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ON AN AFFINE CONNECTION WHICH ADMITS A VOLUME-LIKE FORM

BY
D. P. CHI AND Y. D. YOON

ABSTRACT. A necessary and sufficient condition to obtain a volume-like form from an affine connection is given in terms of the Čech cohomology, after the volume-like form is naturally defined without a Riemannian metric. A necessary condition for an affine connection to be a Riemannian connection for some metric is also given.

1. **Introduction.** When the base manifold M is endowed with a Riemannian metric $ds^2 = \sum g_{ij}\, dx^i \otimes dx^j$, we get an affine connection D, called a Riemannian connection, which is locally expressed by the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} \sum_\alpha g^{i\alpha} \left(\frac{\partial g_{j\alpha}}{\partial x^k} + \frac{\partial g_{k\alpha}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\alpha} \right),$$

and a volume form

$$dV = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n.$$

A simple computation leads us to the following formula [**1**, p. 294]

$$(1) \qquad \frac{\partial}{\partial x^\alpha} \log \sqrt{\det(g_{ij})} = \sum_k \Gamma^k_{\alpha k}.$$

The equation (1) shows the relation between a volume form and a Riemannian connection. Furthermore, the equation (1) is almost independent of the given metric ds^2 and therefore we could obtain a volume form from an affine connection without a metric.

From now on, the affine connection will be expressed by the matrix of connection 1-forms i.e., locally

$$D = d + \omega,$$

where ω is the matrix of connection 1-forms.

Then the equation (1) can be *locally* rewritten as

$$(*) \qquad dG = \operatorname{tr} \omega,$$

where the volume form is $dV = \exp(G)\, dx^1 \wedge \cdots \wedge dx^n$.

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Note that the equation $(*)$ is not global. Actually, if we have

$$\begin{aligned} dV &= \exp(G_\alpha)\, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n \qquad \text{on } (U_\alpha, x_\alpha) \\ &= \exp(G_\beta)\, dx_\beta^1 \wedge \cdots \wedge dx_\beta^n \qquad \text{on } (U_\beta, x_\beta), \end{aligned}$$

then

$$\exp(G_\alpha) = \exp(G_\beta) \left| \frac{\partial x_\beta}{\partial x_\alpha} \right| \qquad \text{on } U_\alpha \cap U_\beta.$$

Therefore we should regard the global solution of the equation $(*)$ as the n-form dV. In this paper we found a local and a global obstructions which seem to be new ones in affine differential geometry.

Also note that, if G is a local solution of $(*)$, so is $G + c$ for any constant c. Hence the volume form is well defined up to a scalar multiple.

DEFINITION. A nowhere vanishing n-form dV defined on a smooth orientable manifold M of dimension n is said to be the affine volume form with respect to an affine connection D, if, when dV is locally expressed by $\pm \exp(G)\, dx^1 \wedge \cdots \wedge dx^n$, the equation $dG = \operatorname{tr} \omega$ is satisfied.

REMARK. When the connection arises from a Riemannian metric on a path connected manifold the Riemannian volume form is just a costant multiple of the affine volume form of the connection because of the equation (*).

2. **Local solvability of $dG = \operatorname{tr} \omega$.** If M is orientable, which we now assume, then we may obtain a locally finite collection Φ of local charts (U_α, x_α) which satisfy

(1) the open sets cover M,

(2) each U_α is simply connected, and

(3) for each two $(U_\alpha, x_\alpha), (U_\beta, x_\beta) \in \Phi$, the transition matrix $A_{\alpha\beta} = (\frac{\partial x_\alpha}{\partial x_\beta})$ has positive determinant.

We choose such a collection Φ Let Ω be the curvature matrix of an affine connection D i.e., $\Omega = d\omega + \omega \wedge \omega$.

THEOREM 1. *Let D be an affine connection with the curvature matrix Ω. Then the equation $(*)$ has a local solution G_α on each $(U_\alpha, x_\alpha) \in \Phi$ if, and only if,* $\operatorname{tr} \Omega \equiv 0$.

PROOF. $(\Longleftarrow)$

$$\begin{aligned} 0 = \operatorname{tr} \Omega &= \operatorname{tr} (d\omega + \omega \wedge \omega) \\ &= \operatorname{tr} (d\omega) \\ &= d(\operatorname{tr} \omega). \end{aligned}$$

Thus $\operatorname{tr} \omega$ is a closed 1-form on each simply connected U_α. Since $H^1_{\text{deRham}}(U_\alpha) \equiv 0$ because U_α is simply connected, $\operatorname{tr} \omega$ is exact on U_α. That is, there is a smooth function G_α on U_α such that $dG_\alpha = \operatorname{tr} \omega_\alpha$.

