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Equations and Complexity for the Dubois–Efroymson Dimension Theorem

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Abstract. Let R be a real closed field, let $X \subset R^n$ be an irreducible real algebraic set and let Z be an algebraic subset of X of codimension ≥ 2. Dubois and Efroymson proved the existence of an irreducible algebraic subset of X of codimension 1 containing Z. We improve this dimension theorem as follows. Indicate by μ the minimum integer such that the ideal of polynomials in $R[x_1, \dots, x_n]$ vanishing on Z can be generated by polynomials of degree $\leq \mu$. We prove the following two results: (1) There exists a polynomial $P \in R[x_1, \dots, x_n]$ of degree $\leq \mu+1$ such that $X \cap P^{-1}(0)$ is an irreducible algebraic subset of X of codimension 1 containing Z. (2) Let F be a polynomial in $R[x_1, \dots, x_n]$ of degree d vanishing on Z. Suppose there exists a nonsingular point x of X such that $F(x) = 0$ and the differential at x of the restriction of F to X is nonzero. Then there exists a polynomial $G \in R[x_1, \dots, x_n]$ of degree $\leq \max\{d, \mu + 1\}$ such that, for each $t \in (-1, 1) \setminus \{0\}$, the set $\{x \in X \mid F(x) + tG(x) = 0\}$ is an irreducible algebraic subset of X of codimension 1 containing Z. Result (1) and a slightly different version of result (2) are valid over any algebraically closed field also.

1 The Theorems

Let R be a fixed real closed field. Let X and Z be algebraic subsets of R^n such that X is irreducible, Z is contained in X and $\dim(X) - \dim(Z) \geq 2$. In [1], Dubois and Efroymson proved the existence of a polynomial P in $R[x_1, \dots, x_n]$ such that the set $X \cap P^{-1}(0)$ is an irreducible algebraic subset of X of codimension 1 containing Z.

In this paper, we give an upper bound for the degree of P and we establish simple conditions for a polynomial $F \in R[x_1, \dots, x_n]$ to be approximated by polynomials $G \in R[x_1, \dots, x_n]$ such that $X \cap G^{-1}(0)$ is an irreducible algebraic subset of X of codimension 1 containing Z. Moreover, we extend these results to higher codimensions.

Let X be a real algebraic set, *i.e.*, an algebraic subset of some R^n. We indicate by $\mathscr{I}_{R^n}(X)$ the ideal of polynomials in $R[x_1, \dots, x_n]$ vanishing on X and by $\mathrm{Nonsing}(X)$ the set of nonsingular points of X of maximum dimension, *i.e.*, of dimension $\dim(X)$. An algebraic subset Z of R^n contained in X is called an algebraic subset of X. The integer $\dim(X) - \dim(Z)$ is called the codimension of Z in X. The empty set is considered to be an algebraic subset of X of codimension $\dim(X)$. Let e be a positive integer and let $F = (F_1, \dots, F_e) \colon X \to R^e$ be a map. Recall that F is said to be polynomial if there exist polynomials $P_1, \dots, P_e$ in $R[x_1, \dots, x_n]$ such that $P_i = F_i$ on X for each $i \in \{1, \dots, e\}$. Suppose F is polynomial. We define the *degree* $\deg(F)$ *of* F as the minimum integer d such that there exist polynomials $P_1, \dots, P_e$ in $R[x_1, \dots, x_n]$ of degree $\leq d$, which coincide with $F_1, \dots, F_e$ on X respectively. If F vanishes on

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whole X, then we consider $\deg(F)$ equal to zero. Let T be a subset of $\mathrm{Nonsing}(X)$. We say that F is *good in* T if $T \cap F^{-1}(0)$ is nonempty and, for each $x \in T \cap F^{-1}(0)$, the rank of the differential of F at x is equal to e. The map F is said to be *admissible* if, for some $x \in \mathrm{Nonsing}(X)$, F is good in $\{x\}$, *i.e.*, $F(x) = 0$ and the rank of the differential of F at x is equal to e.

Let us introduce the notion of μ-complexity of a real algebraic set.

Definition 1.1 Let Z be a proper algebraic subset of R^n. We define the μ-complexity $\mu(Z, R^n)$ of Z in R^n as the minimum integer μ such that there exist generators of $\mathscr{I}_{R^n}(Z)$ in $R[x_1, \dots, x_n]$ of degree $\leq \mu$.

The preceding notions, given over R, can be reformulated identically over any field.

We are now in a position to state the main result of this paper (see Remark 3.3 also).

Theorem 1.2 *Let $X \subset R^n$ be an irreducible real algebraic set, let Z be an algebraic subset of X of codimension $c \geq 2$ and let $e \in \{1, \dots, c-1\}$. Define $\mu := \mu(Z, R^n)$ and $X^* := \mathrm{Nonsing}(X) \setminus Z$.*

(1) *There exists a polynomial map $P\colon X \to R^e$ of degree $\leq \mu + 1$ and good in X^* such that $P^{-1}(0)$ is an irreducible algebraic subset of X of codimension e containing Z.*
(2) *Given an admissible polynomial map $F\colon X \to R^e$ of degree d vanishing on Z, there exists a polynomial map $G\colon X \to R^e$ of degree $\leq \max\{d, \mu + 1\}$ such that, for each $t \in (-1, 1) \setminus \{0\}$, the polynomial map $F_t\colon X \to R^e$ defined by $F_t := F + tG$ is good in X^* and $(F_t)^{-1}(0)$ is an irreducible algebraic subset of X of codimension e containing Z.*

Observe that, if Z is empty, then $\mu = 0$ and the preceding result follows easily from Bertini's theorems applied to X and to the graph of F (see [2, Théorème 6.6, p. 79]). Assume in addition that X is bounded in R^n. Then, if $F\colon X \to R^e$ is a nowhere zero polynomial map, it is easy to see that there does not exist any polynomial map $G\colon X \to R^e$ with the properties required in (2). This fact implies that, in the statement of Theorem 1.2, the adjective "admissible" cannot be omitted.

Kucharz [3] obtained the following interesting version of the Dubois–Efroymson dimension theorem: *Given a nonsingular irreducible algebraic subset X of some $\mathbb{R}^n$, where $\mathbb{R}$ is the field of real numbers, and an algebraic subset Z of X of codimension ≥ 2, there exists an irreducible algebraic subset Y of X of codimension 1 containing Z such that the ideal of regular functions on X vanishing on Y is principal.*

This result can be proved using Theorem 1.2. Let us explain this assertion. By the algebraic Alexandrov compactification and Hironaka's desingularization theorem, we may suppose that X is compact (see Step 3 of the proof of [3, Theorem 1], p. 28). Under this additional condition, Theorem 1.2(1) (with $e = 1$) and [3, Lemma 3] ensure the existence of an algebraic subset Y of X with the required properties.

Theorem 1.2 holds over any algebraically closed field in the following form.

