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Abstract

Let $(R, m)$ be a Noetherian local ring and $U_R = \mathrm{Spec}(R) - \{m\}$ be the punctured spectrum of $R$. Gabber conjectured that if $R$ is a complete intersection of dimension three, then the abelian group $\mathrm{Pic}(U_R)$ is torsion-free. In this note we prove Gabber's statement for the hypersurface case. We also point out certain connections between Gabber's conjecture, Van den Bergh's notion of non-commutative crepant resolutions and some well-studied questions in homological algebra over local rings.

## 1. Introduction

Let $(R, m)$ be a local ring (always Noetherian in this note). Let $U_R = \mathrm{Spec}(R) - \{m\}$ be the punctured spectrum of $R$. In [Gab04] Gabber made the following conjecture.

Conjecture 1.1. Let $R$ be a local complete intersection of dimension three. Then $\mathrm{Pic}(U_R)$ is torsion-free.

The above conjecture is equivalent to the statement that the local flat cohomology group $H^2_{\{m\}}(\mathrm{Spec}(R), \mu_n) = 0$ when $R$ is a local complete intersection of dimension three, and they are both implied by the following conjecture(for more details, see [Gab04]).

Conjecture 1.2. Let $R$ be a strictly henselian local complete intersection of dimension at least four. Then the cohomological Brauer group of $U_R$ vanishes: $Br(U_R) = 0$.

Conjecture 1.1 is known when $R$ contains a field; the characteristic 0 case follows from Grothendieck's techniques on local Lefschetz theorems (cf. [Bad78, Rob76]), and the positive characteristic case can be found in [DLM10]. (It is probably known to experts, though we cannot find an exact reference. It was claimed in [Gab04] that Conjecture 1.2 is known in the positive characteristic case.) We also note that when $U_R$ is replaced by a smooth projective complete intersection the analogous result on the Picard group is contained in [Del73, Theorem 1.8]. In any case, the main difficulty is when $R$ is of mixed characteristic.

In this paper we give a short and relatively self-contained proof of Gabber's Conjecture 1.1 for the case of hypersurfaces, that is, if $\hat{R} \cong T/(f)$ where $T$ is a complete regular local ring. In fact, in this situation we shall prove a stronger result which is a pure commutative algebra statement. To state such a result let us recall a useful notion. For a Noetherian ring $R$ one can define a map $c_1 : G(R) \to \mathrm{CH}^1(R)$ from the Grothendieck group of finitely generated modules over $R$ to the height one component of the Chow group of $\mathrm{Spec}(R)$ (see § 2.2 for more details).

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Given an $R$-module $M$, we shall abuse notation a bit and call $c_1([M])$ the first local Chern class of $M$. Then our main result is the following.

Theorem 1.3. *Let $R$ be local hypersurface of dimension three. Let $N$ be a finitely generated reflexive $R$-module which is locally free on $U_R$. Furthermore, assume that the first local Chern class of $N$ is torsion in $\mathrm{CH}^1(R)$. Then $\mathrm{Hom}_R(N, N)$ is a maximal Cohen–Macaulay $R$-module if and only if $N$ is free.*

It is not hard to see that the above theorem implies Conjecture 1.1 in the hypersurfaces case, by taking $N$ to be the $R$-module generated by the sections of a torsion element in $\mathrm{Pic}(U_R)$; see § 2 and the proof of Corollary 3.5 for more details.

This project actually arises from our attempt to understand a striking definition by Van den Bergh of non-commutative crepant resolutions of a Gorenstein local ring $R$. To explain the connection we recall the following definition.

Definition 1.4 (Van den Bergh [Ber04]). Suppose that there exists a reflexive module $N$ satisfying the following conditions.

(1) $A = \mathrm{Hom}_R(N, N)$ is a maximal Cohen–Macaulay $R$-module.

(2) $A$ has a finite global dimension equal to $d = \dim R$.

Then $A$ is called a non-commutative crepant resolution (henceforth NCCR) of $R$.

In [Dao10] we proved that non-commutative crepant resolutions cannot exist when $R$ is a dimension three, equicharacteristic or unramified hypersurface with isolated singularity and torsion class group. Theorem 1.3 implies the following corollary.

