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STIEFEL-WHITNEY CLASSES OF A SYMMETRIC BILINEAR FORM — A FORMULA OF SERRE

BY

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ABSTRACT. Let K be a field of characteristic different from two. Let L be a finite separable extension of K. If $\bar{K}$ is the separable closure of K, we have a continuous homomorphism $\pi: Ga(\bar{K}/K) \to \Sigma_n (n = [L:K])$. We give a very short proof of Serre's formula which evaluates the Hasse-Witt invariant of a symmetric bilinear form, transferred from L, in terms of the topological Stiefel-Whitney classes of π.

1. Let K be a field of characteristic different from two and suppose that L/K is a finite separable extension. Write $\bar{K}$ for the separable closure of K and $G(M/K)$ for the Galois group of a finite normal, separable extension of K. $G(\bar{K}/K)$ is the profinite group $\varprojlim_M G(M/K)$, always considered with the profinite topology ([11], 1.1).

L/K is equivalent to the following data. Let N/K be the normal closure of L/K then we have, by the normal basis theorem, a map $\lambda: G(\bar{K}/K) \to \Sigma_n$, the symmetric group $n(=[L:K])$ letters. The image of λ acts transitively on $\{1, \ldots, n\}$ and if $H = \lambda^{-1}$ (stabiliser of 1) then $L = N^H$, the fixed field of H. Of course, $[G(\bar{K}/K):H] = [G(N/K):\lambda(H)] = n$.

1.1. Now let (V, β) be a non-singular symmetric, bilinear form over L of rank m. The Scharlau transfer, $\mathrm{Tr}^S_{L/K}$, $[L]$ of (V, β) is the non-singular symmetric, bilinear form obtained by considering V as an nm-dimensional K-vector space and forming the composition $V \times V \xrightarrow{\beta} L \xrightarrow{\text{Trace}} K$.

An important example is $\langle L \rangle$, the Trace Form of L/K, which is $\mathrm{Tr}^S_{L/K}(1)$ where (1) is the form given by the product on L.

Since char $K \neq 2$, any symmetric, bilinear form over K may be diagonalised to look like

$$\langle \alpha_1 \rangle \oplus \ldots \oplus \langle \alpha_m \rangle \quad \text{where } \alpha_j \in K$$

and where $\langle \alpha_1 \rangle : K \times K \to K$ is given by $\langle \alpha_j \rangle (x, y) = \alpha_j xy$. Each α_j defines $(\alpha_j) \in H^1(G(\bar{K}/K); Z/2) = \varinjlim_{M/K} \mathrm{Hom}\,(G(M/K), Z/2)$ which sends $g \in G(\bar{K}/K)$ to $(\sqrt{\alpha_j})^{-1} g(\sqrt{\alpha_j}) \in \{\pm 1\}$.

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1.2. DEFINITION. *The i-th Stiefel-Whitney class of* (V, β) ([3], [6]), *is defined to be*

$$w_i((V, \beta)) = \sigma_i((\alpha_1), (\alpha_2), \ldots, (\alpha_m)) \in H^i(G(\bar{K}/K); Z/2)$$

where σ_i *is the i-th elementary symmetric function.*

1.3. REMARK. When K is a number field, the Witt class of $(V, \beta) \in W(K)$ is entirely determined by rank (V, β), $w_1(V, \beta)$, $w_2(V, \beta)$ and signatures (see [6] for example).

1.4. By diagonalising a symmetric, bilinear form over L, we can consider it as giving rise to a representation $[V, \beta]: G(\bar{K}/L) \rightarrow (Z/2)^m = \{\pm 1\}^m$ defined over any field — for example, $\mathbb{R}$, the real numbers. For if $V \cong \langle\alpha_1\rangle \oplus \ldots \oplus \langle\alpha_m\rangle$ then $(\alpha_1) \oplus \ldots \oplus (\alpha_m)$ is such a representation. Since $[G(\bar{K}/K): G(\bar{K}/L)] = n$ we may form the induced representation

$$\mathrm{Tr}^V_{L/K}([V, \beta]) = \mathbb{R}[G(\bar{K}/K)] \otimes_{\mathbb{R}[G(\bar{K}/L)]} [V, \beta]. \tag{1.5}$$

Of course, a real representation of $G(\bar{K}/K)$ is entitled to Stiefel-Whitney classes in the topological sense ([5], [8]).

