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A SHARP GENERAL OSTROWSKI TYPE INEQUALITY

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Abstract

A new sharp general Ostrowski type inequality in L_∞ norm is established. Some special cases are discussed.

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1. Introduction

In 1998, Cerone *et al.* [1] proved the following Ostrowski type inequality in L_∞ norm.

THEOREM 1.1. *Let $f : [a, b] \to \mathbf{R}$ be such that $f^{(n-1)} (n \geq 1)$ is absolutely continuous on $[a, b]$ and $f^{(n)} \in L_\infty[a, b]$. Then for all $x \in [a, b]$,*

$$\begin{aligned}\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \\ \leq \frac{\|f^{(n)}\|_\infty}{(n+1)!}[(x-a)^{n+1} + (b-x)^{n+1}],\end{aligned} \tag{1.1}$$

where

$$\|f^{(n)}\|_\infty := \operatorname*{ess\,sup}_{x\in[a,b]} |f^{(n)}(x)|$$

is the usual Lebesgue norm on $L_\infty[a, b]$.

In 2008, the author [3] proved the following Ostrowski type inequality in L_∞ norm.

THEOREM 1.2. *Let $f : [a, b] \to \mathbf{R}$ be such that f' is absolutely continuous on $[a, b]$ and $f'' \in L_\infty[a, b]$. Then for all $x \in [a, b]$,*

$$\begin{aligned} \bigg| \int_a^b f(t)\,dt - (b-a)\bigg[(1-\theta) f(x) + \theta \frac{f(a)+f(b)}{2} \\ - (1-\theta)\Big(x - \frac{a+b}{2}\Big) f'(x)\bigg]\bigg| \\ \leq \|f''\|_\infty I(\theta, x), \end{aligned} \tag{1.2}$$

where

$$I(\theta, x) = \begin{cases} \Big(\frac{a+b}{2} - x\Big)\Big[\frac{1}{3}\Big(x - \frac{a+b}{2}\Big)^2 + \Big(\frac{1}{4} - \frac{\theta}{2}\Big)(b-a)^2\Big] \\ \quad + \frac{\theta^3 (b-a)^3}{6}, \quad \text{for } a \leq x \leq a + \theta(b-a), \\ \frac{1-\theta}{2}(b-a)\Big(x - \frac{a+b}{2}\Big)^2 + \Big(\frac{\theta^3}{3} - \frac{\theta}{8} + \frac{1}{24}\Big) \\ \quad \times (b-a)^3, \quad \text{for } a + \theta(b-a) < x < b - \theta(b-a), \\ \Big(x - \frac{a+b}{2}\Big)\Big[\frac{1}{3}\Big(x - \frac{a+b}{2}\Big)^2 + \Big(\frac{1}{4} - \frac{\theta}{2}\Big)(b-a)^2\Big] \\ \quad + \frac{\theta^3 (b-a)^3}{6}, \quad \text{for } b - \theta(b-a) \leq x \leq b, \end{cases} \tag{1.3}$$

for $0 \leq \theta < \frac{1}{2}$, and

$$I(\theta, x) = \begin{cases} \Big(\frac{a+b}{2} - x\Big)\Big[\frac{1}{3}\Big(x - \frac{a+b}{2}\Big)^2 + \Big(\frac{1}{4} - \frac{\theta}{2}\Big)(b-a)^2\Big] \\ \quad + \frac{\theta^3 (b-a)^3}{6}, \quad \text{for } a \leq x \leq b - \theta(b-a), \\ \Big(\frac{\theta}{8} - \frac{1}{24}\Big)(b-a)^3 - \frac{1-\theta}{2}(b-a) \\ \quad \times \Big(x - \frac{a+b}{2}\Big)^2, \quad \text{for } b - \theta(b-a) < x < a + \theta(b-a), \\ \Big(x - \frac{a+b}{2}\Big)\Big[\frac{1}{3}\Big(x - \frac{a+b}{2}\Big)^2 + \Big(\frac{1}{4} - \frac{\theta}{2}\Big)(b-a)^2\Big] \\ \quad + \frac{\theta^3 (b-a)^3}{6}, \quad \text{for } a + \theta(b-a) \leq x \leq b, \end{cases} \tag{1.4}$$

for $\frac{1}{2} \leq \theta \leq 1$. The inequality (1.2) with (1.3) and (1.4) is sharp.

The purpose of this paper is to derive further generalizations of the above inequalities which will also lead to some interesting special cases. We need the following result given by Pearce *et al.* in [4].

LEMMA 1.3. *Let* $\{P_n\}_{n\in\mathbf{N}}$ *and* $\{Q_n\}_{n\in\mathbf{N}}$ *be two sequences of harmonic polynomials, that is,*

$$P_n'(t) = P_{n-1}(t), \quad P_0(t) = 1, \quad t \in \mathbf{R},$$

and

$$Q_n'(t) = Q_{n-1}(t), \quad Q_0(t) = 1, \quad t \in \mathbf{R}.$$

Set

$$S_n(t, x) := \begin{cases} P_n(t), & t \in [a, x], \\ Q_n(t), & t \in (x, b]. \end{cases}$$

Then we have the identity

$$\begin{aligned} \int_a^b f(t)\,dt = \sum_{k=1}^{n} (-1)^{k+1} [Q_k(b) f^{(k-1)}(b) + (P_k(x) - Q_k(x)) f^{(k-1)}(x) \\ - P_k(a) f^{(k-1)}(a)] + (-1)^n \int_a^b S_n(t, x) f^{(n)}(t)\,dt, \end{aligned} \tag{1.5}$$

provided that $f : [a, b] \to \mathbf{R}$ *is such that* $f^{(n-1)}$ *is absolutely continuous on* $[a, b]$.