Theorem 1.3 *Let $\mathbb{K}$ be an algebraically closed field. Let X be an irreducible algebraic subset of $\mathbb{K}^n$, let Z be an algebraic subset of X of codimension $c \geq 2$ and let*

$e \in \{1, \dots, c-1\}$. *Indicate by* μ *the* μ*-complexity of* Z *in* $\mathbb{K}^n$ *and by* X^* *the set of nonsingular points of* X*, which are not contained in* Z*. Then, given any polynomial map* $F\colon X \to \mathbb{K}^e$ *of degree* d *vanishing on* Z*, there exist a polynomial map* $G\colon X \to \mathbb{K}^e$ *of degree* $\leq 1 + \max\{d, \mu\}$ *and a finite subset* E *of* $\mathbb{K}$ *such that: for each* $t \in \mathbb{K} \setminus E$*, the polynomial map* $F_t\colon X \to \mathbb{K}^e$ *defined by* $F_t := F + tG$ *is good in* X^* *and* $(F_t)^{-1}(0)$ *is an irreducible algebraic subset of* X *of codimension* e *containing* Z*. In particular, there exists a polynomial map* $P\colon X \to \mathbb{K}^e$ *of degree* $\leq \mu + 1$ *and good in* X^* *such that* $P^{-1}(0)$ *is an irreducible algebraic subset of* X *of codimension* e *containing* Z*.*

Let us give the idea of the proof of our results. First, we deal with Theorem 1.3. Let $\mathbb{K}$, $X \subset \mathbb{K}^n$, Z, c, e, μ, X^*, $F\colon X \to \mathbb{K}^e$ and d be as in the statement of the mentioned theorem. Let $P_1, \dots, P_e$ be polynomials in $\mathscr{I}_{\mathbb{K}^n}(Z)$ such that $F = (P_1, \dots, P_e)$ on X and $\max_{i\in\{1,\dots,e\}} \deg(P_i) = d$, and let $q_1, \dots, q_\ell$ be generators of $\mathscr{I}_{\mathbb{K}^n}(Z)$ in $\mathbb{K}[x_1, \dots, x_n]$ such that $\max_{i\in\{1,\dots,\ell\}} \deg(q_i) = \mu$. Let us construct the blowing up $\theta\colon X' \to X$ of X with center Z by using the generators $P_1, \dots, P_e, q_1, \dots, q_\ell$ of $\mathscr{I}_{\mathbb{K}^n}(Z)$ and a suitable Segre embedding. Define the polynomial map $q\colon \mathbb{K}^n \to \mathbb{K}^{e+\ell}$ by

$$q := (P_1, \dots, P_e, q_1, \dots, q_\ell),$$

X'' as the Zariski closure in $X \times \mathbb{P}^{e+\ell-1}(\mathbb{K})$ of the set

$$\{(x, [q(x)]) \in X \times \mathbb{P}^{e+\ell-1}(\mathbb{K}) \mid x \in X \setminus Z\},$$

the regular map $\theta'\colon X'' \to X$ as the restriction to X'' of the natural projection of $X \times \mathbb{P}^{e+\ell-1}(\mathbb{K})$ onto X, the integer $m := (n+1)(e+\ell)$ and $\xi\colon X \times \mathbb{P}^{e+\ell-1}(\mathbb{K}) \to \mathbb{P}^{m-1}(\mathbb{K})$ as the regular map, which sends $((x_1, \dots, x_n), [y]) \in X \times \mathbb{P}^{e+\ell-1}(\mathbb{K})$ into $[y, x_1 y, \dots, x_n y] \in \mathbb{P}^{m-1}(\mathbb{K})$. Identifying each point $(x_1, \dots, x_n)$ of X with the point $[1, x_1, \dots, x_n]$ of $\mathbb{P}^n(\mathbb{K})$, we see that ξ coincides with the restriction to $X \times \mathbb{P}^{e+\ell-1}(\mathbb{K})$ of the Segre embedding of $\mathbb{P}^n(\mathbb{K}) \times \mathbb{P}^{e+\ell-1}(\mathbb{K})$ into $\mathbb{P}^{m-1}(\mathbb{K})$. In particular, $X' := \xi(X'')$ is a Zariski locally closed subset of $\mathbb{P}^{m-1}(\mathbb{K})$ and the restriction $\xi'\colon X'' \to X'$ of ξ from X'' to X' is a biregular isomorphism. Define the blowing up $\theta\colon X' \to X$ of X with center Z by setting $\theta := \theta' \circ (\xi')^{-1}$. Observe that, for each $x = (x_1, \dots, x_n) \in X \setminus Z$, the point $\theta^{-1}(x)$ of $\mathbb{P}^{m-1}(\mathbb{K})$ is equal to $[q(x), x_1 q(x), \dots, x_n q(x)]$. Indicate by $Q\colon X \to \mathbb{K}^m$ the polynomial map defined by $Q(x) := (q(x), x_1 q(x), \dots, x_n q(x))$. Let Ω be the set of nonsingular points of Z of some dimension. By simple considerations concerning the blowing up operation in algebraic geometry over an algebraically closed field, we infer that, for each $x \in \Omega$, $\theta^{-1}(x)$ is a Zariski closed subset of $\mathbb{P}^{m-1}(\mathbb{K})$ of dimension $\geq c-1$. Let N be a linear subspace of $\mathbb{P}^{m-1}(\mathbb{K})$ of codimension e. By hypothesis, $e \leq c-1$ and hence, for each $x \in \Omega$, the intersection $\theta^{-1}(x) \cap N$ is nonempty. Since Ω is Zariski dense in Z, it follows that $\theta(X' \cap N)$ is an algebraic subset of X containing Z. Thanks to Bertini's theorems (see [2, Corollaire 6.11, p. 89]), for a generic choice of N, we have that N intersects transversally $\theta^{-1}(X^*)$ in $\mathbb{P}^{m-1}(\mathbb{K})$, $\theta^{-1}(X^*) \cap N \neq \varnothing$ and $X' \cap N$ is an irreducible Zariski closed subset of X' of codimension e. In this way, denoting by $\rho\colon \mathbb{K}^m \setminus \{0\} \to \mathbb{P}^{m-1}(\mathbb{K})$ the natural projection, we can conclude that the restriction of Q to X^* is transverse to $N' := \rho^{-1}(N) \cup \{0\}$ in $\mathbb{K}^m$ and $\theta(X' \cap N)$ is an irreducible algebraic subset of X of codimension e containing Z, which coincides with $Q^{-1}(N')$. Fix N with these properties. Let D be a $(e \times m)$-matrix with coefficients in $\mathbb{K}$ such that the kernel of D is equal to N' and let B be the $(e \times m)$-matrix $(b_{ij})_{i,j}$ such that $b_{ij} = 1$ if $j = i$ for some $i \in \{1, \dots, e\}$ and $b_{ij} = 0$ otherwise. Define $D \cdot Q \colon X \to \mathbb{K}^e$ as the polynomial map that sends $x \in X$ into the standard matrix–vector product $D \cdot Q(x) \in \mathbb{K}^e$. Observe that $D \cdot Q$ is of degree $\leq 1 + \max\{d, \mu\}$, is good in X^* and $(D \cdot Q)^{-1}(0) = Q^{-1}(N')$ is an irreducible algebraic subset of X of codimension e containing Z. Moreover, $B \cdot Q$ is equal to F. It follows that, for a generic choice of t in $\mathbb{K}$, the polynomial map $(B + t(D - B)) \cdot Q \colon X \to \mathbb{K}^e$ has the required properties. Defining the polynomial map $G \colon X \to \mathbb{K}^e$ by $G := (D - B) \cdot Q$, we complete the proof of the first part of Theorem 1.3. The second part is an easy consequence of the first one: it suffices to choose F constantly equal to zero.

Suppose now the ground field is R. Indicate by C the algebraic closure of R. A natural strategy to prove Theorem 1.2 is to apply the preceding argument to the complexifications of X and of Z and to use the fact that the grassmannian of linear subspaces of $\mathbb{P}^{m-1}(R)$ of codimension e is Zariski dense in the corresponding grassmannian over C. A "standard" problem arises: the real part of an irreducible Zariski closed subset of $\mathbb{P}^n(C)$ may be reducible. There is another problem. The upper bound for the degree of G obtained by means of this strategy is $U := 1 + \max\{d, \mu\}$, while the corresponding upper bound stated in Theorem 1.2 is $u := \max\{d, \mu + 1\}$. If $d \leq \mu$, then $u = U = \mu + 1$. However, if $d \geq \mu + 1$, then $u = d < U = d + 1$ and hence the upper bound u is strictly better than U. In order to overcome these difficulties, we need three technical lemmas that we will present in Section 2. Section 3 contains a complete proof of our theorems.

