Proceedings of the Edinburgh Mathematical Society (1990) 33, 79–88 ©

KINEMATIC FORMULA AND TUBE FORMULA IN SPACE OF CONSTANT CURVATURE

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(Received 12th April 1988)

The Euler characteristic of an even dimensional submanifold in a space of constant curvature is given in terms of Weyl's curvature invariants. A derivation of Chern's kinematic formula in non-Euclidean space is completed. As an application of above results Weyl's tube formula about an odd-dimensional submanifold in a space of constant curvature is obtained.

1980 *Mathematics subject classifications* (1985 Revision): 53C20, 53C65.

1. Introduction

Let P be a compact orientable p-dimensional Riemannian manifold which is imbedded in n-dimensional non-Euclidean space $E^n(K)$ of constant curvature K (briefly $P\subset E^n(K)$). Denote by R_P and R_K the curvature tensor fields of P and $E^n(K)$ respectively.

In 1987 Ishihara [3] derived an interesting formula for the Euler characteristic $\chi(P)$. Let $p=\dim P$ be even. He showed that

$$\chi(P)=\frac{1}{(2\pi)^{p/2}}\sum_{0\leq i\leq p/2} K^i(2i-1)!!k_{p-2i}(R_P-R_K), \tag{1.1}$$

where $k_{2c}(R_P-R_K)$ are Weyl's curvature invariants (see Section 2) and $m!!=m(m-2)\dots4\cdot2$ or $m!!=m(m-2)\dots3\cdot1$ according as n is even or odd.

Ishihara's derivation of (1.1) relied on a Teufel's result [6]. In this article we give a simple proof of it using the exterior product of double forms and the contraction operator.

Ishihara [3] also mentioned the following result which generalizes Chern's kinematic formula in Euclidean space [1]. Let $P\subset E^n(K)$ and $Q\subset E^n(K)$ be compact submanifolds of dimensions p and q respectively. Let $E_K(n)$ be the group of proper motions of $E^n(K)$ and dg the standard kinematic density on $E_K(n)$. If $0\leqq e$ even $\leqq p+q-n$ and $g\in E_K(n)$, then

$$\int \mu_e(P\cap gQ,K)\,dg=\sum_{0\leq i\,\text{even}\leq e} c_{e,i}\mu_i(P,K)\mu_{e-i}(Q,K) \tag{1.2}$$

*Partially supported by KOSEF of Korea.

with constants $c_{e,i}$ depending on p, q, n, e and i. Here integral invariants $\mu_e(P,K)$ are related to $k_e(R_P-R_K)$ by

$$\mu_e(P,K)=\frac{2^{e/2}(p-e)!(e/2)!}{p!}k_e(R_P-R_K). \tag{1.3}$$

The second purpose of this article is to determine constants $c_{e,i}$ which Ishihara could not do. In fact $c_{e,i}$ in (1.2) are given by

$$c_{e,i}=O_{n+1}O_n\ldots O_2\frac{\dfrac{O_{p+q-n+1}O_{p+q-n+2}(e/2)!}{O_{p+q-n-e+2}}}{\left(\dfrac{O_{p+1}O_{p+2}(i/2)!}{O_{p-i+2}}\right)\left(\dfrac{O_{q+1}O_{q+2}((e-i)/2)!}{O_{q-e+i+2}}\right)}. \tag{1.4}$$

Here $O_j=2\pi^{j/2}/\Gamma(j/2)$ is the volume of unit sphere in Euclidean j-space.

As an application of the formulas (1.1) and (1.2) we derive Weyl's tube formula for a compact odd-dimensional submanifold $P\subset E^n(K)$. This derivation shows a close relationship among the Gauss–Bonnet theorem, Chern's kinematic formula and Weyl's tube formula.