Corollary 1.5. *Let $R$ be a dimension three hypersurface which has an isolated singularity and a torsion class group (which in this case is equivalent to $R$ being a unique factorization domain, by our main results). Then $R$ has no non-commutative crepant resolution in the sense of Van den Bergh.*

We now briefly describe the organization of the paper. Section 2 deals with preliminary materials. In § 3 we give the proofs of the main results announced above as well as some other interesting applications. Finally, in § 4 we raise some open questions relevant to our approach to Gabber's conjecture.

## 2. Notations and preliminary results

Throughout the note $R$ will be a Noetherian local ring. Recall that a maximal Cohen–Macaulay (MCM) $R$-module $M$ is a finitely generated module satisfying depth $M = \dim R$.

Let $\mathrm{mod}(R)$ and $\mathrm{MCM}(R)$ be the category of finitely generated and finitely generated maximal Cohen–Macaulay $R$-modules, respectively. Suppose $X$ is a Noetherian scheme. Let $\mathfrak{Coh}(X)$ denote the category of coherent sheaves on $X$ and $\mathfrak{Vect}(X)$ the subcategory of vector bundles on $X$. By $G(X), \mathrm{Pic}(X), \mathrm{CH}^i(X), \mathrm{Cl}(X)$ we shall denote the Grothendieck group of coherent sheaves on $X$, the Picard group of invertible sheaves on $X$, the Chow group of codimension $i$ irreducible, closed subschemes of $X$, and the class group of $X$, respectively. When $X = \mathrm{Spec}\, R$ we shall write $G(R), \mathrm{Pic}(R), \mathrm{CH}^i(R), \mathrm{Cl}(R)$. Let $\overline{G}(R) := G(R)/\mathbb{Z}[R]$ be the reduced Grothendieck group and $\overline{G}(R)_{\mathbb{Q}} := \overline{G}(R) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the reduced Grothendieck group of $R$ with rational coefficients.

### 2.1 Vector bundles on $U_R$ and modules over $R$

Let $\Gamma_X$ be the section functor on $X$. We have the following proposition.

Proposition 2.1 (Horrocks [Hor64, § 1]). *Let $R$ be a Noetherian local ring such that* depth $R \geqslant 2$. *Let $X = U_R$. Then $\Gamma_X$ induces an equivalence of categories between $\mathfrak{Vect}(X)$ and the subcategory of* $\mathrm{mod}(R)$ *consisting of finitely generated modules $M$ which is locally free on non-maximal primes with* depth $M \geqslant 2$ *(note that the condition* depth $R \geqslant 2$ *also ensures that $X$ is connected).*

In particular, let $\mathcal{E}$ represent an element in $\mathrm{Pic}(X)$ and let $I = \Gamma_X(\mathcal{E})$. We know that $I$ is a reflexive ideal in $R$ which is locally free of rank 1 on $X$. Furthermore $\mathrm{Hom}_R(I, I) \cong \Gamma_X(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \cong \Gamma_X(\mathcal{O}_X) = R$.

### 2.2 Some maps between Chow, Picard, and Grothendieck groups

In this subsection we assume that $R$ is a local ring such that depth $R \geqslant 2$. For $i = 0, 1$ there are maps $c_i : G(R) \to \mathrm{CH}^i(R)$. These maps admit a very elementary definition as follows. Suppose $M$ is an $R$-module. Pick any prime filtration $\mathcal{F}$ of $M$. Then one can take $c_i([M]) = \sum [R/p]$, where $p$ runs over all prime ideals such that $R/p$ appears in $\mathcal{F}$ and $\mathrm{height}(p) = i$. Note that a prime can occur multiple times in the sum (for a proof that this is well defined see the main theorem of [Cha99]). When $R$ is a normal algebra, essentially of finite type over a field, and $N$ is locally free (i.e. a vector bundle) on $U_R$, $c_1$ agrees with the first Chern class of $N$, as defined in [Ful98, Chapter 3], but we shall not need that fact.

One has the following diagram of maps of abelian groups.

$$\begin{array}{ccc} & & \mathrm{Pic}(U_R) \\ & & \big\downarrow p \\ G(R) & \xrightarrow{c_1} & \mathrm{CH}^1(R) \end{array}$$

Here $p$ is induced by the well-known map between Cartier and Weil divisors (see [Ful98, ch. 2]).