2. The results

LEMMA 2.1. *Let* $f : [a, b] \to \mathbf{R}$ *be such that* $f^{(n-1)} (n \geq 1)$ *is absolutely continuous on* $[a, b]$ *and* $f^{(n)} \in L_\infty[a, b]$. *Then for all* $x \in [a, b]$ *and any* $\theta \in [0, 1]$ *we have the identity*

$$\begin{aligned} \int_a^b f(t)\,dt = {} & \frac{b-a}{2}[\theta f(a) + 2(1-\theta) f(x) + \theta f(b)] \\ & + \sum_{k=1}^{n-1} \left\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \right. \\ & \left. - \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{2k!} \right\} f^{(k)}(x) \\ & + (-1)^n \int_a^b K_n(t, x, \theta) f^{(n)}(t)\,dt, \end{aligned} \tag{2.1}$$

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where

$$K_n(t,x,\theta) := \begin{cases} \dfrac{(t-a)^n}{n!} - \dfrac{\theta(b-a)(t-a)^{n-1}}{2(n-1)!}, & t \in [a,x], \\ \dfrac{(t-b)^n}{n!} + \dfrac{\theta(b-a)(t-b)^{n-1}}{2(n-1)!}, & t \in (x,b]. \end{cases} \tag{2.2}$$

PROOF. The proof is immediate from identity (1.5) in Lemma 1.3.

THEOREM 2.2. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)} (n \geq 1)$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L_\infty[a,b]$. Then for all $x \in [a,b]$ and any $\theta \in [0,1]$,*

$$\begin{aligned} &\bigg| \int_a^b f(t)\,dt - \frac{b-a}{2}[\theta f(a) + 2(1-\theta) f(x) + \theta f(b)] \\ &\quad - \sum_{k=1}^{n-1} \bigg\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \\ &\quad - \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{2k!} \bigg\} f^{(k)}(x) \bigg| \\ &\quad \leq I_n(\theta,x) \| f^{(n)} \|_\infty, \end{aligned} \tag{2.3}$$

where

$$I_n(\theta,x) = \begin{cases} \dfrac{(b-x)^{n+1} - (x-a)^{n+1}}{(n+1)!} - \dfrac{\theta(b-a)[(b-x)^n - (x-a)^n]}{2n!} \\ \quad + \dfrac{n^n \theta^{n+1} (b-a)^{n+1}}{(n+1)! 2^n}, \quad \text{for } a \leq x \leq a + \dfrac{n\theta}{2}(b-a), \\ \dfrac{(x-a)^{n+1} + (b-x)^{n+1}}{(n+1)!} - \dfrac{\theta(b-a)[(x-a)^n + (b-x)^n]}{2n!} \\ \quad + \dfrac{n^n \theta^{n+1} (b-a)^{n+1}}{(n+1)! 2^{n-1}}, \\ \quad \text{for } a + \dfrac{n\theta}{2}(b-a) < x < b - \dfrac{n\theta}{2}(b-a), \\ \dfrac{(x-a)^{n+1} - (b-x)^{n+1}}{(n+1)!} - \dfrac{\theta(b-a)[(x-a)^n - (b-x)^n]}{2n!} \\ \quad + \dfrac{n^n \theta^{n+1} (b-a)^{n+1}}{(n+1)! 2^n}, \quad \text{for } b - \dfrac{n\theta}{2}(b-a) \leq x \leq b, \end{cases} \tag{2.4}$$

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for $0 \le n\theta < 1$, *and*

$$I_n(\theta, x) = \begin{cases} \dfrac{(b-x)^{n+1} - (x-a)^{n+1}}{(n+1)!} - \dfrac{\theta(b-a)[(b-x)^n - (x-a)^n]}{2n!} \\ \quad + \dfrac{n^n\theta^{n+1}(b-a)^{n+1}}{(n+1)!2^n}, \quad \text{for } a \le x \le b - \dfrac{n\theta}{2}(b-a), \\ -\dfrac{(x-a)^{n+1} + (b-x)^{n+1}}{(n+1)!} + \dfrac{\theta(b-a)[(x-a)^n + (b-x)^n]}{2n!}, \\ \quad \text{for } b - \dfrac{n\theta}{2}(b-a) < x < a + \dfrac{n\theta}{2}(b-a), \\ \dfrac{(x-a)^{n+1} - (b-x)^{n+1}}{(n+1)!} - \dfrac{\theta(b-a)[(x-a)^n - (b-x)^n]}{2n!} \\ \quad + \dfrac{n^n\theta^{n+1}(b-a)^{n+1}}{(n+1)!2^n}, \quad \text{for } a + \dfrac{n\theta}{2}(b-a) \le x \le b, \end{cases} \tag{2.5}$$

for $1 \le n\theta < 2$, *and*

$$I_n(\theta, x) = -\frac{(x-a)^{n+1} + (b-x)^{n+1}}{(n+1)!} + \frac{\theta(b-a)[(x-a)^n + (b-x)^n]}{2n!} \tag{2.6}$$

for $n\theta \ge 2$. *Inequality (2.3) with (2.4), (2.5) and (2.6) is sharp.*

PROOF. Using identity (2.1) in Lemma 2.1, we get

$$\begin{aligned} \bigg| \int_a^b f(t)\,dt &- \frac{b-a}{2}[\theta f(a) + 2(1-\theta) f(x) + \theta f(b)] \\ &- \sum_{k=1}^{n-1} \bigg\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \\ &- \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{2k!} \bigg\} f^{(k)}(x) \bigg| \\ &\le I_n(\theta, x) \|f^{(n)}\|_\infty, \end{aligned} \tag{2.7}$$

where

$$I_n(\theta, x) = \int_a^b |K_n(t, x, \theta)|\,dt.$$

Then by (2.2),

$$\begin{aligned} n! I_n(\theta, x) = \int_a^x \left| (t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1} \right| dt \\ + \int_x^b \left| (t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1} \right| dt. \end{aligned} \tag{2.8}$$