2. New derivation of Ishihara's formula

We refer to [2] for basic facts on double forms. We shall say that a double form R of type $(2,2)$ which has the same symmetries as the curvature tensor of a Riemannian manifold P is *curvature-like*. Let R be a curvature-like tensor field on P. The complete contraction $C^{2c}(R^c)$ of $R^c=R\wedge\cdots\wedge R$ (c times) is then given by

$$C^{2c}(R^c)=\sum_{a_1,\ldots,a_{2c}=1}^{p}R^c(e_{a_1},\ldots,e_{a_{2c}})(e_{a_1},\ldots,e_{a_{2c}}) \tag{2.1}$$

where $\{e_1,\ldots,e_p\}$ is any orthonormal frame on P. We put

$$k_{2c}(R)=\frac{1}{c!(2c)!}\int_P C^{2c}(R^c)\,dP \tag{2.2}$$

where dP is the volume element of P. For the case $P\subset E^n(K)$, $k_{2c}(R_P-R_K)$ are Weyl's curvature invariants which appear in (1.1).

If p is even, then the Gauss–Bonnet theorem says that

$$k_p(R_P)=(2\pi)^{p/2}\chi(P). \tag{2.3}$$

In order to prove (1.1) we need the following lemmas.

Lemma 2.1. *Let R and S be curvature-like tensor fields on a Riemannian manifold P. Then*

$$C^{2c}((R+S)^c)=\sum_{i=0}^{c}\binom{c}{i}C^{2c}(R^{c-i}\wedge S^i). \tag{2.4}$$

Proof. This is a simple consequence of the binomial theorem and Bianchi identities.

The ith power R_K^i of the curvature operator R_K of $E^n(K)$ has the following properties.

Lemma 2.2. *For any subset $\{e_1,\dots,e_{2i}\}$ of an orthonormal frame $\{e_1,\dots,e_n\}$ on $E^n(K)$ we have*

$$R_K^i(e_1,\dots,e_{2i})(e_1,\dots,e_{2i})=K^i(2i)!/2^i. \tag{2.5}$$

Furthermore

$$C^{2i}(R_K^i)=K^i((2i)!)^2\binom{p}{2i}\Big/2^i. \tag{2.6}$$

Proof. From the definition of the double form multiplication $\wedge$ ([**2**, p. 158 (2.2)]) and from the property of R_K we have

$$R_K^i(e_1,\dots,e_{2i})(e_1,\dots,e_{2i})=K\sum_{\rho\in Sh(2i-2,2)}R_K^{i-1}(e_{\rho_1},\dots,e_{\rho_{2i-2}})(e_{\rho_1},\dots,e_{\rho_{2i-2}}).$$

Here $Sh(p,r)$ denote the set of all (p,r) shuffles. By an induction it follows that

$$R_K^i(e_1,\dots,e_{2i})(e_1,\dots,e_{2i})=K^i\binom{2i}{2}\binom{2i-2}{2}\cdots\binom{4}{2}=K^i(2i)!/2^i.$$

Then from (2.1) and (2.5) we obtain (2.6).

Lemma 2.3. *For $P\subset E^n(K)$ we have*

$$(R_P^{c-i}\wedge R_K^i)(e_1,\dots,e_{2c})(e_1,\dots,e_{2c})$$

$$=\frac{K^i(2i)!}{2^i}\sum_{\rho\in Sh(2c-2i,2i)}R_P^{c-i}(e_{\rho_1},\dots,e_{\rho_{2c-2i}})(e_{\rho_1},\dots,e_{\rho_{2c-2i}}), \tag{2.7}$$

and

https://doi.org/10.1017/S001309150002890X Published online by Cambridge University Press

$$C^{2c}((R_P-R_K)^{c-i}\wedge R_K^i)$$

$$=\binom{2c}{2i}\binom{p-2c+2i}{2i}C^{2c-2i}(R_P^{c-i})C^{2i}(R_K^i)\Big/\binom{p}{2i}. \tag{2.8}$$

Proof. From the definitions we have

$$R_P^{c-i}\wedge R_K^i(e_1,\ldots,e_{2c})(e_1,\ldots,e_{2c})$$

$$=K\sum_{\rho\in Sh(2c-2,2)}(R_P^{c-i}\wedge R_K^{i-1})(e_{\rho_1},\ldots,e_{\rho_{2c-2}})(e_{\rho_1},\ldots,e_{\rho_{2c-2}}).$$