Note that we do not indicate any map between $\mathrm{Pic}(U_R)$ and $G(R)$. However, the diagram 'commutes' in a weak sense: if $\mathcal{E}$ represents an element in $\mathrm{Pic}(X)$ and $I = \Gamma_X(\mathcal{E})$ then $p([\mathcal{E}]) = c_1([I])$ in $\mathrm{CH}^1(R)$.

Obviously, $c_1([R]) = 0$, so $c_1$ induces a map $q : \overline{G}(R) \to \mathrm{CH}^1(R)$. In particular, if $M$ is a module such that $[M] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$ then $c_1([M])$ is a torsion element in $\mathrm{CH}^1(R)$.

### 2.3 Maximal Cohen–Macaulay approximations

The reference for this subsection is the paper [AB89]. Suppose that $R$ is Cohen–Macaulay and a homomorphic image of a Gorenstein ring. For any $R$-module $N$ there exists a short exact sequence,

$$0 \to W \to M \to N \to 0, \tag{2.1}$$

such that $M \in \mathrm{MCM}(R)$ and $W$ has finite injective dimension. Note that if $R$ is Gorenstein, then $\mathrm{pd}_R W < \infty$. Also, if $R$ is Gorenstein and depth $N \geqslant \dim R - 1$, then, by counting depth and the Auslander–Buchsbaum formula, $W$ must be free.

### 2.4 Hochster's theta function

Let $R$ be a local hypersurface, so $\hat{R} = T/(f)$ where $T$ is a regular local ring. Suppose that $M$ is an $R$ module such that $\mathrm{pd}_{R_p} M_p < \infty$ for any $p \in U_R$. Then for any $R$-module $N$, $\ell(\mathrm{Tor}_i^R(M,N)) < \infty$ for $i \gg 0$; here $\ell(-)$ denotes length. The function $\theta^R(M,N)$ was introduced by Hochster [Hoc81] to be

$$\theta^R(M,N) = \ell(\mathrm{Tor}_{2e+2}^R(M,N)) - \ell(\mathrm{Tor}_{2e+1}^R(M,N))$$

where $e$ is any integer such that $2e \geqslant \dim R$. It is well known (see [Eis80]) that the sequence of modules $\{\mathrm{Tor}_i^R(M,N)\}$ is periodic of period 2 for $i > \operatorname{depth} R - \operatorname{depth} M$, so this function is well defined. The theta function satisfies the following properties.

Proposition 2.2 (Hochster [Hoc81]). (1) *If $M \otimes_R N$ has finite length, then*

$$\theta^R(M,N) = \chi^T(M,N) := \sum_{i \geqslant 0} (-1)^i \ell(\mathrm{Tor}_i^T(M,N)).$$

*Here $\chi^T$ is the well-known Serre's intersection multiplicity. In particular, if $\dim M + \dim N \leqslant \dim R = \dim T - 1$, then $\theta^R(M,N) = 0$ (note that vanishing for $\chi^T$ is proved for all regular local rings; see [Rob98, § 13.1]).*

(2) *$\theta^R(M,N)$ is bi-additive on short exact sequences, assuming it is defined on all pairs. In particular, if $M$ is locally of finite projective dimension on $U_R$, then the rule: $[N] \mapsto \theta^R(M,N)$ induces maps $\overline{G}(R) \to \mathbb{Z}$ and $\overline{G}(R)_{\mathbb{Q}} \to \mathbb{Q}$.*

The following elementary but useful result will be used in the proof of our main theorem.

Lemma 2.3 [Dao10, Lemma 2.3]. *Let $R$ be a Cohen–Macaulay local ring, $M$, $N$ finitely generated $R$-modules and $n > 1$ an integer. Consider the following two conditions.*

(1) $\mathrm{Hom}(M,N)$ *satisfies Serre's condition* $(\mathrm{S}_{n+1})$.

(2) $\mathrm{Ext}_R^i(M,N) = 0$ *for* $1 \leqslant i \leqslant n-1$.

*If $M$ is locally free in codimension $n$ and $N$ satisfies $(\mathrm{S}_n)$, then (1) implies (2). If $N$ satisfies $(\mathrm{S}_{n+1})$, then (2) implies (1).*

Finally we shall need a refined version of the Bourbaki sequence for a module.