2 Preliminary Results

Recall that C indicates the algebraic closure of R, which is equal to $R[t]/(t^2 + 1)$. As is usual, we denote by $\mathbb{P}^n(C)$ the projectivization $\mathbb{P}(C^{n+1})$ of C^{n+1}. Equip each projective space $\mathbb{P}^n(C)$ with its natural structure of algebraic variety over C and each Zariski locally closed subset of $\mathbb{P}^n(C)$ with the structure of algebraic subvariety of $\mathbb{P}^n(C)$ (see [5]). Identify C^n with a Zariski open subset of $\mathbb{P}^n(C)$ by the affine chart which sends $(x_1, \dots, x_n) \in C^n$ into $[1, x_1, \dots, x_n] \in \mathbb{P}^n(C)$. In this way, an algebraic subset of C^n can be regarded as a Zariski locally closed subset of $\mathbb{P}^n(C)$. Let X_C be such a subset of $\mathbb{P}^n(C)$. We denote by $\dim_C(X_C)$ the complex dimension of X_C and by $\mathrm{Nonsing}_C(X_C)$ the set of nonsingular points of X_C of maximum complex dimension, *i.e.*, of complex dimension $\dim_C(X_C)$.

Lemma 2.1 *Let X_C be an irreducible algebraic subset of C^n, let Z_C be an algebraic subset of X_C of complex codimension $c \geq 1$ and let $\theta_C \colon X'_C \to X_C$ be the blowing up of X_C with center Z_C. Then, for each irreducible component Z^*_C of Z_C of complex dimension d, there exists an irreducible component Z'_C of $\theta_C^{-1}(Z_C)$ of complex dimension v such that $\theta_C(Z'_C) = Z^*_C$ and $v - d \geq c - 1$.*

Proof Let $\Theta_C \colon B'_C \to C^n$ be the blowing up of C^n with center Z_C. We may suppose that X'_C is an irreducible Zariski closed subset of B'_C and θ_C is the restriction

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of Θ_C from X'_C to X_C. Let Z^*_C be an irreducible component of Z_C of complex dimension d, and let $\bar{Z}_C$ be the union of irreducible components of Z_C different from Z^*_C. Indicate by $Z'_{C,1},\dots,Z'_{C,s}$ the irreducible components of $\theta_C^{-1}(Z_C)$, by $\nu_1,\dots,\nu_s$ the complex dimensions of $Z'_{C,1},\dots,Z'_{C,s}$ respectively and by I the set of all indices $i\in\{1,\dots,s\}$ such that $Z'_{C,i}\cap\theta_C^{-1}(Z^*_C\setminus\bar{Z}_C)\neq\varnothing$. Since θ_C is surjective and $Z^*_C\setminus\bar{Z}_C=Z_C\setminus\bar{Z}_C$ is a Zariski open subset of Z_C, it follows that I is nonempty and $Z^*_C\setminus\bar{Z}_C\subset\bigcup_{i\in I}\theta_C(Z'_{C,i})\subset Z^*_C$. Observe that the set $Z^*_C\setminus\bar{Z}_C$ is Zariski dense in Z^*_C and the map θ_C is Zariski closed, *i.e.*, it sends Zariski closed subsets of X'_C into Zariski closed subsets of X_C. These facts imply that $\bigcup_{i\in I}\theta_C(Z'_{C,i})=Z^*_C$. Let $J:=\{i\in I\mid\theta_C(Z'_{C,i})=Z^*_C\}$. Since $\theta_C(Z'_{C,i})$ is an irreducible algebraic subset of Z^*_C for each $i\in I$, it follows that J is nonempty and $V_C:=\bigcup_{i\in I\setminus J}\theta_C(Z'_{C,i})$ is a proper algebraic subset of Z^*_C. For each $i\in J$, denote by $\theta_{C,i}\colon Z'_{C,i}\to Z^*_C$ the restriction of θ_C from $Z'_{C,i}$ to Z^*_C. By applying Theorem 7 of [6, p. 76] to each $\theta_{C,i}$, we infer the existence of a point $z\in\mathrm{Nonsing}_C(Z^*_C)\setminus(\bar{Z}_C\cup V_C)$ such that $\dim_C\theta_{C,i}^{-1}(z)=\nu_i-d$ for each $i\in J$. Since $\theta_C^{-1}(z)=\bigcup_{i\in J}\theta_{C,i}^{-1}(z)$, we have that $\dim_C\theta_C^{-1}(z)=\max_{i\in J}\{\nu_i-d\}$. We will show that $\dim_C\theta_C^{-1}(z)\geq c-1$, completing the proof. Let $r:=\dim_C(X_C)$. The point z is a nonsingular point of Z_C of complex dimension $d\leq\dim_C(Z_C)=r-c$, so there exists a Zariski open neighborhood U_C of z in C^n such that $Z_C\cap U_C$ is a nonsingular Zariski closed subset of U_C of complex dimension d. Observe that $\dim_C(B'_C)=n$, $\Theta_C^{-1}(U_C)$ is a Zariski open subset of $\mathrm{Nonsing}_C(B'_C)$, $\dim_C(X'_C)=r$, $\Theta_C^{-1}(z)$ is a (nonsingular) irreducible Zariski closed subset of B'_C of complex dimension $n-d-1$ and $\theta_C^{-1}(z)$ is equal to $X'_C\cap\Theta_C^{-1}(z)$. Thanks to [4, Proposition 3.28], we have that

$$\dim_C(\theta_C^{-1}(z))\geq r+(n-d-1)-n\geq c-1. \qquad\blacksquare$$

For each non–negative integer n, we indicate by $\alpha_n\colon\mathbb{P}^n(C)\to\mathbb{P}^n(C)$ the complex conjugation map and identify $\mathbb{P}^n(R)=\mathbb{P}(R^{n+1})$ with the fixed point set of α_n. Let S be a subset of $\mathbb{P}^n(C)$. Define the real part $S(R)$ of S as the intersection $S\cap\mathbb{P}^n(R)$. The set S is said to be defined over R if it is σ_n-invariant. Suppose S is defined over R and let T be a subset of $\mathbb{P}^m(C)$ defined over R. A map $f\colon S\to T$ is said to be defined over R if $f\circ\sigma_n=\sigma_m\circ f$ on S. Observe that, if f has this property, then it sends $S(R)$ into $T(R)$. Identify R^n with the real part of C^n. Equip each Zariski locally closed subset of $\mathbb{P}^n(R)$ with its natural structure of algebraic variety over R. Unless otherwise indicated, all the topological notions related to these real algebraic varieties are refered to the euclidean topology.

Let V_C be a Zariski locally closed subset of $\mathbb{P}^n(C)$. A map $\varphi_C\colon V_C\to\mathbb{P}^m(C)$ is said to be a complex biregular embedding if $\varphi_C(V_C)$ is a Zariski locally closed subset of $\mathbb{P}^m(C)$ and the restriction of φ_C from V_C to $\varphi_C(V_C)$ is a complex biregular isomorphism. In the real setting, the notion of biregular embedding can be defined in the same way.