The last two integrals in (2.8) can be calculated as follows. For brevity, we put

$$P(t) := (t-a)^{n-1}\left[t-a-\frac{n\theta}{2}(b-a)\right], \quad t\in[a,b],$$
$$Q(t) := (t-b)^{n-1}\left[t-b+\frac{n\theta}{2}(b-a)\right], \quad t\in[a,b],$$

where $\theta\in[0,1]$. It is clear that both $P(t)$ and $Q(t)$ have only one zero in (a,b) for $0<n\theta<2$. Let $t_1=a+n\theta(b-a)/2$ and $t_2=b-n\theta(b-a)/2$. It is easy to see that $a\le t_1<(a+b)/2<t_2\le b$ if and only if $0\le n\theta<1$ as well as $a\le t_2\le(a+b)/2\le t_1\le b$ if and only if $1\le n\theta\le 2$. Thus we have: for an odd n with $0\le n\theta<1$,

$$\begin{aligned} n!I_n(\theta,x) &= -\int_a^x\left[(t-a)^n-\frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\ &\quad -\int_x^{t_2}\left[(t-b)^n+\frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \\ &\quad +\int_{t_2}^b\left[(t-b)^n+\frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \\ &= \frac{(b-x)^{n+1}-(x-a)^{n+1}}{n+1}-\frac{\theta(b-a)[(b-x)^n-(x-a)^n]}{2} \\ &\quad +\frac{n^n\theta^{n+1}(b-a)^{n+1}}{(n+1)2^n} \end{aligned} \tag{2.9}$$

when $a\le x\le a+n\theta(b-a)/2$,

$$\begin{aligned} n!I_n(\theta,x) &= -\int_a^{t_1}\left[(t-a)^n-\frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\ &\quad +\int_{t_1}^x\left[(t-a)^n-\frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\ &\quad -\int_x^{t_2}\left[(t-b)^n+\frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \\ &\quad +\int_{t_2}^b\left[(t-b)^n+\frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \\ &= \frac{(x-a)^{n+1}+(b-x)^{n+1}}{n+1}-\frac{\theta(b-a)[(x-a)^n+(b-x)^n]}{2} \\ &\quad +\frac{n^n\theta^{n+1}(b-a)^{n+1}}{(n+1)2^{n-1}} \end{aligned} \tag{2.10}$$

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when $a + n\theta(b-a)/2 < x < b - n\theta(b-a)/2$,

$$\begin{aligned} n!I_n(\theta, x) &= -\int_a^{t_1}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\ &\quad + \int_{t_1}^{x}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\ &\quad + \int_x^{b}\left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \\ &= \frac{(x-a)^{n+1}-(b-x)^{n+1}}{n+1} - \frac{\theta(b-a)[(x-a)^n-(b-x)^n]}{2} \\ &\quad + \frac{n^n\theta^{n+1}(b-a)^{n+1}}{(n+1)2^n} \end{aligned} \tag{2.11}$$

when $b - n\theta(b-a)/2 \le x \le b$; for an even n with $0 \le n\theta < 1$,

$$\begin{aligned} n!I_n(\theta, x) &= -\int_a^{x}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\ &\quad + \int_{x}^{t_2}\left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \\ &\quad - \int_{t_2}^{b}\left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \\ &= \frac{(b-x)^{n+1}-(x-a)^{n+1}}{n+1} - \frac{\theta(b-a)[(b-x)^n-(x-a)^n]}{2} \\ &\quad + \frac{n^n\theta^{n+1}(b-a)^{n+1}}{(n+1)2^n} \end{aligned} \tag{2.12}$$

when $a \le x \le a + n\theta(b-a)/2$,

$$\begin{aligned} n!I_n(\theta, x) &= -\int_a^{t_1}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\ &\quad + \int_{t_1}^{x}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\ &\quad + \int_x^{t_2}\left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \\ &\quad - \int_{t_2}^{b}\left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \\ &= \frac{(x-a)^{n+1}+(b-x)^{n+1}}{n+1} - \frac{\theta(b-a)[(x-a)^n+(b-x)^n]}{2} \\ &\quad + \frac{n^n\theta^{n+1}(b-a)^{n+1}}{(n+1)2^{n-1}} \end{aligned} \tag{2.13}$$