Theorem 2.4 [HWJ01, Theorem 1.4]. *Let $R$ be a commutative, Noetherian ring satisfying condition $(\mathrm{S}_2)$. Let $M$ be a torsion-free $R$-module and $S$ be a finite set of prime ideals of $R$. Assume that $M$ is free and of constant rank on both $S$ and the set of primes in $R$ that have height at most 1. Then there is a Bourbaki sequence $0 \to F \to M \to I \to 0$ such that $I \nsubseteq \bigcup_{P \in S} P$.*

## 3. Main results

Throughout this section, $R$ will be a local hypersurface of dimension three. All modules are finitely generated. Note that since $\operatorname{depth} R > 2$, $U_R$ is connected, so any module which is locally free on $U_R$ also has constant rank.

Proposition 3.1. *Let $M$ be reflexive $R$-module which is locally free of constant rank on $U_R$. Let $N$ be an $R$-module which is locally free of constant rank on the minimal primes of $R$ and such that $c_1([N])$ is a torsion in $\mathrm{CH}^1(R)$. Then $\theta^R(M,N) = 0$.*

*Proof.* Without loss of generality one can assume $c_1([N])=0$ by replacing $N$ with a direct sum of copies of $N$ if necessary. First we claim that in $\overline{G}(R)_{\mathbb{Q}}$, the reduced Grothendieck group with rational coefficients, we have an equality $[N]=\sum a_i[R/P_i]$ such that each $P_i \in \operatorname{Spec} R$ has height at least 2. Since $N$ has constant rank $a$ we have a short exact sequence:

$$0 \to R^a \to N \to N' \to 0$$

where $N'$ is a torsion module. Let $\mathcal{F}$ be a prime filtration of $N'$. Clearly $\mathcal{F}$ involves only primes of height at least 1. Let $s$ be the formal sum of all height 1 primes in $\mathcal{F}$. Since $c_1([N']=c_1([N])=0$ we have formally (see § 2.2)

$$s=\sum n_j \operatorname{div}(f_j, R/q_j).$$

Here the $n_j$ are integers, each $q_j$ is a minimal prime of $R$ and $f_j$ is a regular element in $R/q_j$ and by definition

$$\operatorname{div}(f_j, R/q_j)=\sum \ell(R/(q_j, f_j)_p)[R/p]$$

(the sum runs over all primes of height 1 in $\operatorname{Supp}(R/(q_j, f_j))$. The above formal equality shows that in $G(R)$ one has

$$[N']=\sum n_j[R/(q_j, f_j)]+\sum a_i[R/p_i]$$

such that all the primes $p_i$ are of height at least 2 and the $a_i$ are integers. However, the exact sequence $0 \to R/q_j \to R/q_j \to R/(f_j, q_j) \to 0$ shows that each $[R/(q_j, f_j)]=0$ in $G(R)$, so our claim follows.

Because of the claim above we will have completed the proof by showing that $\theta^R(M, R/P)=0$ for each $P \in \operatorname{Spec} R$ such that height $P \geqslant 2$.

By Theorem 2.4 one can construct a Bourbaki sequence for $M$:

$$0 \to F \to M \to I \to 0$$

such that $I \subsetneq P$. Obviously $\theta^R(M, R/P)=\theta^R(I, R/P)$. However, $R/I \otimes_R R/P$ has finite length, and $\dim R/I+\dim R/P \leqslant 3=\dim R$. By Proposition 2.2 $\theta^R(R/I, R/P)=0$. Since $\theta^R(I, R/P)=-\theta^R(R/I, R/P)$ we are done. □

PROPOSITION 3.2. *Let $M \in \operatorname{MCM}(R)$ such that $M$ is locally free on $U_R$ and $N$ be any finitely generated $R$-module. Suppose that $\theta^R(M^*, N)=0$. If $\operatorname{Ext}^1_R(M, N)=0$ then $M$ is free or $\operatorname{pd}_R N<\infty$.*

*Proof.* One has the following short exact sequence (see [Har98, 3.6] or [Jor08, Jot75]):

$$\operatorname{Tor}_2^R(M_1, N) \to \operatorname{Ext}^1_R(M, R) \otimes_R N \to \operatorname{Ext}^1_R(M, N) \to \operatorname{Tor}_1^R(M_1, N) \to 0.$$

Here $M_1$ is the cokernel of $F_1^* \to F_2^*$, where $\mathbf{F}: \cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ is a minimal resolution of $M$. Since $\operatorname{Ext}^1_R(M, N)=0$ it follows that $\operatorname{Tor}^1_R(M_1, N)=0$.