By an induction it follows that

$$R_P^{c-i}\wedge R_K^i(e_1,\ldots,e_{2c})(e_1,\ldots,e_{2c})$$

$$=K^i\binom{2i}{2}\binom{2i-2}{2}\cdots\binom{4}{2}\sum_{\rho\in Sh(2c-2i,2i)}R_P^{c-i}(e_{\rho_1},\ldots,e_{\rho_{2c-2i}})(e_{\rho_1},\ldots,e_{\rho_{2c-2i}}).$$

This gives (2.7). Next from (2.1) and (2.7) we obtain

$$C^{2c}(R_P^{c-i}\wedge R_K^i)=\sum_{e_1,\ldots,e_{2c}=1}^{p}R_P^{c-i}\wedge R_K^i(e_1,\ldots,e_{2c})(e_1,\ldots,e_{2c})$$

$$=\sum_{e_1,\ldots,e_{2c}=1}^{p}\sum_{\rho\in Sh(2c-2i,2i)}R_P^{c-i}(e_{\rho_1},\ldots,e_{\rho_{2c-2i}})(e_{\rho_1},\ldots,e_{\rho_{2c-2i}})$$

$$\times R_K^i(e_{\rho_{2c-2i+1}},\ldots,e_{\rho_{2c}})(e_{\rho_{2c-2i+1}},\ldots,e_{\rho_{2c}})$$

$$=\sum_{\rho\in Sh(2c-2i,2i)}\left\{\sum_{e_{\rho 1},\ldots,e_{\rho 2c-2i}=1}^{p}R_P^{c-i}(e_{\rho_1},\ldots,e_{\rho_{2c-2i}})(e_{\rho_1},\ldots,e_{\rho_{2c-2i}})\right\}$$

$$\times\left\{\sum_{e\rho_{2c-2i+1},\ldots,e\rho_{2c}=1}^{p-2c+2i}R_K^i(e_{\rho_{2c-2i+1}},\ldots,e_{\rho_{2c}})(e_{\rho_{2c-2i+1}},\ldots,e_{\rho_{2c}})\right\}$$

$$=\binom{2c}{2i}\binom{p-2c+2i}{2i}C^{2c-2i}(R_P^{c-i})C^{2i}(R_K^i)\Big/\binom{p}{2i}.$$

Now we are ready to prove (1.1).

Proof of Ishihara's formula (1.1). Let $P \subset E^n(K)$ and $\dim P = 2c$ be even. Applying the Gauss–Bonnet theorem (2.3), and Lemmas 2.1, 2.2, 2.3 we have

$$
\begin{aligned}
(2\pi)^c \chi(P) &= k_{2c}(R_P) \\
&= \frac{1}{c!(2c)!} \int_P C^{2c}(R_P^c)\, dP \\
&= \frac{1}{c!(2c)!} \sum_{i=0}^{c} \binom{c}{i} \int_P C^{2c}((R_P - R_K)^{c-i} \wedge R_K^i)\, dP \\
&= \frac{1}{c!(2c)!} \sum_{i=0}^{c} \binom{c}{i} \int_P C^{2c-2i}((R_P - R_K)^{c-i}) C^{2i}(R_K^i)\, dP \\
&= \sum_{i=0}^{c} K^i (2i-1)!!\, k_{2c-2i}(R_P - R_K).
\end{aligned}
$$

3. Determination of constants $c_{e,i}$

We refer to [5] for basic facts on non-Euclidean integral geometry and use the notations of [1, 5]

In this section we determine the constants $c_{e,i}$ of (1.2) by evaluating the integral

$$A_e = \int \mu_e(S_o^p(a) \cap g S^q(b), K)\, dg, \tag{3.1}$$

where $S_o^p(a)$ is a fixed p-dimensional geodesic sphere of radius a in a $(p+1)$-plane $E^{p+1}(K)$ (briefly E^{p+1}) and $S^q(b)$ is a q-dimensional geodesic sphere of radius b in $E^n(K)$. Note that

$$\mu_e(S^{n-1}(a), K) = O_n \left(\frac{\sin \sqrt{K} a}{\sqrt{K}} \right)^{n-e-1} (\cos \sqrt{K} a)^e. \tag{3.2}$$

We prove the result for the elliptic space $E^n(K)$ where $K > 0$. The proof for the hyperbolic space $E^n(K)$ where $K < 0$ is similar. Assume $b < a < (\pi/6\sqrt{K})$. We begin with the following lemma.