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when $a + n\theta(b-a)/2 < x < b - n\theta(b-a)/2$,

$$
\begin{aligned}
n!I_n(\theta, x) = &-\int_a^{t_1}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\
&+ \int_{t_1}^{x}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\
&- \int_x^{b}\left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \qquad (2.14)\\
= &\frac{(x-a)^{n+1}-(b-x)^{n+1}}{n+1} - \frac{\theta(b-a)[(x-a)^n-(b-x)^n]}{2} \\
&+ \frac{n^n\theta^{n+1}(b-a)^{n+1}}{(n+1)2^n}
\end{aligned}
$$

when $b - n\theta(b-a)/2 \le x \le b$; for an odd n with $1 \le n\theta < 2$,

$$
\begin{aligned}
n!I_n(\theta, x) = &-\int_a^{x}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\
&- \int_{x}^{t_2}\left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \\
&+ \int_{t_2}^{b}\left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \qquad (2.15)\\
= &\frac{(b-x)^{n+1}-(x-a)^{n+1}}{n+1} - \frac{\theta(b-a)[(b-x)^n-(x-a)^n]}{2} \\
&+ \frac{n^n\theta^{n+1}(b-a)^{n+1}}{(n+1)2^n}
\end{aligned}
$$

when $a \le x \le b - n\theta(b-a)/2$,

$$
\begin{aligned}
n!I_n(\theta, x) = &-\int_a^{x}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\
&+ \int_{x}^{b}\left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \qquad (2.16)\\
= &-\frac{(x-a)^{n+1}+(b-x)^{n+1}}{n+1} + \frac{\theta(b-a)[(x-a)^n+(b-x)^n]}{2}
\end{aligned}
$$

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when $b - n\theta(b-a)/2 < x < a + n\theta(b-a)/2$,

$$\begin{aligned}
n! I_n(\theta, x) &= -\int_a^{t_1} \left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1} \right] dt \\
&\quad + \int_{t_1}^{x} \left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1} \right] dt \\
&\quad + \int_x^b \left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1} \right] dt \qquad (2.17)\\
&= \frac{(x-a)^{n+1} - (b-x)^{n+1}}{n+1} - \frac{\theta(b-a)[(x-a)^n - (b-x)^n]}{2} \\
&\quad + \frac{n^n \theta^{n+1} (b-a)^{n+1}}{(n+1)2^n}
\end{aligned}$$

when $a + n\theta(b-a)/2 \leq x \leq b$; for an even n with $1 \leq n\theta < 2$,

$$\begin{aligned}
n! I_n(\theta, x) &= -\int_a^{x} \left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1} \right] dt \\
&\quad + \int_{x}^{t_2} \left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1} \right] dt \\
&\quad - \int_{t_2}^b \left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1} \right] dt \qquad (2.18)\\
&= \frac{(b-x)^{n+1} - (x-a)^{n+1}}{n+1} - \frac{\theta(b-a)[(b-x)^n - (x-a)^n]}{2} \\
&\quad + \frac{n^n \theta^{n+1} (b-a)^{n+1}}{(n+1)2^n}
\end{aligned}$$

when $a \leq x \leq a + n\theta(b-a)/2$,

$$\begin{aligned}
n! I_n(\theta, x) &= -\int_a^{x} \left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1} \right] dt \\
&\quad - \int_x^b \left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1} \right] dt \qquad (2.19)\\
&= -\frac{(x-a)^{n+1} + (b-x)^{n+1}}{n+1} + \frac{\theta(b-a)[(x-a)^n + (b-x)^n]}{2}
\end{aligned}$$

when $a + n\theta(b-a)/2 < x < b - n\theta(b-a)/2$,

$$\begin{aligned} n!I_n(\theta, x) = &-\int_a^{t_1}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\ &+ \int_{t_1}^{x}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\ &- \int_x^{b}\left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \qquad (2.20)\\ = &\frac{(x-a)^{n+1} - (b-x)^{n+1}}{n+1} - \frac{\theta(b-a)[(x-a)^n - (b-x)^n]}{2} \\ &+ \frac{n^n\theta^{n+1}(b-a)^{n+1}}{(n+1)2^n} \end{aligned}$$

when $b - n\theta(b-a)/2 \le x \le b$; for an odd n with $n\theta \ge 2$,

$$\begin{aligned} n!I_n(\theta, x) = &-\int_a^{x}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\ &+ \int_x^{b}\left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \qquad (2.21)\\ = &-\frac{(x-a)^{n+1} + (b-x)^{n+1}}{n+1} + \frac{\theta(b-a)[(x-a)^n + (b-x)^n]}{2} \end{aligned}$$

when $a \le x \le b$; and for an even n with $n\theta \ge 2$,

$$\begin{aligned} n!I_n(\theta, x) = &-\int_a^{x}\left[(t-a)^n - \frac{n\theta}{2}(b-a)(t-a)^{n-1}\right]dt \\ &- \int_x^{b}\left[(t-b)^n + \frac{n\theta}{2}(b-a)(t-b)^{n-1}\right]dt \qquad (2.22)\\ = &-\frac{(x-a)^{n+1} + (b-x)^{n+1}}{n+1} + \frac{\theta(b-a)[(x-a)^n + (b-x)^n]}{2} \end{aligned}$$

when $a \le x \le b$. Consequently, inequality (2.3) with (2.4), (2.5) and (2.6) follows from (2.7) and (2.9)–(2.22).

