different from them because:

- $F_0(L^*)$ is a constant; therefore all Poisson brackets including F_0 vanish;
- F_1 is zero; therefore all Poisson brackets including F_1 vanish;
- The partial Poisson brackets $\{.,.\}_1$ lower the degree of homogeneity in $\sqrt{\varepsilon}$ and x by one.

Thus, we obtain the equations for resonant systems:

$$\begin{array}{ll} F_0^* = & F_0(L^*) \\ F_1^* = & 0 \\ F_2^* = & F_2(x^*, L_\rho^*, \ell_i^*) \\ F_3^* = & F_3 + \{F_2, T_2^*\}_1 \\ F_4^* = & F_4 + \{F_3, T_2^*\}_1 + \{F_2, T_2^*\}_\rho + \frac{1}{2}\{\{F_2, T_2^*\}_1, T_2^*\}_1 + \{F_2, T_3^*\}_1 \\ \cdots & \cdots \\ F_k^* = & F_k + \{F_{k-1}, T_2^*\}_1 + \{F_{k-2}, T_2^*\}_\rho + \frac{1}{2}\{\{F_{k-2}, T_2^*\}_1, T_2^*\}_1 + \\ & \frac{1}{2}\{\{F_{k-3}, T_2^*\}_1, T_2^*\}_\rho + \frac{1}{2}\{\{F_{k-3}, T_2^*\}_\rho, T_2^*\}_1 + \\ & \frac{1}{2}\{\{F_{k-4}, T_2^*\}_\rho, T_2^*\}_\rho + \cdots + \{F_2, T_{k-1}^*\}_1 \end{array} \tag{26}$$

These equations are very similar to the perturbation equations of the general non-resonant theory. However, besides the intrinsic differences due to the separation of each Poisson bracket into two parts, two major differences are to be emphasized:

1. The first non-trivial equation only appears for the subscript $k = 3$ of F_k and F_k^*; that is, two units more than the corresponding subscripts in the first non-trivial equation of the non-resonant case;
2. The subscripts of the unknowns T_k^* in the non-trivial equations are one unit less than the corresponding subscripts of F_k^* (they are equal in the homological equation of the general non-resonant case). If F is known only up to the degree of homogeneity p, T can be determined only up to the degree $p-1$.

The non-trivial equations obtained by the identification of the orders have the homological form

$$\{F_2, T_{k-1}^*\}_1 = F_k^* - \Psi_k(x^*, L_\rho^*, \ell_i^*) \tag{27}$$

or, in explicit terms,

$$\frac{\partial F_2}{\partial \ell_1^*}\frac{\partial T_{k-1}^*}{\partial x^*} - \frac{\partial T_{k-1}^*}{\partial \ell_1^*}\frac{\partial F_2}{\partial x^*} = F_k^* - \Psi_k(x^*, L_\rho^*, \ell_i^*). \tag{28}$$

We note that, at each order, Ψ_k is a function which is known provided the equations of the previous orders have been solved.

The next steps are identical to the general Hori's theory (see Ferraz-Mello, 1990, 1997) with the only difference that all functions are expanded and ordered following the degree of homogeneity with respect to the variables $\sqrt{\varepsilon}$ and x^*.

It is worthwhile mentioning that, as in the method of Bohlin, it is necessary to solve the equations up to the degree of homogeneity $p+1$, to have the generating function of the canonical transformation T determined up to the degree p.

The characteristic equations of this partial differential equation consist of the equations of the Hori's kernel:

$$\frac{dx^*}{du} = -\frac{\partial F_2(x^*, L_\rho^*, \ell_i^*)}{\partial \ell_1^*} \tag{29}$$

$$\frac{d\ell_1^*}{du} = \frac{\partial F_2(x^*, L_\rho^*, \ell_i^*)}{\partial x^*} \tag{30}$$

and the equation for T_{k-1}^*:

$$\frac{dT_{k-1}^*}{du} = \Psi_k(x^*, L_\rho^*, \ell_i^*) - F_k^*. \tag{31}$$

The function F_2 plays a special role in the solution of the resulting equations since it is the new Hori's kernel. In the new variables, it is

$$F_2(x^*, L_\rho^*, \ell_i^*) = \frac{1}{2}\frac{d^2F_0(L_1^*)}{dL_1^{*2}}x^{*2} + \varepsilon R_2(x^*, L_\rho^*, \ell_i^*). \tag{32}$$

F_2 is not just a function of the variables x^*, ℓ_1^*, but it also depends on the variables L_ρ^*, ℓ_ρ^*. In fact, x, L_ρ, ℓ_i are action-angle variables associated with the Hamiltonian F_0 and have no particular meaning in the dynamical system defined by the Hamiltonian F_2. However, the homological partial differential equation (eq. 27) involves only the variables x^*, ℓ_1^* and the corresponding Hori's kernel has only one degree of freedom. Thus, the variables L_ρ^*, ℓ_ρ^* are constants with respect to u. This Hori's kernel has only one degree of freedom, is integrable and, at least in theory, we may proceed with the averaging and the construction of formal solutions. In fact, the integration of actual Hori's kernels in resonant systems is very difficult. Indeed, from the kernel equations we easily obtain

$$du = \int \frac{d\ell_1^*}{\sqrt{2\frac{d^2F_0(L_1^*)}{dL_1^{*2}}[F_2 - \varepsilon R_2(L_1^*, L_\rho^*, \ell_i^*)]}}. \tag{33}$$

Accordingly, $u(\ell_1^*)$ is given by an elliptic integral making almost impossible to proceed beyond the equation for T_2^*, that is, beyond the order $\mathcal{O}(\varepsilon)$, unless some additional approximations are made. But this difficulty is only of a practical nature and we have to remember that in some simple cases such as in the Ideal Resonance Problem, the system of equations may be completely solved with the aid of some recurrence relations introduced by Garfinkel *et al.* (1971).