The following attractive formulae are originally due to J-P. Serre [13]. I learnt of it from conversations with Pierre Conner. My proof, which is extremely short, is a product of a more general framework which I developed in order simultaneously to tackle (i) these formulae in higher dimensions and (ii) to obtain similar formulae for Milnor's K-theory characteristic classes. It seemed a good idea to isolate this result — as my other material is monolithic and incomplete.

1.6. THEOREM. *Let* L/K *be a finite separable field extension of characteristic not equal to two. Let* (V, β) *be a non-singular, symmetric, bilinear form over L. Then*

(*i*) $\quad w_1(\mathrm{Tr}^S_{L/K}(V, \beta)) = w_1(\mathrm{Tr}^V_{L/K}[V, \beta]) \in H^1(G(\bar{K}/K); Z/2).$

(*ii*) $\quad w_2(\mathrm{Tr}^S_{L/K}(V, \beta)) = w_2(\mathrm{Tr}^V_{L/K}[V, \beta]) + \mathrm{rank}\,(V, \beta)\{(2)w_1(\mathrm{Tr}^V_{L/K}\langle 1\rangle)\}$
$\in H^2(G(\bar{K}/K); Z/2).$

(*iii*) $\quad w_3(\mathrm{Tr}^S_{L/K}(V, \beta)) = w_3(\mathrm{Tr}^V_{L/}[V, \beta]) + \mathrm{rank}\,(V, \beta)\,\{(2)w_1(\mathrm{Tr}^V_{L/K}\langle 1\rangle)$
$\times [w_1(\mathrm{Tr}^V_{L/K}\langle 1\rangle) + w_1(\mathrm{Tr}^V_{L/K}[V, \beta])]\}$

in $H^3(G(\bar{K}/K); Z/2)$.

1.7. In addition to the references given above, other references concerning symmetric, bilinear forms are [2], [7] and [9].

2. Proof of Theorem 1.6.

2.1. Recall, by Galois descent theory, that non-singular, symmetric, bilinear forms of rank m are classified by $H^1(G(\bar{K}/K); O_m(\bar{K}))$ ([12], pp. 152–153). This in turn coincides with continuous homomorphisms $f: G(\bar{K}/K) \rightarrow G(\bar{K}/K) \propto O_m(\bar{K})$ of the form $f(g) = (g, \phi(g))$, up to composition with an inner automorphism given by an element of $O_m(\bar{K})$. If (V, β) is a bilinear form, we choose a basis so that (V, β) is given by a symmetric $m \times m$ matrix $B \in GL_m K$,then choose $A \in GL_m \bar{K}$ so that $B = AA^T$ and set $\phi(g) = A^{-1}g(A)$. Notice that if $(V, \beta) = \langle\alpha_1\rangle \oplus \ldots \oplus \langle\alpha_m\rangle$, then ϕ is the diagonal homomorphism, diag $((\alpha_1), \ldots, (\alpha_m))$. For us, $O_m(M) = \{X \in GL_m M | XX^T = I_m\}$.

2.2. LEMMA. *Under the conditions of 1.6, suppose* (V, β) *is represented by a diagonal homomorphism,* $\phi_{(V,\beta)}$, *as in 2.1, then* $\mathrm{Tr}^S_{L/K}(V, \beta)$ *is represented by* $f(s) = (g, \mathrm{Tr}^V_{L/K}[V, \beta](g))$, *in the notation of (1.5).*

PROOF. This is very straightforward — probably well-known to experts in quadratic forms — so I will give merely the key details.

In general if, as in 2.1, (V, β) is represented by $B = AA^T$, let $v_1, \ldots, v_n$ be a basis for L/K. Form an $nm \times nm$ matrix $\tilde{A}$ consisting of $n^2 m \times m$ blocks. The (i, j)-th $m \times m$ block in $\tilde{A}$ is $\hat{g}_j(v_i A)$ where $\hat{g}_1, \ldots, \hat{g}_n$ are coset representatives for $G(\bar{K}/K)/G(\bar{K}/L)$. If C has (i, j)-th $m \times m$ block $\hat{g}_i(v_j A^{-1})$ and $\langle L\rangle$ is the $n \times n$ matrix of the trace form of L/K, then $C(\langle L\rangle^{-1} \otimes I_m) = \tilde{A}^{-1}$. One easily computes the representing cocycle for $\mathrm{Tr}^S_{L/K}(V, \beta)$ as this has matrix $\tilde{A}\tilde{A}^T$, using the consequences of the equation $C(\langle L\rangle^{-1} \otimes I_m)\tilde{A} = I_{nm}$ to smooth the apparently complicated algebra.