We now prove that inequality (2.3) with (2.4), (2.5) and (2.6) is sharp. Indeed, we can choose f to attain equality in (2.3) with (2.4), (2.5) and (2.6). If n is odd, we may construct f such that:

$$f^{(n-1)}(t) = \begin{cases} -t, & a \le t < t_2, \\ t - 2t_2, & t_2 \le t \le b, \end{cases}$$

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when $a \le x \le t_1$,

$$f^{(n-1)}(t) = \begin{cases} -t, & a \le t < t_1, \\ t - 2t_1, & t_1 \le t < x, \\ -t + 2(x - t_1), & x \le t < t_2, \\ t + 2(x - t_1 - t_2), & t_2 \le t \le b, \end{cases}$$

when $t_1 < x < t_2$, and

$$f^{(n-1)}(t) = \begin{cases} -t, & a \le t < t_1, \\ t - 2t_1, & t_1 \le t \le b, \end{cases}$$

when $t_2 \le x \le b$ for $0 \le n\theta < 1$;

$$f^{(n-1)}(t) = \begin{cases} -t, & a \le t < t_2, \\ t - 2t_2, & t_2 \le t \le b, \end{cases}$$

when $a \le x \le t_2$,

$$f^{(n-1)}(t) = \begin{cases} -t, & a \le t < x, \\ t - 2x, & x \le t \le b, \end{cases}$$

when $t_2 < x < t_1$,

$$f^{(n-1)}(t) = \begin{cases} -t, & a \le t < t_1, \\ t - 2t_1, & t_1 \le t \le b, \end{cases}$$

when $t_1 \le x \le b$ for $1 \le n\theta < 2$; and

$$f^{(n-1)}(t) = \begin{cases} -t, & a \le t < x, \\ t - 2x, & x \le t \le b, \end{cases}$$

when $a \le x \le b$ for $n\theta \ge 2$.

If n is even, we may construct f such that:

$$f^{(n-1)}(t) = \begin{cases} -t, & a \le t < x, \\ t - 2x, & x \le t < t_2, \\ -t + 2(t_2 - x), & t_2 \le t \le b, \end{cases}$$

when $a \le x \le t_1$,

$$f^{(n-1)}(t) = \begin{cases} -t, & a \le t < t_1, \\ t - 2t_1, & t_1 \le t < t_2, \\ -t + 2(t_2 - t_1), & t_2 \le t \le b, \end{cases}$$

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when $t_1 < x < t_2$, and

$$f^{(n-1)}(t) = \begin{cases} -t, & a \le t < t_1, \\ t - 2t_1, & t_1 \le t \le x, \\ -t + 2(x - t_1), & x \le t \le b, \end{cases}$$

when $t_2 \le x \le b$ for $0 \le n\theta < 1$;

$$f^{(n-1)}(t) = \begin{cases} -t, & a \le t < x, \\ t - 2x, & x < t \le t_2, \\ -t + 2(t_2 - x), & t_2 \le t \le b, \end{cases}$$

when $a \le x \le t_2$,

$$f^{(n-1)}(t) = -t, \quad a \le t \le b,$$

when $t_2 < x < t_1$, and

$$f^{(n-1)}(t) = \begin{cases} -t, & a \le t < t_1, \\ t - 2t_1, & t_1 \le t < x, \\ -t + 2(x - t_1), & x \le t \le b, \end{cases}$$

when $t_1 \le x \le b$ for $1 \le n\theta < 2$;

$$f^{(n-1)}(t) = -t, \quad a \le t \le b,$$

when $a \le x \le b$ for $n\theta \ge 2$.

Clearly, all the above $f^{(n-1)}$ are absolutely continuous on $[a, b]$, and then, if n is odd,

$$f^{(n)}(t) = \begin{cases} -1, & a < t < t_2, \\ 1, & t_2 < t < b, \end{cases}$$

when $a \le x \le t_1$,

$$f^{(n)}(t) = \begin{cases} -1, & a < t < t_1, \\ 1, & t_1 < t < x, \\ -1, & x < t < t_2, \\ 1, & t_2 < t < b, \end{cases}$$

when $t_1 < x < t_2$, and

$$f^{(n)}(t) = \begin{cases} -1, & a < t < t_1, \\ 1, & t_1 < t < b, \end{cases}$$

when $t_2 \le x \le b$ for $0 \le n\theta < 1$;

$$f^{(n)}(t) = \begin{cases} -1, & a < t < t_2, \\ 1, & t_2 < t < b, \end{cases}$$

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when $a \le x \le t_2$,

$$f^{(n)}(t) = \begin{cases} -1, & a < t < x, \\ 1, & x < t < b, \end{cases}$$

when $t_2 < x < t_1$, and

$$f^{(n)}(t) = \begin{cases} -1, & a < t < t_1, \\ 1, & t_1 < t < b, \end{cases}$$

when $t_1 \le x \le b$, for $1 \le n\theta < 2$; and

$$f^{(n)}(t) = \begin{cases} -1, & a < t < x, \\ 1, & x < t < b, \end{cases}$$

when $a \le x \le b$ for $n\theta \ge 2$. If n is even:

$$f^{(n)}(t) = \begin{cases} -1, & a < t < x, \\ 1, & x < t < t_2, \\ -1, & t_2 < t < b, \end{cases}$$

when $a \le x \le t_1$,

$$f^{(n)}(t) = \begin{cases} -1, & a < t < t_1, \\ 1, & t_1 < t < t_2, \\ -1, & t_2 < t < b, \end{cases}$$

when $t_1 < x < t_2$, and

$$f^{(n)}(t) = \begin{cases} -1, & a < t < t_1, \\ 1, & t_1 < t < x, \\ -1, & x < t < b, \end{cases}$$

when $t_2 \le x \le b$, for $0 \le n\theta < 1$;

$$f^{(n)}(t) = \begin{cases} -1, & a < t < x, \\ 1, & x < t < t_2, \\ -1, & t_2 < t < b, \end{cases}$$

when $a \le x \le t_2$,

$$f^{(n)}(t) = -1, \quad a < t < b,$$

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when $t_2 < x < t_1$,

$$f^{(n)}(t) = \begin{cases} -1, & a < t < t_1, \\ 1, & t_1 < t < x, \\ -1, & x < t < b \end{cases}$$

when $t_1 \le x \le b$, for $1 \le n\theta < 2$;

$$f^{(n)}(t) = -1, \quad a < t < b,$$

when $a \le x \le b$, for $n\theta \ge 2$, which satisfy the condition of Theorem 2.2 with $\|f^{(n)}\|_\infty = 1$. The proof is complete. □

REMARK 2.3. It is clear that Theorem 1.1 is just the special case $\theta = 0$ of Theorem 2.2 without a proof on sharpness of inequality (1.1), and Theorem 1.2 is just the special case $n = 2$ of Theorem 2.2.