Since $M$ is MCM and $R$ is a hypersurface we know that the minimal resolution $\mathbf{F}$ is periodic of period at most 2 (see [Eis80] and $\operatorname{Ext}^i_R(M, R)=0$ for $i>0$). It follows that the dual complex $F^*$ is also exact and periodic of period at most 2. Thus $M_1$ is isomorphic to the first syzygy of $M^*$. In particular, $M_1$ is maximal Cohen–Macaulay or zero. Since $\theta^R(M_1, N)=-\theta^R(M^*, N)=0$, it now follows that $\operatorname{Tor}_i^R(M_1, N)=0$ for all $i>0$ (as $M_1$ is maximal Cohen–Macaulay, the sequence of modules $\{\operatorname{Tor}_i^R(M_1, N)\}$ is periodic of period 2 for $i>0$). So either $M_1$ or $N$ has finite projective dimension by [HW97, Theorem 1.9] or [Mil98, 1.1]. However, if $M_1$ has finite projective dimension and is non-zero, it must be free by the Auslander–Buchsbaum formula, contradicting the minimality of $\mathbf{F}$. Thus $M_1$ is zero and $M$ must be free. □

Corollary 3.3. *Let* $(R, \mathfrak{m})$ *be local hypersurface of dimension three. Let* $N$ *be a reflexive* $R$*-module which is locally free on* $U_R$*. Assume that* $\theta^R(N^*, N) = 0$*. Then* $\mathrm{Hom}_R(N, N) \in \mathrm{MCM}(R)$ *if and only if* $N$ *is free.*

*Proof.* The sufficient direction is trivial. Suppose that $\mathrm{Hom}_R(N, N)$ is maximal Cohen–Macaulay. Then by Lemma 2.3 $\mathrm{Ext}^1_R(N, N) = 0$. We look at the MCM approximation of $N$ as in § 2.3:

$$0 \to W \to M \to N \to 0.$$

As the discussion in § 2.3 indicates, $W$ is free. Applying $\mathrm{Hom}_R(-, N)$ we obtain $\mathrm{Ext}^1_R(M, N) = 0$. Also, applying $\mathrm{Hom}_R(-, R)$ yields

$$0 \to N^* \to M^* \to W^* \to \mathrm{Ext}^1_R(N, R) \to 0,$$

since $\mathrm{Ext}^1_R(M, R) = 0$ because its Matlis dual is $\mathrm{H}^2_{\mathfrak{m}}(M) = 0$. Note that $\theta^R(L, -)$ is always defined if $L$ is any of the four modules in the above exact sequence, and it is 0 when $L = W^*$ or $L = \mathrm{Ext}^1_R(N, R)$ (the latter is because $\mathrm{Ext}^1_R(N, R)$ has finite length and Proposition 2.2). So $\theta^R(M^*, N) = \theta^R(N^*, N) = 0$.

Proposition 3.2 shows that either $M$ is free or $\mathrm{pd}_R N < \infty$. Both possibilities imply that $\mathrm{pd}_R N < \infty$. As $N$ is reflexive and $\dim R = 3$, $\mathrm{pd}_R N \leqslant 1$. However, $\mathrm{Ext}^1_R(N, N) = 0$, so $\mathrm{pd}_R N$ cannot be 1 by Nakayama's Lemma, and thus $N$ is free. □

We have derived enough to prove our main result.

Theorem 3.4. *Let* $R$ *be local hypersurface of dimension three. Let* $N$ *be a reflexive* $R$*-module which is locally free on* $U_R$*. Furthermore, assume that the image* $c_1([N])$ *(of* $N$ *as an element in* $G(R)$*) is a torsion in* $\mathrm{CH}^1(R)$*. Then* $\mathrm{Hom}_R(N, N) \in \mathrm{MCM}(R)$ *if and only if* $N$ *is free.*

*Proof.* A combination of Proposition 3.1 and Corollary 3.3 give the desired result. □

Corollary 3.5. *Let* $R$ *be local hypersurface of dimension three. Then* $\mathrm{Pic}\, U_R$ *is torsion-free.*

*Proof.* Let $\mathcal{E}$ represent a torsion element in $\mathrm{Pic}\, U_R$. By § 2.1 $I = \Gamma_X(\mathcal{E})$ is a reflexive ideal which is locally free of rank 1 on $U_R$. By the diagram in § 2.2 we know that $c_1([I])$ is torsion in $\mathrm{CH}^1(R)$. Theorem 3.4 now applies directly to give the desired result. □

Finally we note some interesting consequences of the main results above in the following theorem.