2.3. Let $\theta : O_m(M) \to M^*/M^{**}$ denote the Spinor norm ([9], p. 137), where M^{**} denotes the non-zero squares in M. We will need to recall that θ is a homomorphism and on a permutation matrix, σ, $\theta(\sigma)$ is trivial if σ is even while $\theta(\sigma) = 2$ if σ is odd. If $\sigma \in \Sigma_m \int Z/2$, the wreath product of Σ_m with the diagonal group $\{\mp 1\}^m$, then $\theta(\sigma)$ equals the Spinor norm of the image of σ in Σ_m. Finally, (c.f. [1], [9] and [4], p. 99), there are central extensions on which $G(\bar{K}/K)$ acts through $G(M/K)$,

$$Z/2 \to \mathrm{Pin}_m(M) \xrightarrow{\pi} NO_m(M), \tag{2.4}$$

where $NO_m M = \{X \in O_m(M) | \theta(X) \equiv 1 \bmod M^{**}\}$. $G(M/K)$ acts trivial on ker (π).

2.5. Choose a section $f: O(\bar{K}) \to \mathrm{Pin}(\bar{K})$ (so that $\pi f(X) = X$) such that if $X \in O_m(\bar{K})$, then $f(X) \in \mathrm{Pin}_m(K(\sqrt{\theta(X)}))$. Define a 2-cochain

$$\hat{w}_2 \in \mathrm{Map}((G(\bar{K}/K) \propto O(\bar{K}))^2, Z/2)$$

by

$$\hat{w}_2((x, X), (y, Y)) = f(X)x(f(Y))\,[f(Xx(Y))]^{-1} \in \ker \pi \cong Z/2. \tag{2.6}$$

Here $X, Y \in O(\bar{K})$, $x, y \in G(\bar{K}/K)$. Note that if $X, Y \in O_m(M)$, the expression (2.6) depends only on the images of x, y in $G(M(\sqrt{\theta(X)}, \sqrt{\theta(Y)})/K)$.

It is straightforward to verify that $\hat{w}_2$ is a 2-cocyle. In addition, changing the section, f, changes $\hat{w}_2$ only by the boundary of a 1-cochain of the form $g : G(\bar{K}/K) \propto O(\bar{K}) \to Z/2$ for which $g(x, X)$ depends only on X.

2.7. COMPLETION OF THE PROOF OF 1.6. Firstly $w_1 : (x, X) \to \det X \in Z/2$ is a 1-cocycle in Map $(G(\bar{K}/K) \propto O(\bar{K}), Z/2)$ which can be used to define w_1. For if (V, β) is classified by $(1, \phi) : G(\bar{K}/K) \to G(\bar{K}/K) \propto O_m(\bar{K})$, then $w_1(1, \phi)$ clearly represents $w_1(V, \beta)$ in $H^1(G(\bar{K}/K); Z/2)$. From 2.2, 1.6(i) follows at once.

In addition, 1.6(*iii*) follows by applying Sq^1 to the formula for $w_2(\mathrm{Tr}^S_{L/K}(V, \beta))$. Here we use $Sq^1(w_2) = w_3 + w_1 w_2$ for both types of Stiefel-Whitney classes and that $(2)^2 = 0$ since 2 is a norm from $K(\sqrt{2})$.

Next I claim that assigning (V, β) to $\hat{w}_2(1, \phi)$ defines $w_2(V, \beta) \in H^2(G(\bar{K}/K):Z/2)$. Observe that, by 2.5, $\hat{w}_2(1, \phi)$ is indeed a continuous cocyle. To verify the claim, we may assume $\phi(g) \in \{\pm 1\}^m \subset O_m(\bar{K})$, then as $G(\bar{K}/K)$ acts trivially on $\phi(g)$, (2.6) shows us that

$$\hat{w}_2(1, \phi)(g_1, g_2) = f(\phi(g_1))f(\phi(g_2))[f(\phi(g_1 g_2))]^{-1}.$$

This is ϕ^* of the class in $H^2(\{\pm 1\}^m; Z/2)$ which classifies the restriction of (2.4) to $\{\pm 1\}^m$. However, in ([10], 4), the 2-cocyle of this extra-special 2-extension is explicitly computed, from which we see that if $\phi(g) = \mathrm{diag}\,(\alpha_1(g), \ldots, \alpha_m(g))$, then $\hat{w}_2(1, \phi)$ represents $\sum_{i<j} (\alpha_i)(\alpha_j)$.