COROLLARY 2.4. *Let the assumptions of Theorem 2.2 hold. Then for $n = 1, 2$ we have sharp trapezoid inequalities*

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}[f(a) + f(b)] \right| \le \frac{(b-a)^2}{4}\|f'\|_\infty \tag{2.23}$$

and

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}[f(a) + f(b)] \right| \le \frac{(b-a)^3}{12}\|f''\|_\infty, \tag{2.24}$$

and for $n \ge 3$ we have sharp trapezoid type inequalities

$$\begin{aligned} &\Bigg| \int_a^b f(t)\,dt - \frac{b-a}{2}[f(a) + f(b)] \\ &\quad - \sum_{k=1}^{n-1} \Bigg\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \\ &\quad - \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{2k!} \Bigg\} f^{(k)}(x) \Bigg| \\ &\quad \le \|f^{(n)}\|_\infty \left[\frac{(x-a)^n + (b-x)^n}{2n!}(b-a) - \frac{(x-a)^{n+1} + (b-x)^{n+1}}{(n+1)!} \right]. \end{aligned} \tag{2.25}$$

PROOF. Letting $\theta = 1$ in (2.3) with (2.4), (2.5) and (2.6) readily produces the results (2.23)–(2.25). □

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COROLLARY 2.5. *Let the assumptions of Theorem 2.2 hold. Then for* $n = 1, 2, 3, 4, 5$, *we have sharp Simpson type inequalities*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{6}[f(a)+4f(x)+f(b)]\right|$$

$$\leq \|f'\|_\infty \times \begin{cases} \frac{2}{3}(b-a)\left(\frac{a+b}{2}-x\right)+\frac{(b-a)^2}{36}, \\ \quad \text{for } a \leq x \leq a+\frac{5a+b}{6}, \\ \left(x-\frac{a+b}{2}\right)^2+\frac{5(b-a)^2}{36}, \\ \quad \text{for } \frac{5a+b}{6} < x < \frac{a+5b}{6}, \\ \frac{2}{3}(b-a)\left(x-\frac{a+b}{2}\right)+\frac{(b-a)^2}{36}, \\ \quad \text{for } \frac{a+5b}{6} \leq x \leq b, \end{cases} \tag{2.26}$$

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{6}[f(a)+4f(x)+f(b)] + \frac{2(b-a)}{3}\left(x-\frac{a+b}{2}\right)f'(x)\right|$$

$$\leq \|f''\|_\infty \times \begin{cases} \frac{1}{3}\left(\frac{a+b}{2}-x\right)^3+\frac{(b-a)^2}{12}\left(\frac{a+b}{2}-x\right)+\frac{(b-a)^3}{162}, \\ \quad \text{for } a \leq x \leq a+\frac{2a+b}{3}, \\ \frac{b-a}{3}\left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^3}{81}, \\ \quad \text{for } \frac{2a+b}{3} < x < \frac{a+2b}{3}, \\ \frac{1}{3}\left(x-\frac{a+b}{2}\right)^3+\frac{(b-a)^2}{12}\left(x-\frac{a+b}{2}\right)+\frac{(b-a)^3}{162}, \\ \quad \text{for } \frac{a+2b}{3} \leq x \leq b, \end{cases} \tag{2.27}$$

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \frac{b-a}{6}[f(a)+4f(x)+f(b)]\right.\\
&\quad\left. + \frac{2(b-a)}{3}\left(x-\frac{a+b}{2}\right)f'(x) - \frac{b-a}{3}\left(x-\frac{a+b}{2}\right)^2 f''(x)\right|\\
&\qquad \le \|f'''\|_\infty \times \begin{cases} \dfrac{b-a}{9}\left(\dfrac{a+b}{2}-x\right)^3 + \dfrac{(b-a)^4}{576}, & \text{for } a \le x \le \dfrac{a+b}{2},\\[2mm] \dfrac{b-a}{9}\left(x-\frac{a+b}{2}\right)^3 + \dfrac{(b-a)^4}{576}, & \text{for } \dfrac{a+b}{2} \le x \le b, \end{cases}
\end{aligned} \tag{2.28}$$