Lemma 2.2 *Let $X\subset R^n$ be a real algebraic set, and let Z be a proper algebraic subset of X. Let $q_1,\dots,q_\ell$ be generators of $\mathscr{I}_{R^n}(Z)$ in $R[x_1,\dots,x_n]$, let $q\colon X\to R^\ell$ be the polynomial map defined by $q(x):=(q_1(x),\dots,q_\ell(x))$ and let $F\colon X\to R^k$ be a polynomial map vanishing on Z. Define the integer $m:=\ell(n+1)+k$, the linear subspace H_C*

of $\mathbb{P}^{m-k-1}(C)$ *by*

$$H_C := \{[y_1, y_2, \dots, y_{\ell n+\ell}] \in \mathbb{P}^{m-k-1}(C) \mid y_1 = y_2 = \dots = y_\ell = 0\},$$

and the linear subspace H *of* $\mathbb{P}^{m-k-1}(R)$ *as the real part of* H_C. *Define the regular maps* $\lambda\colon X \setminus Z \to \mathbb{P}((R^\ell)^{n+1}) \setminus H = \mathbb{P}^{m-k-1}(R) \setminus H$ *and* $\sigma\colon X \setminus Z \to \mathbb{P}((R^\ell)^{n+1} \times R^k) = \mathbb{P}^{m-1}(R)$ *by setting*

$$\lambda(x) := [q(x), x_1 q(x), \dots, x_n q(x)],$$
$$\sigma(x) := [q(x), x_1 q(x), \dots, x_n q(x), F(x)]$$

for each $x = (x_1, \dots, x_n) \in X \setminus Z$. *Let* G_C *be a finite subset of* $\mathbb{P}^{m-k-1}(C) \setminus H_C$. *Then there exist a Zariski open subset* E_C *of* $\mathbb{P}^{m-k-1}(C)$ *defined over* R *and a complex biregular embedding* $\eta_C\colon E_C \to \mathbb{P}^{m-1}(C)$ *defined over* R *with the following properties:*

(1) $E_C(R) = \mathbb{P}^{m-k-1}(R) \setminus H$ *and* $G_C \subset E_C$.
(2) *Indicating by* $\eta\colon \mathbb{P}^{m-k-1}(R) \setminus H \to \mathbb{P}^{m-1}(R)$ *the biregular embedding defines as the restriction of* η_C *from* $E_C(R) = \mathbb{P}^{m-k-1}(R) \setminus H$ *to* $\mathbb{P}^{m-1}(R)$, *we have that* $\sigma = \eta \circ \lambda$.

Proof Let $p_1, \dots, p_h$ be the points of $\mathbb{P}^{m-k-1}(C)$ such that $G_C = \{p_1, \dots, p_h\}$. By hypothesis, the intersection $G_C \cap H_C$ is empty. In this way, for each $i \in \{1, \dots, h\}$, we can write $p_i = [p_{i1}, p_{i2}, \dots, p_{i,\ell n+\ell}]$, where $(p_{i1}, p_{i2}, \dots, p_{i\ell}) \in C^\ell \setminus \{0\}$. Choose positive elements $r_1, \dots, r_\ell$ of R such that $\sum_{j=1}^{\ell} r_j p_{ij}^2 \neq 0$ for each $i \in \{1, \dots, h\}$. Let $P_1, \dots, P_k$ be polynomials in $R[x_1, \dots, x_n]$ such that $F = (P_1, \dots, P_k)$ on X. For each $s \in \{1, \dots, k\}$, P_s vanishes on Z, so there exist polynomials $a_{s1}, \dots, a_{s\ell}$ in $R[x_1, \dots, x_n]$ such that $P_s = \sum_{j=1}^{\ell} a_{sj} q_j$. For each $s \in \{1, \dots, k\}$ and for each $j \in \{1, \dots, \ell\}$, indicate by $a_{sj,C}$ the polynomial a_{sj}, viewed as an element of $C[x_1, \dots, x_n]$. Let E_C be the Zariski open subset of $\mathbb{P}^{m-k-1}(C)$ defined by

$$E_C := \{[y_1, y_2, \dots, y_{\ell n+\ell}] \in \mathbb{P}^{m-k-1}(C) \mid \sum_{j=1}^{\ell} r_j y_j^2 \neq 0\}$$

and let $\varphi_C = (\varphi_{C,1}, \dots, \varphi_{C,n})\colon E_C \to C^n$ be the complex regular map whose i-th component $\varphi_{C,i}\colon E_C \to C$ is defined as follows:

$$\varphi_{C,i}([y_1, \dots, y_{\ell n+\ell}]) := \frac{\sum_{j=1}^{\ell} r_j y_j y_{\ell i+j}}{\sum_{j=1}^{\ell} r_j y_j^2}.$$

Denote points of $\mathbb{P}^{m-k-1}(C) = \mathbb{P}(C^{\ell n+\ell})$ and of $\mathbb{P}^{m-1}(C) = \mathbb{P}(C^{\ell n+\ell} \times C^k)$ by $[\hat{y}] = [y_1, \dots, y_\ell, y_{\ell+1}, \dots, y_{\ell n+\ell}]$ and $[\hat{y}, y_{\ell n+\ell+1}, \dots, y_m]$ respectively. Define the Zariski open subset T_C of $\mathbb{P}^{m-1}(C)$ by

$$T_C := \{[y_1, y_2, \dots, y_m] \in \mathbb{P}^{m-1}(C) \mid \sum_{j=1}^{\ell} r_j y_j^2 \neq 0\},$$

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the nonsingular Zariski closed subset D_C of T_C as the following intersection

$$\bigcap_{s=1}^{k}\{[\hat{y}, y_{\ell n+\ell+1}, \dots, y_m] \in T_C \mid y_{\ell n+\ell+s} = \sum_{j=1}^{\ell} a_{sj,C}(\varphi_C([\hat{y}])) \cdot y_j\},$$

the complex regular map $\eta'_C \colon E_C \to D_C$ by

$$\eta'_C([\hat{y}]) := [\hat{y}, \sum_{j=1}^{\ell} a_{1j,C}(\varphi_C([\hat{y}])) \cdot y_j, \dots, \sum_{j=1}^{\ell} a_{kj,C}(\varphi_C([\hat{y}])) \cdot y_j]$$

and the complex regular map $\eta_C \colon E_C \to \mathbb{P}^{m-1}(C)$ as the composition of η'_C with the inclusion map $D_C \hookrightarrow \mathbb{P}^{m-1}(C)$. The map η'_C is a complex biregular isomorphism. In fact, the complex regular map from D_C to E_C which sends $[\hat{y}, y_{\ell n+\ell+1}, \dots, y_m] \in D_C$ into $[\hat{y}] \in E_C$, is the inverse of η'_C. It follows that η_C is a complex biregular embedding defined over R. (1) follows immediately from the definition of E_C. Let us prove (2). Let $x \in X \setminus Z$ and let $\eta \colon \mathbb{P}^{m-k-1}(R) \setminus H \to \mathbb{P}^{m-1}(R)$ be the restriction of η_C from $E_C(R) = \mathbb{P}^{m-k-1}(R) \setminus H$ to $\mathbb{P}^{m-1}(R)$. Since $\varphi_C(\lambda(x)) = x$, we obtain that

$$\eta(\lambda(x)) = [q(x), x_1 q(x), \dots, x_n q(x), P_1(x), \dots, P_k(x)] = \sigma(x).$$

This completes the proof. ∎

Let e and m be positive integers with $e \leq m$, and let $\mathcal{M}_{e,m}(R)$ be the vector space of $(e \times m)$-matrices with coefficients in R. Equip $\mathcal{M}_{e,m}(R)$ with its natural structure of affine irreducible algebraic variety over R and indicate by $\mathcal{M}^*_{e,m}(R)$ the nonempty Zariski open subset of $\mathcal{M}_{e,m}(R)$ formed by all matrices of rank e. For each $A \in \mathcal{M}_{e,m}(R)$, we denote by $\pi_A \colon R^m \to R^e$ the linear map associated with A (which sends $v \in R^m$ into the standard product $A \cdot v \in R^e$) and by L_A the kernel of π_A.

Let us fix a notation.

Notation Let X be a real algebraic set, let m be a positive integer, let $Q \colon X \to R^m$ be a polynomial map and let $e \in \{1, \dots, m\}$. We denote by $\mathcal{A}_e(Q)$ the set of all matrices $A \in \mathcal{M}^*_{e,m}(R)$ such that $\pi_A \circ Q \colon X \to R^e$ is admissible.