$(\Longrightarrow)$

Conversely, if $(*)$ has a local solution G_α on each $(U_\alpha, x_\alpha) \in \Phi$,

$$0 = ddG_\alpha = d(\operatorname{tr} \omega) = \operatorname{tr} \Omega.$$

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Since U_α's cover M, tr $\Omega \equiv 0$ on M. ∎

Theorem 1 shows the local solvability of the equation $(*)$.

Now we pass from the local solutions to a global solution dV which will be a special affine volume form.

Let A be an $n \times n$ non-singular matrix of smooth functions.

Then the following identity is well known.

$$\operatorname{tr}(A^{-1}dA) = |A|^{-1}d|A|.$$

And, if $|A| > 0$, $|A|^{-1}dA = d(\log|A|)$.

Using this identity, we obtain

$$\begin{aligned}\operatorname{tr}\omega_\beta &= \operatorname{tr}(A_{\alpha\beta}^{-1}\omega_\alpha A_{\alpha\beta} + A_{\alpha\beta}^{-1}dA_{\alpha\beta})\\ &= \operatorname{tr}(\omega_\alpha) + \operatorname{tr}(A_{\alpha\beta}^{-1}dA_{\alpha\beta})\\ &= \operatorname{tr}(\omega_\alpha) + |A_{\alpha\beta}|^{-1}d|A_{\alpha\beta}|\\ &= \operatorname{tr}(\omega_\alpha) + d(\log|A_{\alpha\beta}|), \qquad \text{if } |A_{\alpha\beta}| > 0.\end{aligned}$$

From now on, we assume that tr $\Omega = 0$ i.e., the equation $(*)$ is locally solvable.

Choose a solution G_α on each $(U_\alpha, x_\alpha) \in \Phi$, and consider a set $\{G_\alpha\}$ of such solutions.

On the intersection $U_\alpha \cap U_\beta$,

$$\begin{aligned}dG_\alpha &= \operatorname{tr}\omega_\alpha\\ dG_\beta &= \operatorname{tr}\omega_\beta\\ &= \operatorname{tr}\omega_\alpha + d(\log|A_{\alpha\beta}|)\\ &= dG_\alpha + d(\log|A_{\alpha\beta}|)\end{aligned}$$

Hence we get, on $U_\alpha \cap U_\beta$,

$$G_\beta - G_\alpha - \log|A_{\alpha\beta}| \equiv \text{constant} \quad \text{on } U_\alpha \cap U_\beta.$$

We denote this constant $c_{\alpha\beta}$, i.e.,

$$c_{\alpha\beta} \equiv G_\beta - G_\alpha - \log|A_{\alpha\beta}|.$$

LEMMA 1. *The set $\{c_{\alpha\beta}\}$ is a 1-cocycle whose coefficients are in the constant sheaf $M \times R$ in the Čech cohomology sense.*

PROOF. (i)

$$\begin{aligned}c_{\beta\alpha} &= G_\alpha - G_\beta - \log|A_{\beta\alpha}|\\ &= -G_\beta + G_\alpha + \log|A_{\alpha\beta}|\\ &= -c_{\alpha\beta}\end{aligned}$$

Therefore $\{c_{\alpha\beta}\}$ is a 1-cochain in the Čech sense.

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(ii) On $U_\alpha \cap U_\beta \cap U_\gamma$,

$$\begin{aligned} c_{\alpha\beta} &= G_\beta - G_\alpha - \log|A_{\alpha\beta}| \\ c_{\beta\gamma} &= G_\gamma - G_\beta - \log|A_{\beta\gamma}| \\ c_{\gamma\alpha} &= G_\alpha - G_\gamma - \log|A_{\gamma\alpha}| \\ c_{\alpha\beta} + c_{\beta\gamma} + c_{\gamma\alpha} &= -\log|A_{\alpha\beta}A_{\beta\gamma}A_{\gamma\alpha}| \\ &= 0. \end{aligned}$$

Therefore $\delta\{c_{\alpha\beta}\} = 0$ i.e., $\{c_{\alpha\beta}\}$ is a 1-cocycle in the Čech sense. ∎

Lemma 1 means that : $\{c_{\alpha\beta}\}| \in \check{\mathrm{H}}^1(\Phi, M \times R)$.
Since $\check{\mathrm{H}}^1(U_\alpha) \cong 0$ for all α,

$$\check{\mathrm{H}}^1(\Phi, M \times R) \cong \check{\mathrm{H}}^1(M, R).$$

Thus we obtain an obstruction $\theta \overset{\text{def}}{\equiv} [c_{\alpha\beta}] \in \check{\mathrm{H}}^1(M, R)$.