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \frac{b-a}{6}[f(a)+4f(x)+f(b)] + \frac{2(b-a)}{3}\left(x-\frac{a+b}{2}\right)f'(x)\right.\\
&\quad\left. - \frac{b-a}{3}\left(x-\frac{a+b}{2}\right)^2 f''(x) + \frac{b-a}{9}\left(x-\frac{a+b}{2}\right)^3 f'''(x)\right|\\
&\qquad \le \|f^{(4)}\|_\infty \times \begin{cases} \dfrac{1}{60}\left(\dfrac{a+b}{2}-x\right)^5 + \dfrac{(b-a)^2}{72}\left(\dfrac{a+b}{2}-x\right)^3\\ \quad - \dfrac{(b-a)^4}{576}\left(\dfrac{a+b}{2}-x\right) + \dfrac{2(b-a)^5}{3645},\\ \quad \text{for } a \le x \le a + \dfrac{2a+b}{3},\\[2mm] \dfrac{(b-a)^5}{2880} - \dfrac{b-a}{36}\left(x-\dfrac{a+b}{2}\right)^4,\\ \quad \text{for } \dfrac{2a+b}{3} < x < \dfrac{a+2b}{3},\\[2mm] \dfrac{1}{60}\left(x-\dfrac{a+b}{2}\right)^5 + \dfrac{(b-a)^2}{72}\left(x-\dfrac{a+b}{2}\right)^3\\ \quad - \dfrac{(b-a)^4}{576}\left(x-\dfrac{a+b}{2}\right) + \dfrac{2(b-a)^5}{3645},\\ \quad \text{for } \dfrac{a+2b}{3} \le x \le b, \end{cases}
\end{aligned} \tag{2.29}$$

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and

$$
\begin{aligned}
&\bigg| \int_a^b f(t)\,dt - \frac{b-a}{6}[f(a) + 4f(x) + f(b)] \\
&\quad + \frac{2(b-a)}{3}\left(x - \frac{a+b}{2}\right) f'(x) - \frac{b-a}{3}\left(x - \frac{a+b}{2}\right)^2 f''(x) \\
&\quad + \frac{b-a}{9}\left(x - \frac{a+b}{2}\right)^3 f'''(x) \\
&\quad + \left[\frac{(b-a)^5}{2880} - \frac{b-a}{36}\left(x - \frac{a+b}{2}\right)^4\right] f^{(4)}(x) \bigg| \\
&\quad \le \| f^{(5)} \|_\infty \times \begin{cases} \frac{b-a}{180}\left(\frac{a+b}{2} - x\right)^5 - \frac{(b-a)^5}{2880}\left(\frac{a+b}{2} - x\right) \\ \quad + \frac{625(b-a)^6}{3359232}, \quad \text{for } a \le x \le \frac{5a+b}{6}, \\ -\frac{1}{360}\left(x - \frac{a+b}{2}\right)^6 - \frac{(b-a)^2}{288}\left(x - \frac{a+b}{2}\right)^4 \\ \quad + \frac{(b-a)^4}{1152}\left(x - \frac{a+b}{2}\right)^2 + \frac{(b-a)^6}{23040}, \\ \quad \text{for } \frac{5a+b}{6} < x < \frac{a+5b}{6}, \\ \frac{b-a}{180}\left(x - \frac{a+b}{2}\right)^5 - \frac{(b-a)^5}{2880}\left(x - \frac{a+b}{2}\right) \\ \quad + \frac{625(b-a)^6}{3359232}, \quad \text{for } \frac{a+5b}{6} \le x \le b, \end{cases}
\end{aligned} \tag{2.30}
$$

and for $n \ge 6$ we have the sharp Simpson type inequality

$$
\begin{aligned}
&\bigg| \int_a^b f(t)\,dt - \frac{b-a}{6}[f(a) + 4f(x) + f(b)] \\
&\quad - \sum_{k=1}^{n-1} \left\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} \right. \\
&\quad \left. - \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{6k!} \right\} f^{(k)}(x) \bigg| \\
&\quad \le \| f^{(n)} \|_\infty \left[\frac{(x-a)^n + (b-x)^n}{6n!}(b-a) - \frac{(x-a)^{n+1} + (b-x)^{n+1}}{(n+1)!} \right].
\end{aligned} \tag{2.31}
$$

PROOF. Putting $\theta = \frac{1}{3}$ in (2.3) with (2.4), (2.5) and (2.6) readily produces the results (2.26)–(2.31). □

COROLLARY 2.6. *Let the assumptions of Theorem 2.2 hold. Then for* $n = 1, 2, 3$ *we have sharp averaged midpoint-trapezoid type inequalities*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{4}[f(a)+2f(x)+f(b)]\right| \le \|f'\|_\infty \times \begin{cases} \frac{1}{2}(b-a)\left(\frac{a+b}{2}-x\right)+\frac{(b-a)^2}{16}, \\ \quad \text{for } a \le x \le a+\frac{3a+b}{4}, \\ \left(x-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{8}, \\ \quad \text{for } \frac{3a+b}{4} < x < \frac{a+3b}{4}, \\ \frac{1}{2}(b-a)\left(x-\frac{a+b}{2}\right)+\frac{(b-a)^2}{16}, \\ \quad \text{for } \frac{a+3b}{4} \le x \le b, \end{cases} \tag{2.32}$$

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{4}[f(a)+2f(x)+f(b)] + \frac{b-a}{2}\left(x-\frac{a+b}{2}\right)f'(x)\right| \le \|f''\|_\infty \times \begin{cases} \frac{1}{3}\left(\frac{a+b}{2}-x\right)^3+\frac{(b-a)^3}{48}, & \text{for } a \le x \le \frac{a+b}{2}, \\ \frac{1}{3}\left(x-\frac{a+b}{2}\right)^3+\frac{(b-a)^3}{48}, & \text{for } \frac{a+b}{2} \le x \le b, \end{cases} \tag{2.33}$$

and

$$\Bigg|\int_a^b f(t)\,dt - \frac{b-a}{4}[f(a)+2f(x)+f(b)] + \frac{b-a}{2}\left(x-\frac{a+b}{2}\right)f'(x) + \left[\frac{(b-a)^3}{48} - \frac{b-a}{4}\left(x-\frac{a+b}{2}\right)^2\right]f''(x)\Bigg|$$