Observe that, using the preceding terminology, a matrix $A \in \mathcal{M}^*_{e,m}(R)$ belongs to $\mathcal{A}_e(Q)$ if and only if, for some $x \in \mathrm{Nonsing}(X)$, $Q(x) \in L_A$ and Q is transverse to L_A in R^m at x.

Lemma 2.3 *Let $X \subset R^n$ be a real algebraic set, let Z be an algebraic subset of X of codimension $c \geq 2$ and let $X^* := \mathrm{Nonsing}(X) \setminus Z$. Let m be a positive integer and let $Q \colon X \to R^m$ be a polynomial map such that $Q(X^*) \subset R^m \setminus \{0\}$. Indicate by $\rho \colon R^m \setminus \{0\} \to \mathbb{P}^{m-1}(R)$ the natural projection and by $Q^* \colon X^* \to R^m \setminus \{0\}$ the restriction of Q from X^* to $R^m \setminus \{0\}$. Suppose there exists a point $q \in X^*$ such that the differential at q of the composition map $\rho \circ Q^* \colon X^* \to \mathbb{P}^{m-1}(R)$ is injective. Then, for each $e \in \{1, \dots, c-1\}$, $\mathcal{A}_e(Q)$ is a nonempty open semialgebraic subset of $\mathcal{M}^*_{e,m}(R)$ and there exists a proper Zariski closed subset $\mathcal{V}_e$ of $\mathcal{M}^*_{e,m}(R)$ such that, for each $A \in \mathcal{M}^*_{e,m}(R) \setminus \mathcal{V}_e$, Q^* is transverse to L_A in R^m.*

Proof Let $e \in \{1, \dots, c-1\}$. Define the polynomial map $\psi_e \colon X^* \times \mathcal{M}^*_{e,m}(R) \to R^e$ by $\psi_e(x, A) := (\pi_A \circ Q)(x)$. It is easy to verify that the origin 0 of R^e is a regular value of ψ_e, so $V_e := (\psi_e)^{-1}(0)$ is a nonempty nonsingular Zariski closed subset of $X^* \times \mathcal{M}^*_{e,m}(R)$ of codimension e. Indicate by $\nu_e \colon V_e \to \mathcal{M}^*_{e,m}(R)$ the restriction to V_e of the natural projection of $X^* \times \mathcal{M}^*_{e,m}(R)$ onto $\mathcal{M}^*_{e,m}(R)$. Let Σ_e be the set of regular points of ν_e. Combining standard considerations of Linear Algebra with the Implicit Function Theorem (for Nash maps), it follows immediately that a point (x, A) of V_e belongs to Σ_e if and only if the rank of the differential of $\pi_A \circ Q$ at x is equal to e. Since $e < c$, $\mathcal{A}_e(Q)$ is equal to $\nu_e(\Sigma_e)$ and hence $\mathcal{A}_e(Q)$ is an open semialgebraic subset of $\mathcal{M}^*_{e,m}(R)$. Moreover, by applying Sard's theorem to ν_e, we find a proper Zariski closed subset $\mathcal{V}_e$ of $\mathcal{M}^*_{e,m}(R)$ with the required property: Q^* is transverse to L_A in R^m for each $A \in \mathcal{M}^*_{e,m}(R) \setminus \mathcal{V}_e$. It remains to prove that $\mathcal{A}_e(Q)$ is nonempty. Let $\sigma^* \colon X^* \to \mathbb{P}^{m-1}(R)$ be the composition $\rho \circ Q^*$. Indicate by $\mathrm{d}_q Q^* \colon T_q(X^*) \to R^m$ the differential of Q^* at q and by N the vector subspace $\mathrm{d}_q Q^*(T_q(X^*))$ of R^m. Observe that the kernel of the differential $\mathrm{d}_{Q(q)}\rho$ of ρ at $Q(q)$ is equal to the vector line of R^m generated by $Q(q)$. Since $\mathrm{d}_q\sigma^* = \mathrm{d}_{Q(q)}\rho \circ \mathrm{d}_q Q^*$ is injective, it follows that $\mathrm{d}_q Q^*$ is injective and $Q(q) \notin N$. In particular, we have that $\dim(N) = \dim(X)$. Since $e \leq \dim(X)$, there exists a vector subspace L of R^m of codimension e which contains $Q(q)$ and is transverse to N in R^m. Let D be a matrix in $\mathcal{M}^*_{e,m}(R)$ such that $L = L_D$. Evidently, D is an element of $\mathcal{A}_e(Q)$. ■

3 Proof of the Theorems

We begin proving a "more constructive" version of Theorem 1.2.

Theorem 3.1 *Let $X \subset R^n$ be an irreducible real algebraic set, let Z be an algebraic subset of X of codimension $c \geq 2$ and let $X^* := \mathrm{Nonsing}(X) \setminus Z$. Let $q_1, \dots, q_\ell$ be generators of $\mathscr{I}_{R^n}(Z)$ in $R[x_1, \dots, x_n]$ and let $F \colon X \to R^k$ be a polynomial map vanishing on Z. Define the polynomial map $q \colon X \to R^\ell$ by $q(x) := (q_1(x), \dots, q_\ell(x))$. Let $m := \ell(n+1) + k$ and define the polynomial map $Q \colon X \to (R^\ell)^{n+1} \times R^k = R^m$ by setting*

$$Q(x) := (q(x), x_1 q(x), \dots, x_n q(x), F(x))$$

for each $x = (x_1, \dots, x_n) \in X$. Then, for each $e \in \{1, \dots, c-1\}$, $\mathcal{A}_e(Q)$ is a nonempty open semialgebraic subset of $\mathcal{M}^*_{e,m}(R)$ and there exists a proper Zariski closed subset $\mathcal{B}_e(Q)$ of $\mathcal{M}^*_{e,m}(R)$ with the following property: for each $A \in \mathcal{A}_e(Q) \setminus \mathcal{B}_e(Q)$, the polynomial map $\pi_A \circ Q \colon X \to R^e$ is good in X^* and $(\pi_A \circ Q)^{-1}(0)$ is an irreducible algebraic subset of X of codimension e containing Z.

Proof We subdivide the proof into five steps.

Step I. Indicate by $q_{1,C}, \dots, q_{\ell,C}$ the polynomials $q_1, \dots, q_\ell$, viewed as elements of $C[x_1, \dots, x_n]$. Define the polynomial map $q_C \colon C^n \to C^\ell$ by $q_C := (q_{1,C}, \dots, q_{\ell,C})$; let X_C and Z_C be the Zariski closures of X and of Z in C^n respectively; define X''_C to be the Zariski closure in $X_C \times \mathbb{P}(C^\ell) = X_C \times \mathbb{P}^{\ell-1}(C)$ of the set

$$\{(x, [q_C(x)]) \in X_C \times \mathbb{P}^{\ell-1}(C) \mid x \in X_C \setminus Z_C\};$$