Lemma 3.1. *Let $S_0^{m-1}(a)$ be a fixed geodesic sphere of radius a in $E^m(K)$ and $S^{m-1}(b)$ the geodesic sphere of radius b with center x. Let*

$$B_e = \int \mu_e(S_o^{m-1}(a) \cap S^{m-1}(b), K)\, dx, \tag{3.3}$$

where dx is the volume element of $E^m(K)$. *Then*

$$B_e = K^{(e/2)-m+1} \sum_{\substack{0 \leq i \leq e \\ i \text{ even}}} b_{e,m-1-i} (\cos\sqrt{K}a)^{e-i} (\sin\sqrt{K}a)^{m-e-1+i}$$

$$\times (\cos\sqrt{K}b)^i (\sin\sqrt{K}b)^{m-1+i}, \tag{3.4}$$

where $b_{e,m-1-i}$ *are given by*

$$b_{e,m-1-i} = \frac{O_{m-1} O_m O_{m+1-i} O_{m+1-e+i}}{O_{m+1} O_{m-e}} \binom{e/2}{i/2}.$$

Proof. Let ρ be the distance of x from the centre of $S_o^{m-1}(a)$. Then the radius y of $S_o^{m-1}(a) \cap S^{m-1}(b) = S^{m-2}(y)$ is given by

$$\cos^2\sqrt{K}y = \frac{1}{\sin^2\sqrt{K}\rho} (\cos^2\sqrt{K}a + \cos^2\sqrt{K}b - 2\cos\sqrt{K}a \cos\sqrt{K}b \cos\sqrt{K}\rho).$$

Applying (3.2) and (17.46) in [**5**, p. 307] we obtain

$$B_e = K^{(e/2)-m+1} O_m O_{m-1} \int_{-\sin\sqrt{K}a \sin\sqrt{K}b}^{\sin\sqrt{K}a \sin\sqrt{K}b} (\cos^2\sqrt{K}a \sin^2\sqrt{K}b + \cos^2\sqrt{K}b \sin^2\sqrt{K}a$$

$$- 2u \cos\sqrt{K}a \cos\sqrt{K}b)^{e/2} (\sin^2\sqrt{K}a \sin^2\sqrt{K}b - u^2)^{(m-e-2)/2}\, du.$$

Now from the calculations in [**4**, pp. 478–479] we get (3.4).

Theorem 3.2. *We have*

$$A_e = K^{(e-p-q)/2} \frac{O_{n+1} \cdots O_1 O_{p+q-n+3}}{O_1 O_{p+2} O_{q+2}} \sum_{\substack{0 \leq i \leq e \\ i \text{ even}}} b_{e,p+q-n+1-i} \frac{O_{q-i+2} O_{p+i-e+2}}{O_{p+q-n+3-i} O_{p+q-n+i-e+3}}$$

$$\times (\cos\sqrt{K}a)^{e-i} (\sin\sqrt{K}a)^{p+i-e} (\cos\sqrt{K}b)^i (\sin\sqrt{K}b)^{q-i}. \tag{3.5}$$

Proof. We apply Chern's argument [**1**, pp. 115–117] to $E^n(K)$ and evaluate the integral by iterations. Let E^{p+1} and E^{q+1} be planes which contain $S_o^p(a)$ and $gS^q(b)$ respectively, and let x be the centre of $gS^q(b)$. First from the fibering

$$E_K(n) \xrightarrow{\pi} \{(x, E^{q+1}) \mid x \in E^{q+1} \text{ unoriented}\}$$

it is not difficult to see that

$$A_e = 2O_{q+1} \cdots O_2 O_{n-q-1} \cdots O_2 \int \mu_e(S_o^p(a) \cap gS^q(b), K) d(x, E^{q+1}), \tag{3.6}$$

where

$$d(x, E^{q+1}) = dx(E^{q+1}) \wedge dE^{q+1(n)}$$

and $dE^{q+1(n)}$ is the density for $(q+1)$-planes in $E^n(K)$.