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$$\leq \|f'''\|_\infty \times \begin{cases} \dfrac{b-a}{12}\left(\dfrac{a+b}{2}-x\right)^3 - \dfrac{(b-a)^3}{48}\left(\dfrac{a+b}{2}-x\right) \\ \quad + \dfrac{9(b-a)^4}{1024}, \quad \textit{for } a \leq x \leq a + \dfrac{3a+b}{4}, \\ -\dfrac{1}{12}\left(x-\dfrac{a+b}{2}\right)^4 + \dfrac{(b-a)^4}{192}, \\ \quad \textit{for } \dfrac{3a+b}{4} < x < \dfrac{a+3b}{4}, \\ \dfrac{b-a}{12}\left(x-\dfrac{a+b}{2}\right)^3 - \dfrac{(b-a)^3}{48}\left(x-\dfrac{a+b}{2}\right) \\ \quad + \dfrac{9(b-a)^4}{1024}, \quad \textit{for } \dfrac{a+3b}{4} \leq x \leq b, \end{cases} \tag{2.34}$$

and for $n \geq 4$ we have the sharp averaged midpoint-trapezoid type inequality

$$\begin{aligned} &\bigg| \int_a^b f(t)\,dt - \frac{b-a}{4}[f(a)+2f(x)+f(b)] \\ &\quad - \sum_{k=1}^{n-1}\bigg\{ \frac{(-1)^k(x-a)^{k+1}+(b-x)^{k+1}}{(k+1)!} \\ &\quad - \frac{\theta(b-a)[(-1)^k(x-a)^k+(b-x)^k]}{4k!}\bigg\} f^{(k)}(x)\bigg| \\ &\quad \leq \|f^{(n)}\|_\infty \left[\frac{(x-a)^n+(b-x)^n}{4n!}(b-a) - \frac{(x-a)^{n+1}+(b-x)^{n+1}}{(n+1)!}\right]. \end{aligned} \tag{2.35}$$

PROOF. Putting $\theta = \frac{1}{2}$ in (2.3) with (2.4), (2.5) and (2.6) readily produces the results (2.32)–(2.35). □

COROLLARY 2.7. *Let the assumptions of Theorem 2.2 hold. Then for any $\theta \in [0, 1]$,*

$$\begin{aligned} &\bigg| \int_a^b f(x)\,dx - \frac{b-a}{2}\left[\theta f(a) + 2(1-\theta) f\left(\frac{a+b}{2}\right) + \theta f(b)\right] \\ &\quad - \sum_{k=1}^{[(n-1)/2]} \frac{[1-(n+1)\theta](b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right)\bigg| \\ &\quad \leq \|f^{(n)}\|_\infty \times \begin{cases} \dfrac{[1-(n+1)\theta + 2n^n\theta^{n+1}](b-a)^{n+1}}{(n+1)!2^n}, & n < 1/\theta, \\ \dfrac{[(n+1)\theta - 1](b-a)^{n+1}}{(n+1)!2^n}, & n \geq 1/\theta, \end{cases} \end{aligned} \tag{2.36}$$

where $[(n-1)/2]$ denotes the integer part of $(n-1)/2$.

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PROOF. Putting $x = \frac{1}{2}(a+b)$ in (2.3) with (2.4), (2.5) and (2.6) readily produces the result (2.36). □

REMARK 2.8. If we take $\theta = 0$ in (2.36), we get the sharp midpoint type inequality

$$\left|\int_a^b f(x)\,dx - (b-a)f\left(\frac{a+b}{2}\right) - \sum_{k=1}^{[(n-1)/2]} \frac{(b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| \\ \le \frac{(b-a)^{n+1}}{(n+1)!2^n}\|f^{(n)}\|_\infty$$

which appeared in [1] without a proof of sharpness.

If we take $\theta = 1$ in (2.36), we get the sharp trapezoid type inequality

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] + \sum_{k=1}^{[(n-1)/2]} \frac{k(b-a)^{2k+1}}{(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| \\ \le \frac{n(b-a)^{n+1}}{(n+1)!2^n}\|f^{(n)}\|_\infty.$$

If we take $\theta = \frac{1}{3}$ in (2.36), we get the sharp Simpson type inequality

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \right. \\ \left. + \sum_{k=1}^{[(n-1)/2]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| \\ \le \|f^{(n)}\|_\infty \times \begin{cases} \frac{5}{36}(b-a)^2, & n=1, \\ \frac{1}{81}(b-a)^2, & n=2, \\ \frac{(n-2)(b-a)^{n+1}}{3n+1)!2^n}, & n \ge 3 \end{cases}$$

which appeared in [2] without a proof of sharpness.

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If we take $\theta = \frac{1}{2}$ in (2.36), we get the sharp averaged midpoint-trapezoid type inequality

$$\begin{aligned}\bigg|\int_a^b f(x)\,dx - \frac{b-a}{4}\bigg[f(a) + 2f\bigg(\frac{a+b}{2}\bigg) + f(b)\bigg] \\ + \sum_{k=1}^{[(n-1)/2]} \frac{(2k-1)(b-a)^{2k+1}}{(2k+1)!2^{2k+1}} f^{(2k)}\bigg(\frac{a+b}{2}\bigg)\bigg| \\ \le \|f^{(n)}\|_\infty \times \begin{cases} \dfrac{1}{8}(b-a)^2, & n = 1, \\ \dfrac{(n-1)(b-a)^{n+1}}{(n+1)!2^{n+1}}, & n \ge 2. \end{cases}\end{aligned}$$

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