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and define the complex regular map $\theta'_C\colon X''_C \to X_C$ to be the restriction to X''_C of the natural projection of $X_C \times \mathbb{P}^{\ell-1}(C)$ onto X_C. Let $\xi_C\colon X_C \times \mathbb{P}^{\ell-1}(C) \to \mathbb{P}((C^\ell)^{n+1}) = \mathbb{P}^{m-k-1}(C)$ be the complex regular map that sends $((x_1,\dots,x_n),[y]) \in X_C \times \mathbb{P}^{\ell-1}(C)$ into $[y, x_1 y, \dots, x_n y] \in \mathbb{P}((C^\ell)^{n+1})$. Observe that ξ_C coincides with the restriction to $X_C \times \mathbb{P}^{\ell-1}(C)$ of the Segre embedding of $\mathbb{P}^n(C) \times \mathbb{P}^{\ell-1}(C)$ into $\mathbb{P}^{m-k-1}(C)$. Define the irreducible Zariski locally closed subset X'_C of $\mathbb{P}^{m-k-1}(C)$ by $X'_C := \xi_C(X''_C)$, the complex biregular isomorphism $\xi'_C\colon X''_C \to X'_C$ as the restriction of ξ_C from X''_C to X'_C and the complex regular map $\theta_C\colon X'_C \to X_C$ by $\theta_C := \theta'_C \circ (\xi'_C)^{-1}$. Since $q_{1,C},\dots,q_{\ell,C}$ generate $\mathscr{I}_{C^n}(Z_C)$ in $C[x_1,\dots,x_n]$ (see [7]), θ_C is the blowing up of X_C with center Z_C. Let $Z_{C,1},\dots,Z_{C,h}$ be the irreducible components of Z_C and let $d_1,\dots,d_h$ be the complex dimensions of $Z_{C,1},\dots,Z_{C,h}$, respectively. Thanks to Lemma 2.1, we have that, for each $i \in \{1,\dots,h\}$, there exists an irreducible component $Z'_{C,i}$ of $\theta_C^{-1}(Z_C)$ of complex dimension v_i such that

$$\theta_C(Z'_{C,i}) = Z_{C,i} \text{ and } v_i - d_i \geq c - 1. \tag{3.1}$$

Step II. For each $i \in \{1,\dots,h\}$, choose a point p_i in $Z'_{C,i}$. Indicate by G_C the finite subset $\{p_1,\dots,p_h\}$ of X'_C. As in the statement of Lemma 2.2, define the linear subspace H_C of $\mathbb{P}^{m-k-1}(C)$ by

$$H_C := \{[y_1,y_2,\dots,y_{\ell n+\ell}] \in \mathbb{P}^{m-k-1}(C) \mid y_1 = y_2 = \dots = y_\ell = 0\},$$

the linear subspace H of $\mathbb{P}^{m-k-1}(R)$ as the real part of H_C and the regular map $\lambda\colon X \setminus Z \to \mathbb{P}((R^\ell)^{n+1}) \setminus H = \mathbb{P}^{m-k-1}(R) \setminus H$ and the regular map $\sigma\colon X \setminus Z \to \mathbb{P}((R^\ell)^{n+1} \times R^k) = \mathbb{P}^{m-1}(R)$ as follows:

$$\lambda(x) := [q(x), x_1 q(x), \dots, x_n q(x)],$$
$$\sigma(x) := [q(x), x_1 q(x), \dots, x_n q(x), F(x)]$$

for each $x = (x_1,\dots,x_n) \in X \setminus Z$. Observe that $X'_C \subset \xi_C(X_C \times \mathbb{P}^{\ell-1}(C))$ and $\xi_C(X_C \times \mathbb{P}^{\ell-1}(C)) \cap H_C = \varnothing$. In this way, we have that $X'_C \cap H_C = \varnothing$ and hence $G_C \subset \mathbb{P}^{m-k-1}(C) \setminus H_C$. By applying Lemma 2.2, we obtain a Zariski open subset E_C of $\mathbb{P}^{m-k-1}(C)$ defined over R and a complex biregular embedding $\eta_C\colon E_C \to \mathbb{P}^{m-1}(C)$ defined over R such that $E_C(R) = \mathbb{P}^{m-k-1}(R) \setminus H$, $G_C \subset X'_C \cap E_C$ and, denoting by $\eta\colon \mathbb{P}^{m-k-1}(R) \setminus H \to \mathbb{P}^{m-1}(R)$ the biregular embedding defined as the restriction of η_C from $E_C(R) = \mathbb{P}^{m-k-1}(R) \setminus H$ to $\mathbb{P}^{m-1}(R)$, it holds that

$$\sigma = \eta \circ \lambda. \tag{3.2}$$

Indicate by X' the real part of X'_C and by $\theta\colon X' \to X$ the restriction of θ_C from X' to X. The map θ is the blowing up of X with center Z and, for each $x \in X \setminus Z$, we have:

$$\text{the point } \theta^{-1}(x) \text{ of } \mathbb{P}^{m-k-1}(R) \text{ coincides with } \lambda(x). \tag{3.3}$$

It follows that X' is an irreducible Zariski locally closed subset of $\mathbb{P}^{m-k-1}(R)$ and λ is a biregular embedding. Combining the latter fact with (3.2), we obtain that

$$\sigma \text{ is a biregular embedding.} \tag{3.4}$$

Define X^* and $Q\colon X \to R^m$ as in the statement of the theorem we are proving. Let $Q^*\colon X^* \to R^m \setminus \{0\}$ be the restriction of Q from X^* to $R^m \setminus \{0\}$ and let $\rho\colon R^m \setminus \{0\} \to \mathbb{P}^{m-1}(R)$ be the natural projection. Since $\rho \circ Q^*$ coincides with the restriction of σ to X^*, (3.4) implies that $\rho \circ Q^*$ is a biregular embedding. In this way, the hypotheses of Lemma 2.3 are satisfied, so we have that, for each $e \in \{1, \dots, c-1\}$, $\mathcal{A}_e(Q)$ is a nonempty open semialgebraic subset of $\mathcal{M}^*_{e,m}(R)$ and there exists a proper Zariski closed subset $\mathcal{V}_e$ of $\mathcal{M}^*_{e,m}(R)$ such that

(3.5) $\quad Q^*$ is transverse to L_A in R^m for each $A \in \mathcal{M}^*_{e,m}(R) \setminus \mathcal{V}_e$.

Step III. Define the irreducible Zariski locally closed subset $\widetilde{X}_C$ of $\mathbb{P}^{m-1}(C)$ by $\widetilde{X}_C := \eta_C(X'_C \cap E_C)$; define the complex biregular isomorphism $\eta'_C\colon X'_C \cap E_C \to \widetilde{X}_C$ as the restriction of η_C from $X'_C \cap E_C$ to $\widetilde{X}_C$; define $i'_C\colon X'_C \cap E_C \hookrightarrow X'_C$ as the inclusion map of $X'_C \cap E_C$ into X'_C; define the complex regular map $\widetilde{\theta}_C\colon \widetilde{X}_C \to X_C$ by $\widetilde{\theta}_C := \theta_C \circ i'_C \circ (\eta'_C)^{-1}$; and, for each $i \in \{1, \dots, h\}$, define the Zariski closed subset $\widetilde{Z}_{C,i}$ of $(\widetilde{\theta}_C)^{-1}(Z_C)$ by $\widetilde{Z}_{C,i} := \eta_C(Z'_{C,i} \cap E_C)$. Observe that each set $\widetilde{Z}_{C,i}$ contains the point $\eta_C(p_i)$. This fact and (3.1) imply that, for each $i \in \{1, \dots, h\}$, $\widetilde{Z}_{C,i}$ is a (nonempty) irreducible component of $(\widetilde{\theta}_C)^{-1}(Z_C)$ of complex dimension ν_i such that

(3.6) $\quad \widetilde{\theta}_C(\widetilde{Z}_{C,i})$ is a Zariski dense subset of $Z_{C,i}$ and $\nu_i - d_i \geq c-1$.

Fix $i \in \{1, \dots, h\}$. Let $\widetilde{\theta}_{C,i}\colon \widetilde{Z}_{C,i} \to Z_{C,i}$ be the restriction of $\widetilde{\theta}_C$ from $\widetilde{Z}_{C,i}$ to $Z_{C,i}$. By (3.6) and Sard's theorem, there exists a point $z_i \in \operatorname{Nonsing}(Z_{C,i})$ such that, denoting by $W_{C,i}$ the set $\operatorname{Nonsing}(\widetilde{Z}_{C,i}) \cap (\widetilde{\theta}_{C,i})^{-1}(z_i)$, $W_{C,i}$ is a nonempty nonsingular Zariski locally closed subset of $\widetilde{Z}_{C,i}$ of complex dimension $\nu_i - d_i \geq c-1$ such that

(3.7) $\quad$ the rank of the differential of $\widetilde{\theta}_{C,i}$ at each point of $W_{C,i}$ is equal to d_i.