Next we fix $E^{q+1} \subset E^n(K)$ and integrate over $dx(E^{q+1})$. By transversality we may assume $\dim(E^{p+1} \cap E^{q+1}) = p+q-n+2$. Let $E^{p+1} \cap E^{q+1} = E^{p+q-n+2}$, and let $E^{n-p-1} = (E^{p+q-n+2})^{\perp}(E^{q+1})$ be the complement of $E^{p+q-n+2}$ in E^{q+1} through x. A point $x \in E^{q+1}$ can be coordinatized by its projections $x_1 \in E^{p+q-n+2}$ and $x_2 \in E^{n-p-1}$. Let s be the distance from x to $E^{p+q-n+2}$. The intersection $gS^q(b) \cap E^{p+q-n+2} = S^{p+q-n+1}(\rho) \subset E^{q+1}$ is a geodesic sphere of radius ρ, where $\cos\sqrt{K}\rho = \cos\sqrt{K}b / \cos\sqrt{K}s$. Then we have

$$\begin{aligned} dx(E^{q+1}) &= (\cos\sqrt{K}s)^{p+q-n+2} dx_1(E^{p+q-n+2}) \wedge dx_2(E^{n-p-1}) \\ &= K^{(-n+p+2)/2} (\cos\sqrt{K}s)^{p+q-n+2} (\sin\sqrt{K}s)^{n-p-2} dx_1 \wedge ds \wedge du_{n-p-1} \end{aligned} \tag{3.7}$$

where du_{n-p-1} is the solid angle element such that $\int du_{n-p-1} = O_{n-p-1}$. Using (3.7) we again calculate the integral by iteration. It follows that

$$\begin{aligned} A_e = {} & K^{(-n+p+2)/2} O_{q+1} \cdots O_1 O_{n-q-1} \cdots O_2 O_{n-p-1} \\ & \times \int dE^{q+1} \int_0^b (\cos\sqrt{K}s)^{p+q-n+2} (\sin\sqrt{K}s)^{n-p-2} ds \\ & \times \int \mu_e(S_o^{p+q-n+1}(r) \cap S^{p+q-n+1}(\rho), K)\, dx_1(E^{p+q-n+2}). \end{aligned}$$

Here $S_o^{p+q-n+1}(r) = S_o^p(a) \cap E^{p+q-n+2}$ is a geodesic sphere of radius r in $E^{p+q-n+2}$, $0 \leqq r \leqq a$. According to (3.4), we obtain, after simple computations,

$$\begin{aligned} A_e = {} & K^{(n+e-p-2q-1)/2} O_{q+1} \cdots O_1 O_{n-q-1} \cdots O_2 \\ & \times \sum_{\substack{0 \leqq i \leqq e \\ \text{even}}} b_{e,p+q-n+1-i} \frac{O_{q-i+2}}{O_{p+q-n-i+3}} (\cos\sqrt{K}b)^i (\sin\sqrt{K}b)^{q-i} \\ & \times \int (\cos\sqrt{K}r)^{e-i} (\sin\sqrt{K}r)^{p+q-n+1+i-e} dE^{q+1}. \end{aligned} \tag{3.8}$$

In order to evaluate the last integral we recall the following lemma due to Chern [1, p. 106].