Theorem 3.6. *Let* $R$ *be a local hypersurface with an isolated singularity and* $\dim R = 3$*. The following statements are equivalent.*

(1) $\theta^R(M, N) = 0$ *for all* $M, N \in \mathrm{mod}(R)$.

(2) $R$ *is a unique factorization domain (equivalently,* $\mathrm{CH}^1(R) = \mathrm{Cl}(R) = 0$*).*

(3) *The class group* $\mathrm{Cl}(R)$ *is a torsion.*

*Proof.* First, since $R$ is local and normal (by Serre's criterion), it is a domain (see [Mat86, Theorem 23.8]). Assume (1). Let $I$ be a reflexive ideal representing an element of $\mathrm{Cl}(R)$. Then $\mathrm{Hom}_R(I, I) \cong R$, and Corollary 3.3 implies $I$ is principal, so $\mathrm{Cl}(R) = 0$.

The implication (2) $\Rightarrow$ (1) follows from Proposition 3.1. The equivalence (2) $\Leftrightarrow$ (3) is implied by the Corollary 3.5. □

*Remark.* If $\hat{R}$ is a hypersurface in an equicharacteristic or unramified regular local ring then the above result follows from [Dao, Corollary 3.5] and [Dao08, Theorem 3.4]. We also note that the equivalence (1) $\Leftrightarrow$ (3) when $k=\mathbb{C}$ and $R$ is graded is derived in [MPSW10, 3.10, 6.1].

## 4. Open questions

In this section we discuss some open questions motivated by the results obtained previously. Clearly, the most important question is whether or not Theorem 1.3 is true when $R$ is a local complete intersection of dimension three. Affirmation of such a statement would immediately prove Gabber's Conjecture 1.1. Since for an $R$-module $M$, $c_1([M])$ will be a torsion in $\mathrm{CH}^1(R)$ if $[M]=0$ in $\overline{G}(R)_{\mathbb{Q}}$, a possibly weaker but somewhat less technical version would be the following conjecture.

Conjecture 4.1. Let $R$ be local complete intersection of dimension three. Let $N$ be a reflexive $R$-module which is locally free of constant rank on $U_R$. Furthermore, assume that $[N]=0$ in $\overline{G}(R)_{\mathbb{Q}}$, the reduced Grothendieck group of $R$ with rational coefficients. Then $\mathrm{Hom}_R(N,N)$ is a maximal Cohen–Macaulay $R$-module if and only if $N$ is free.

In view of the proof of the key Proposition 3.2 and previously known results for regular and hypersurface rings, we feel it is reasonable to make the following conjecture.

Conjecture 4.2. Let $R$ be local complete intersection (of arbitrary dimension). Let $M,N$ be $R$-modules such that $M$ is locally free of constant rank on $U_R$ and $[N]=0$ in $\overline{G}(R)_{\mathbb{Q}}$. Then $(M,N)$ is Tor-rigid, in the sense that for any $i>0$, $\mathrm{Tor}_i^R(M,N)=0$ forces $\mathrm{Tor}_j^R(M,N)=0$ for $j\geqslant i$.

By going through the proofs of Propositions 3.2 and 3.3 one can see easily that an affirmative answer to Conjecture 4.2 (in dimension three) would imply Conjecture 4.1.

Tor-rigidity has been a subject of active investigation in commutative algebra. For more in-depth discussion and references, we refer the reader to the introduction of [Dao] and the bibliography there. It is well known that if $R$ is regular then Tor-rigidity holds for any pair of modules from the work of Auslander and Lichtenbaum [Aus61, Lic66]. Furthermore, Conjecture 4.2 is known when $\hat{R}$ is a hypersurface in an equicharacteristic or unramified regular local ring, see [Dao, Dao08]. A simple unknown case is when $M,N$ are zero-dimensional; in such a situation, the conditions on $M$ and $N$ are automatic, so the above conjecture would just say that any pair of finitely generated $R$-modules of finite length over a local complete intersection is Tor-rigid. The case of Conjecture 4.2 when one of the modules has finite length is discussed in the last section of [Dao08], and is still open to the best of our knowledge.

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