Indicate by $\widetilde{X}$ the real part of $\widetilde{X}_C$ and by $\widetilde{\theta}\colon \widetilde{X} \to X$ the restriction of $\widetilde{\theta}_C$ from $\widetilde{X}$ to X. From (3.2) and (3.3), we infer that

(3.8) $\quad$ the restriction of $\widetilde{\theta}$ from $(\widetilde{\theta})^{-1}(X \setminus Z)$ to $X \setminus Z$ is a biregular isomorphism.

Moreover, for each $x \in X \setminus Z$, we have

(3.9) $\quad$ the point $(\widetilde{\theta})^{-1}(x)$ of $\mathbb{P}^{m-1}(R)$ coincides with $\sigma(x)$.

Step IV. For each $e \in \{1, \dots, c-1\}$, indicate by $\mathbb{G}_{m,m-e}(C)$ the grassmannian of linear subspaces of $\mathbb{P}^{m-1}(C)$ of complex codimension e, equipped with the usual structure of irreducible algebraic variety over C. By point 1b) of Corollaire 6.11 of [2], by the proof of point 2) of the same corollary and by the second part of (3.6), for each $e \in \{1, \dots, c-1\}$, there exists a proper Zariski closed subset $\mathcal{G}'_{C,e}$ of $\mathbb{G}_{m,m-e}(C)$ such that, for each $N_C \in \mathbb{G}_{m,m-e}(C) \setminus \mathcal{G}'_{C,e}$ and for each $i \in \{1, \dots, h\}$, the following holds:

(3.10) $\quad W_{C,i} \cap N_C$ is nonempty and N_C is transverse to $W_{C,i}$ in $\mathbb{P}^{m-1}(C)$.

We will show that, for each $e \in \{1, \dots, c-1\}$ and for each $N_C \in \mathbb{G}_{m,m-e}(C) \setminus \mathcal{G}'_{C,e}$, the following holds:

(3.11) $\qquad Z_C$ is contained in the Zariski closure of $\widetilde{\theta}_C(\widetilde{X}_C \cap N_C)$ in X_C.

Let e and N_C be as above. For each $i \in \{1, \dots, h\}$, choose a point q_i in $W_{C,i} \cap N_C$ (which exists by the first part of (3.10)) and indicate by $\theta^*_{C,i} \colon \widetilde{Z}_{C,i} \cap N_C \to Z_{C,i}$ the restriction of $\widetilde{\theta}_{C,i}$ to $\widetilde{Z}_{C,i} \cap N_C$. Fix $i \in \{1, \dots, h\}$. By the second part of (3.10), N_C is transverse to $W_{C,i}$ in $\mathbb{P}^{m-1}(C)$ at q_i and, by (3.7), the rank of the differential $\mathrm{d}_{q_i}\widetilde{\theta}_{C,i}$ of $\widetilde{\theta}_{C,i}$ at q_i is equal to $d_i = \dim_C(Z_{C,i})$. Since the kernel of $\mathrm{d}_{q_i}\widetilde{\theta}_{C,i}$ coincides with the tangent space of $W_{C,i}$ at q_i, it follows that q_i is a nonsingular point of $\widetilde{Z}_{C,i} \cap N_C$ of complex dimension $v_i - e$ and the rank of the differential of $\theta^*_{C,i}$ at q_i is equal to d_i. In particular, we have that, for each $i \in \{1, \dots, h\}$, $\theta^*_{C,i}$ is dominating. This fact and the inclusion $\bigcup_{i=1}^{h} \theta^*_{C,i}(\widetilde{Z}_{C,i} \cap N_C) \subset \widetilde{\theta}_C(\widetilde{X}_C \cap N_C)$ imply (3.11).

Step V. Fix $e \in \{1, \dots, c-1\}$. Since $\widetilde{X}_C$ is irreducible, point 3) of Corollaire 6.11 of [2] ensures the existence of a proper Zariski closed subset $\mathcal{H}'_{C,e}$ of $\mathbb{G}_{m,m-e}(C)$ such that, for each $N_C \in \mathbb{G}_{m,m-e}(C) \setminus \mathcal{H}'_{C,e}$, it holds:

(3.12) $\qquad \widetilde{X}_C \cap N_C$ is irreducible and of complex codimension e in $\widetilde{X}_C$.

Indicate by $\mathcal{M}^*_{e,m}(C)$ the vector space of $(e \times m)$-matrices with coefficients in C and rank e, equipped with its natural structure of algebraic variety over C. Observe that the real part of $\mathcal{M}^*_{e,m}(C)$ coincides with $\mathcal{M}^*_{e,m}(R)$. Define the surjective complex regular map $\Phi_C \colon \mathcal{M}^*_{e,m}(C) \to \mathbb{G}_{m,m-e}(C)$ as follows: $\Phi_C(A_C) := \rho_C(\ker(A_C) \setminus \{0\})$, where $\rho_C \colon C^m \setminus \{0\} \to \mathbb{P}^{m-1}(C)$ is the natural projection and $\ker(A_C)$ is the kernel of the complex matrix A_C. Let $\mathcal{G}_{C,e} := (\Phi_C)^{-1}(\mathcal{G}'_{C,e})$ and $\mathcal{H}_{C,e} := (\Phi_C)^{-1}(\mathcal{H}'_{C,e})$. Define the proper Zariski closed subset $\mathcal{B}_e(Q)$ of $\mathcal{M}^*_{e,m}(R)$ by

$$\mathcal{B}_e(Q) := \mathcal{G}_{C,e}(R) \cup \mathcal{H}_{C,e}(R) \cup \mathcal{V}_e.$$

Let $A \in \mathcal{A}_e(Q) \setminus \mathcal{B}_e(Q)$. Since $A \in \mathcal{A}_e(Q)$, $(\pi_A \circ Q)^{-1}(0) \cap X^*$ is nonempty and hence, thanks to (3.5), the polynomial map $\pi_A \circ Q$ is good in X^*. In particular, we have that $(\pi_A \circ Q)^{-1}(0) \cap X^*$ is a nonempty (nonsingular) Zariski closed subset of X^* of codimension e. Let A_C be the matrix A, viewed as an element of $\mathcal{M}^*_{e,m}(C)$. Let $N_C := \Phi_C(A_C)$ and let N be the real part of N_C. By (3.9), we have:

$$\widetilde{X} \cap N \supset \sigma(X \setminus Z) \cap N = \sigma(\sigma^{-1}(N)) \supset \sigma((\pi_A \circ Q)^{-1}(0) \cap X^*).$$

In particular, the codimension of $\widetilde{X} \cap N$ in $\widetilde{X}$ is $\leq e$. On the other hand, $\widetilde{X} \cap N$ is the real part of $\widetilde{X}_C \cap N_C$ and hence, by (3.12), $\widetilde{X} \cap N$ is a (nonempty) irreducible Zariski closed subset of $\widetilde{X}$ of codimension e and is Zariski dense in $\widetilde{X}_C \cap N_C$. Let Y be the Zariski closure of $\widetilde{\theta}(\widetilde{X} \cap N)$ in X. From (3.8) and (3.9), it follows that Y is an irreducible algebraic subset of X of codimension e and $\widetilde{\theta}(\widetilde{X} \cap N) \setminus Z = \sigma^{-1}(N)$. In particular, we have:

(3.13) $$Y \setminus Z = \sigma^{-1}(N) = (\pi_A \circ Q)^{-1}(0) \setminus Z.$$

Let Y_C be the Zariski closure of $\widetilde{\theta}_C(\widetilde{X}_C \cap N_C)$ in X_C. Since $\widetilde{X} \cap N$ is Zariski dense in $\widetilde{X}_C \cap N_C$, it is easy to verify that Y is a Zariski dense subset of Y_C, and hence $Y_C(R) = Y$. Thanks to (3.11), Z_C is contained in Y_C, so we have that $Z = Z_C(R) \subset Y_C(R) = Y$. On the other hand, by definition of Q, Z is contained in $(\pi_A \circ Q)^{-1}(0)$. In this way, (3.13) implies that $Y = (\pi_A \circ Q)^{-1}(0)$ and the proof is complete. ∎

A byproduct of the argument used in the preceding proof is as follows.