Lemma 3.3 (Chern). *Let $p+q\geqq n$ and $F(E^p)$ be an integrable function which depends only on $E^{p+q-n(q)}=E^p\cap E^q$, where E^q is a fixed q-plane. Then we have*

$$\int F(E^p)\,dE^p=\frac{O_{n+1}\cdots O_1 O_{p+q-n+1}\cdots O_1}{O_{p+1}\cdots O_1 O_{q+1}\cdots O_1}\int F(E^{p+q-n(q)})\,dE^{p+q-n(q)}. \tag{3.9}$$

Applying (3.9) we obtain

$$\int(\cos\sqrt{K}r)^{e-i}(\sin\sqrt{K}r)^{p+q-n+1+i-e}\,dE^{q+1}$$
$$=\frac{O_{n+1}\cdots O_{p+3}}{O_{q+2}\cdots O_{p+q-n+4}}\int(\cos\sqrt{K}r)^{e-i}(\sin\sqrt{K}r)^{p+q-n+1+i-e}$$
$$\times\,dE^{p+q-n+2(p+1)}. \tag{3.10}$$

To integrate over $dE^{p+q-n+2(p+1)}$ let u be the distance of $E^{p+q-n+2}$ from the centre of $S_o^p(a)$. Then $\cos\sqrt{K}u=(\cos\sqrt{K}a/\cos\sqrt{K}r)$. Since we have

$$dE^{p+q-n+2(p+1)}=K^{(-n+q+2)/2}(\cos\sqrt{K}u)^{p+q-n+2}(\sin\sqrt{K}u)^{n-q-2}$$
$$\times\,du\wedge du_{n-q-1}\wedge dE_{[0]}^{n-q-1(p+1)},$$

where $dE_{[0]}^{n-q-1(p+1)}$ is the density for $(n-q-1)$-planes through the origin in E^{p+1}, we find, from (17.53b) in [5, p. 309],

$$\int(\cos\sqrt{K}r)^{e-i}(\sin\sqrt{K}r)^{p+q-n+1+i-e}\,dE^{p+q-n+2(p+1)}$$
$$=K^{(-n+q+1)/2}\frac{O_{p+1}\cdots O_1 O_{p+i-e+2}}{O_{n-q-1}\cdots O_1 O_{p+q-n+2}\cdots O_1 O_{p+q-n+i-e+3}}$$
$$\times(\cos\sqrt{K}a)^{e-i}(\sin\sqrt{K}a)^{p+i-e}. \tag{3.11}$$

Combining (3.8), (3.10) and (3.11) we obtain the final result (3.5).

Corollary 3.4.

$$c_{e,e-i}=\frac{O_{n+1}\cdots O_1 O_{p+q-n+3}O_{q+2-i}O_{p+2-e+i}}{O_1 O_{p+1}O_{p+2}O_{q+1}O_{q+2}O_{p+q-n+3-i}O_{p+q-n+3-e+i}}\,b_{e,p+q-n+1-i}. \tag{3.12}$$

Remark. (3.5) and (3.12) give (1.4).

4. New derivation of Weyl's tube formula for odd-dimensional manifolds in $E^n(K)$

Let M be a compact odd-dimensional imbedded submanifold of $E^n(K)$. Denote by $V_M^{E^n(K)}(r)$ the n-dimensional volume of the tube $T(M,r)$ of radius r about M and by $A_M^{E^n(K)}(r)$ the $(n-1)$-dimensional volume of the boundary of $T(M,r)$. In this section we will derive Weyl's tube formula for $A_M^{E^n(K)}(r)$ (see (4.4) below) by employing Chern's kinematic formula (1.2) in $E^n(K)$ and Ishihara's formula (1.1).

Let $\dim M = 2p+1$. We apply (1.2) with M as a stationary submanifold and with $S^{n-1}(r)$ as the moving submanifold of $E^n(K)$. Here $S^{n-1}(r)$ is a geodesic sphere of radius r, and $r>0$ is less than or equal to the distance from M to its nearest focal point. Let x be the centre of $gS^{n-1}(r)$, $g\in E_K(n)$. Since $E_K(n)$ is the semidirect product $E^n(K)\dot{\times} SO(n)$ we can write $gS^{n-1}(r)=g_0S_x^{n-1}(r)$, where $g_0\in SO(n)$ and $S_x^{n-1}(r)$ denotes the geodesic sphere of radius r with centre x. If $d(x,M)>r$, then $M\cap gS^{n-1}(r)$ is empty. Hence we can say that for $0\leqq e$ even $\leqq 2p$