The equations of the characteristics of the homological equation need to be completed with the averaging condition

$$F_k^* = <\Psi_k> = \oint \Psi_k[x^*(u), L_\rho^*, \ell_1^*(u), \ell_\rho^*]du \tag{34}$$

where the integral is performed over one period of variation of u.

5. Conclusion

We have presented here an extension of the method of Hori to resonant systems. The essential modification with respect to Hori's general theory is the fact that the identification of all series is done following the degree of homogeneity with respect to $\sqrt{\varepsilon}$ and $x = L_1 - L_1^*$ (L_1^* is the exact resonant value of L_1). The continuation is a standard application of Hori's theory. For the sake of simplicity, we have considered one generic case in which the resonance predominates over long-period perturbations, but cases in which they concur both on an equal footing may be studied following the same steps. The application will be successful in all cases in which we can obtain a separable Hori kernel.

From a formal point of view, the extension of Hori's method to problems with two or more independent resonances acting simultaneously is

https://doi.org/10.1017/S0252921100046352 Published online by Cambridge University Press

easy. However, the Hori kernel will have, in this case, respectively, two or more degrees of freedom and, even in the simplest cases, this kernel is not integrable. We may, at most, study the solutions close to an stable equilibrium point of F_2 by adopting a linear approximation for $R_2(L_1^*, L_\rho^*, \ell_i^*)$.

Acknowledgements. The author thanks Drs. A. Jupp and E. Wnuk for their comments. This investigation was supported by the Research Foundation of São Paulo State (FAPESP).

References

Bohlin, K.: 1888, "Über eine neue Annäherungsmetode in der Störungstheorie", *Bihang Svenska Vet.-Akad.* **14** Afd.I No.5, 1–26.
Brouwer, D.: 1959, "Solution of the problem of artificial satellite theory without drag", *Astron. J.* **64**, 378–397.
Brouwer, D. and Clemence, G.M.: 1961, *Methods of Celestial Mechanics*, Academic Press, New York.
Campbell, J.A. and Jefferys, W.H.: 1970, "Equivalence of the perturbation theories of Hori and Deprit", *Celest. Mech.* **2**, 467–473.
Carathéodory, C.: 1965, *Calculus of Variations and Partial Differential Equations of the First Order*, Holden-Day, San Francisco.
Deprit. A.: 1969, "Canonical transformation depending on a small parameter", *Celest. Mech.* **1**, 12–30.
Ferraz-Mello, S.: 1990, "Averaging Hamiltonian systems", in: *Modern Methods in Celestial Mechanics* (D.Benest, Cl.Froeschlé, eds), Ed. Frontières, pp.151–211.
Ferraz-Mello, S.: 1997, *Hamiltonian Averaging Theories and Resonance in Celestial Mechanics*, to be published.
Garfinkel, B., Jupp, A. and Williams, C.: 1971, "A recursive Von Zeipel algorithm for the ideal resonance problem", *Astron. J.* **76**, 157.
Gröbner, W.: 1960, *Die Lie Reihen und ihre Anwendungen*, Deutsche Verl. Wiss., Berlin.
Hori, G.-I.: 1966, "Theory of general perturbations with unspecified canonical variables", *Publ. Astron. Soc. Japan*, **18**, 287–296.
Jupp, A.: 1970, "On the ideal resonance problem - II", *Mon. Not. Roy. Astron. Soc.* **148**, 197–210.
Jupp, A.: 1982, "A comparison of the Brouwer-Von Zeipel and Brouwer-Lie series methods in resonant systems", *Celest. Mech.* **26**, 413–422.
Jupp, A.: 1992, "Chapter XIX – Bohlin Methods", *Not. Scient. Techn. Bur. Longit.* No. S 036.
Mersman, W. A.: 1970, "A new algorithm for the Lie transforms" *Celest. Mech.* **3**, 81–89.
Poincaré, H.: 1893, *Les Méthodes Nouvelles de la Mécanique Celeste*, Gauthier-Villars, Paris, Vol. II.
Sérsic, J.L.: 1956, *Aplicaciones de un Certo Tipo de Transformaciones Canónicas a la Mecánica Celeste*, Dr. Thesis, Univ. Nac. La Plata. (repr. *Serie Astronomica, Obs. Astron. Univ. La Plata* **35**, 1969).
Sessin, W.: 1986, "A theory of integration for the extended Delaunay method", *Celest. Mech.* **39**, 173–180.
Von Zeipel, H.: 1916, "Recherches sur le mouvement des petites planètes", *Arkiv. Mat. Astron. Fysik*, **11–13**.