Theorem 3.2 *Let $\mathbb{K}$ be an algebraically closed field. Let X be an irreducible algebraic subset of $\mathbb{K}^n$, let Z be an algebraic subset of X of codimension $c \geq 2$, let $q_1, \dots, q_\ell$ be generators of $\mathscr{I}_{\mathbb{K}^n}(Z)$ in $\mathbb{K}[x_1, \dots, x_n]$ and let X^* be the set of nonsingular points of X which are not contained in Z. Define: the polynomial map $q \colon X \to \mathbb{K}^\ell$ by $q(x) := (q_1(x), \dots, q_\ell(x))$, the integer $m := \ell(n+1)$ and the polynomial map $Q \colon X \to (\mathbb{K}^\ell)^{n+1} = \mathbb{K}^m$ by setting*

$$Q(x) := (q(x), x_1 q(x), \dots, x_n q(x))$$

for each $x = (x_1, \dots, x_n) \in X$. Indicate by $\mathcal{M}_{e,m}(\mathbb{K})$ the vector space of $(e \times m)$-matrices with coefficients in $\mathbb{K}$, equipped with its natural structure of algebraic variety over $\mathbb{K}$. Then, for each $e \in \{1, \dots, c-1\}$, there exists a proper Zariski closed subset $\mathcal{D}_e(Q)$ of $\mathcal{M}_{e,m}(\mathbb{K})$ with the following property: for each $A \in \mathcal{M}_{e,m}(\mathbb{K}) \setminus \mathcal{D}_e(Q)$, the polynomial map $\pi_A \circ Q \colon X \to \mathbb{K}^e$ is good in X^* and $(\pi_A \circ Q)^{-1}(0)$ is an irreducible algebraic subset of X of codimension e containing Z.

Proof of Theorem 1.2 Fix $e \in \{1, \dots, c-1\}$. Let $q_1, \dots, q_\ell$ be generators of $\mathscr{I}_{R^n}(Z)$ in $R[x_1, \dots, x_n]$ such that $\max_{i \in \{1, \dots, \ell\}} \deg(q_i) = \mu$, let $q \colon X \to R^\ell$ be the polynomial map defined by $q(x) := (q_1(x), \dots, q_\ell(x))$ and let $F' \colon X \to R^0 = \{0\}$ be the polynomial map constantly equal to 0. Define the integer $m' := \ell(n+1)$ and indicate by $Q' \colon X \to R^{m'}$ the polynomial map which sends $x = (x_1, \dots, x_n) \in X$ into $(q(x), x_1 q(x), \dots, x_n q(x), F'(x)) = (q(x), x_1 q(x), \dots, x_n q(x)) \in R^{m'}$. Define the integer $m := m' + e$ and indicate by $Q \colon X \to R^m$ the polynomial map which sends $x = (x_1, \dots, x_n) \in X$ into $(q(x), x_1 q(x), \dots, x_n q(x), F(x)) \in R^m$. By applying Theorem 3.1 to Q', we obtain a matrix $B' \in \mathcal{M}^*_{e,m'}(R)$ such that the polynomial map $P \colon X \to R^e$ defined by $P := \pi_{B'} \circ Q'$ has the properties required in (1). Let us prove (2). Indicate by B the matrix $(b_{ij})_{i,j}$ in $\mathcal{M}^*_{e,m}(R)$ such that $b_{ij} = 1$ if $j = m' + i$ for some $i \in \{1, \dots, e\}$ and $b_{ij} = 0$ otherwise. Since $\pi_B \circ Q = F$ and F is admissible, we have that $B \in \mathcal{A}_e(Q)$. By Theorem 3.1 applied to Q, it follows that $\mathcal{A}_e(Q)$ is an open (semialgebraic) subset of $\mathcal{M}^*_{e,m}(R)$ and there exists a proper Zariski closed subset $\mathcal{B}_e(Q)$ of $\mathcal{M}^*_{e,m}(R)$ such that, for each $A \in \mathcal{A}_e(Q) \setminus \mathcal{B}_e(Q)$, $\pi_A \circ Q$ is good in X^* and $(\pi_A \circ Q)^{-1}(0)$ is irreducible, has codimension e in X and contains Z. Choose a matrix D in $\mathcal{M}_{e,m}(R) \setminus (\mathcal{B}_e(Q) \cup \{B\})$. The affine line of $\mathcal{M}_{e,m}(R)$ containing B and D intersects $\mathcal{B}_e(Q)$ in a (possibly empty) finite set. In this way, since $\mathcal{A}_e(Q)$ is open in $\mathcal{M}_{e,m}(R)$, there exists a positive element ε of R such that $B + t\varepsilon(D - B) \in \mathcal{A}_e(Q) \setminus \mathcal{B}_e(Q)$ for each $t \in (-1, 1) \setminus \{0\}$. The polynomial map $G \colon X \to R^e$ defined by $G := \pi_{\varepsilon(D-B)} \circ Q$ has the desired properties. ∎

Proof of Theorem 1.3 The second part of the theorem follows immediately from the first one by choosing F constantly equal to 0. Let us prove the first part. Let $q_1, \dots, q_\ell$

be generators of $\mathscr{I}_{\mathbb{K}^n}(Z)$ in $\mathbb{K}[x_1,\dots,x_n]$ such that $\max_{i\in\{1,\dots,\ell\}}\deg(q_i) = \mu$, and let $P_1,\dots,P_e$ be polynomials in $\mathbb{K}[x_1,\dots,x_n]$ such that $F = (P_1,\dots,P_e)$ on X and $\max_{i\in\{1,\dots,e\}}\deg(P_i) = d$. It is now sufficient to repeat the preceding proof of point (2) of Theorem 1.2, using Theorem 3.2 instead of Theorem 3.1 with $P_1,\dots,P_e$, $q_1,\dots,q_\ell$ as generators of $\mathscr{I}_{\mathbb{K}^n}(Z)$ in $\mathbb{K}[x_1,\dots,x_n]$. ■

Remark 3.3. Let $X \subset R^n$ be a real algebraic set, let $F = (F_1,\dots,F_e)\colon X \to R^e$ be a polynomial map and let T be a nonempty Zariski open subset of Nonsing(X). For each $i \in \{1,\dots,e\}$, we denote by $F'_i\colon T \to R$ the restriction of F_i to T. We say that F is *very good in T* if the origin 0 of R is a regular value of each F'_i,

$$\bigcap_{i=1}^{e}(F'_i)^{-1}(0) = T \cap F^{-1}(0)$$

is nonempty and the nonsingular algebraic hypersurfaces $(F'_1)^{-1}(0),\dots,(F'_e)^{-1}(0)$ of T are in general position. Evidently, if F is very good in T, then it is good in T also. However, it is easy to construct examples of good polynomial maps that are not very good. The notion of very good polynomial map can be defined similarly over any field. In the statements of all our theorems, one can replace the adjective "good" with "very good". This can be done by slightly improving point (3.5) in the proof of Theorem 3.1.

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