$$\int \mu_e(M\cap gS^{n-1}(r),K)\,dg=\int_{T(M,r)}\left\{\int_{SO(n)}\mu_e(M\cap g_0S_x^{n-1}(r),K)\,dg_0\right\}dx, \tag{4.1}$$

where dg_0 is the Haar measure on $SO(n)$ normalized so that $\int_{SO(n)}dg_0=O_nO_{n-1}\dots O_2$, and $T(M,r)=\{x\in E^n(K)\,|\,d(x,M)\leqq r\}$. To evaluate the integral (4.1) we may assume $d(x,M)<r$ since the measure of the boundary of $T(M,r)$ is equal to 0. Then $\chi(M\cap g_0S_x^{n-1}(r))$ is 2 since $M\cap g_0S_x^{n-1}(r)$ is homeomorphic to an even-dimensional sphere, and

$$\int_{T(M,r)} dx = V_M^{E^n(K)}(r)=\int_0^r A_M^{E^n(K)}(s)\,ds.$$

Putting the formulas (1.1) and (1.2) together we obtain from (4.1)

$$V_M^{E^n(K)}(r)=\frac{1}{2O_n\dots O_2(2\pi)^p}\sum_{i=0}^{p}\left\{\sum_{k=0}^{p-i}K^k(2k-1)!!(2p-2k-1)!!\binom{2p}{2p-2k}c_{2p-2k,2i}\right.$$
$$\left.\times\,\mu_{2p-2k-2i}(S^{n-1}(r),K)\right\}\mu_{2i}(M,K). \tag{4.2}$$

According to (3.3), differentiating (4.2) with respect to r and applying

$$(2k-1)!!(2p-2k-1)!!\binom{2p}{2p-2k}(2p-2k-2i)c_{2p-2k,2i}$$
$$=K(2k+1)!!(2p-2k-3)!!(n+1-2p+2k+2i)c_{2p-2k-2,2i} \tag{4.3}$$

for $0\leqq k\leqq p-i-1$, we have

$$A_M^{E^n(K)}(r)=\frac{1}{2O_{n-1}\cdots O_2(2\pi)^p}\sum_{i=0}^{p}(2p-1)!!(n-1-2p+2i)c_{2p,2i}\mu_{2i}(M,K)$$
$$\times\left(\frac{\sin\sqrt{K}r}{\sqrt{K}}\right)^{n-2-2p+2i}(\cos\sqrt{K}r)^{2p+1-2i}.$$

By a simple calculation we finally get

$$A_M^{E^n(K)}(r)=\sum_{i=0}^{p}\frac{O_{n-(2p+1)+2i}k_{2i}(R_P-R_K)}{(2\pi)^i}\left(\frac{\sin\sqrt{K}r}{\sqrt{K}}\right)^{n-1-(2p+1)+2i}(\cos\sqrt{K}r)^{2p+1-2i}. \quad (4.4)$$

REFERENCES

1. S. S. Chern, On the kinematic formula in integral geometry, *J. Math. Mech.* **16** (1966), 101–118.

2. A. Gray, Some relations between curvature and characteristic classes, *Math. Ann.* **184** (1970), 257–267.

3. T. Ishihara, The Euler characteristics and Weyl's curvature invariants of submanifolds in spheres, *J. Math. Soc. Japan* **39**(2) (1987), 247–256.

4. A. Nijenhuis, On Chern's kinematic formula in integral geometry, *J. Differential Geom.* **9** (1974), 475–482.

5. L. A. Santalo, *Integral Geometry and Geometric Probability* (Encyclopedia of Mathematics and its Applications, Vol. 1, Addison-Wesley, 1976).

6. E. Teufel, Anwendung der differentialtopologischen Berechnung der Totalen Krümmung und Totalen Absolutkrümmung in der sphärischen Differentialgeometrie, *Manuscripta Math.* **32** (1980), 239–262.

7. H. Weyl, On the volume of tubes, *Amer. J. Math.* **61** (1939), 461–472.

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