To complete the proof, it remains, by 2.2, only to evaluate $\hat{w}_2(1, \mathrm{Tr}^V_{L/K}[V, \beta])$. To do this, we observe that $\hat{w}_2$ defines a class, $\tilde{w}_2$, in

$$H^2(G(M/K) \times O(K); Z/2) \equiv \bigoplus_{a=0}^{2} H^a(G(M/K); Z/2) \otimes H^{2-a}(O(K); Z/2),$$

for any finite Galois extension M/K and we wish to evaluate $\Delta^*(1 \otimes \mathrm{Tr}^V_{L/K}[V, \beta])\tilde{w}_2$ where Δ^* is the cup product.

Consider the $H^2(O(K); Z/2)$ component of $\tilde{w}_2$. If K were $\mathbb{R}$, the real numbers, then from (2.6) we see that $\tilde{w}_2$ has $H^2(O(K); Z/2)$-component equal to $1 \otimes w_2^{\mathrm{top}}$ where w_2^{top} is the topological 2nd Stiefel-Whitney class (here we appeal again to the fact that (2.6) restricts on $(1) \times \{\pm 1\}^m$ to the 2-cocyle of ([10], 4)). However, K is not, in general, equal to $\mathbb{R}$. Nevertheless, the homomorphism $\mathrm{Tr}^V_{L/K}[V, \beta]$ lands in the monoidal subgroup $S = \Sigma_{nm} \int (\pm 1)$ in $O_{nm}(K)$ and the pullback of the $H^2(O(K); Z/2)$-component of $\tilde{w}_2$ to S is equal to the restriction of w_2^{top} to S. Hence this component contributes $w_2(\mathrm{Tr}^V_{L/K}[V, \beta])$.

Next observe that (2.6) implies that the $H^2(G(M/K), Z/2)$-component of $\tilde{w}_2$ is trivial. Finally, we come to the $H^1(G(M/K); Z/2) \otimes H^1(O(K); Z/2)$ component — the (1, 1)-component — of $\tilde{w}_2$. Suppose in (2.6) that $X = I$, $y = 1$ and $Y \in \Sigma_t \int \{\pm 1\}$ is a monoidal matrix (like $\mathrm{Tr}^V_{L/K}[V, \beta](g)$). We may write Y as a product of transpositions of the canonical basis of K^t — a reflection in the plane perpendicular to a unit vector of the form $1/\sqrt{2}(e_i - e_j)$ and of reflections in planes perpendicular to some e_j. If T_x denotes the reflection in the plane perpendicular to a unit vector x, then $Y = T_{x_1}T_{x_2} \ldots T_{x_n}$ lifts to $f(Y) = x_1 \circ x_2 \circ \ldots \circ x_n \in \mathrm{Pin}_t(K(\sqrt{2}))$ (see [1], or [4], p. 73) where $(- \circ -)$ is Clifford multiplication. Hence, by (2.6), $\tilde{w}_2((x, I), (1, Y)) = (x(\sqrt{2})/\sqrt{2})^{\epsilon}$ where ϵ is the determinant of the image of $Y \in \Sigma_t \int \{\pm 1\}$ in Σ_t. In the case of $Y = \mathrm{Tr}^V_{L/K}[V, \beta](g)$, ϵ is given by $\epsilon = (\det \mathrm{Tr}^V_{L/K}\langle 1\rangle\ (g))^{\mathrm{rank}(V,\beta)}$, which completes the proof.

2.8. Remark. A recent result of Merkurjev-Suslin states that, for a K of characteristic not equal to two, the norm residue symbol [6],

$$K_2(K) \otimes Z/2 \to H^2(G(\bar{K}/K); Z/2),$$

is an isomorphism. Consequently, the 2-dimensional formula of 1.6 holds also for the K-theory Stiefel-Whitney classes which were introduced in [6].

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