LEMMA 2. *The obstruction θ is independent of the choice of the solutions G_α's.*

PROOF. Let $\tilde{G}_\alpha$ be another choice of the local solutions and let $\tilde{c}_{\alpha\beta} = \tilde{G}_\beta - \tilde{G}_\alpha - \log|A_{\alpha\beta}|$.
Then $\tilde{G}_\alpha = G_\alpha + c_\alpha$ for some constant c_α on U_α because $d\tilde{G}_\alpha = \operatorname{tr}\omega = dG_\alpha$, and

$$\begin{aligned} \tilde{c}_{\alpha\beta} &= G_\beta - G_\alpha - \log|A_{\alpha\beta}| + c_\beta - c_\alpha \\ &= c_{\alpha\beta} + c_\beta - c_\alpha \end{aligned}$$

i.e.,

$$\{\tilde{c}_{\alpha\beta}\} = \{c_{\alpha\beta}\} + \delta\{c_\alpha\}.$$

Hence $[\{\tilde{c}_{\alpha\beta}\}] = [\{c_{\alpha\beta}\}]$ in $\check{\mathrm{H}}^1(\Phi, M \times R)$.
Therefore they give the same $\theta \in \check{\mathrm{H}}^1(M, R)$. ∎

LEMMA 3. *θ is also independent of the choice of the locally finite collection Φ.*

PROOF. Step 1: Let $\Phi = \{(U_j, x_j) \mid j \in J\}, \tilde{\Phi} = \{(V_\alpha, x_\alpha) \mid \alpha \in \Lambda\}$ be two collections as above, and let $f : \Lambda \longrightarrow J$ be a map such that

$$V_\alpha \subset U_{f(\alpha)},\ x_\alpha = x_{f(\alpha)}\,|_{V_\alpha}.$$

Then, taking $G_\alpha \equiv G_{f(\alpha)}$ on V_α, we easily find that the two $[\{c_{ij}\}]$, $[\{\tilde{c}_{\alpha\beta}\}]$, which are computed from $\Phi, \tilde{\Phi}$ respectively are the same in the cohomology group $\check{\mathrm{H}}^1(M, R)$.
Step 2: Let $\mathcal{U} = \{U_\alpha | \ \alpha \in \Lambda\}$ be a locally finite open covering of M, and $\Phi, \tilde{\Phi}$ two collections of local charts as above such that

$$\begin{aligned} \Phi &= \{(U_\alpha, x_\alpha) | \ \alpha \in \Lambda\} \\ \tilde{\Phi} &= \{(U_\alpha, \tilde{x}_\alpha) | \ \alpha \in \Lambda\}. \end{aligned}$$

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On U_α, take G_α and $\tilde{G}_\alpha$ such that

$$dG_\alpha = \operatorname{tr} \omega_\alpha \quad \text{w.r.t. } (U_\alpha, x_\alpha)$$
$$d\tilde{G}_\alpha = \operatorname{tr} \tilde{\omega}_\alpha \quad \text{w.r.t. } (U_\alpha, \tilde{x}_\alpha).$$

And define two 1-cocyles $\{c_{\alpha\beta}\}$, and $\{\tilde{c}_{\alpha\beta}\}$, respectively. Here,

$$\tilde{\omega}_\alpha = P_\alpha^{-1}\omega_\alpha P_\alpha + P_\alpha^{-1} dP_\alpha,$$

where $P_\alpha = (\frac{\partial x_\alpha}{\partial \tilde{x}_\alpha})$ on U_α. Hence we get $\tilde{G}_\alpha = G_\alpha + \log |P_\alpha| + c_\alpha$ for some constant c_α on U_α. Then

$$\begin{aligned}
\tilde{c}_{\alpha\beta} &= \tilde{G}_\beta - \tilde{G}_\alpha - \log |\tilde{A}_{\alpha\beta}| \\
&= G_\beta + \log |P_\beta| + c_\beta - G_\alpha - \log |P_\alpha| - c_\alpha - \log |\tilde{A}_{\alpha\beta}| \\
&= G_\beta - G_\alpha - \log |P_\beta^{-1}\tilde{A}_{\alpha\beta}P_\alpha| + c_\beta - c_\alpha \\
&= c_{\alpha\beta} + c_\beta - c_\alpha,
\end{aligned}$$

where $\tilde{A}_{\alpha\beta} = (\frac{\partial \tilde{x}_\alpha}{\partial \tilde{x}_\beta})$.

Thus $[\{\tilde{c}_{\alpha\beta}\}] = [\{c_{\alpha\beta}\}] \in \check{H}^1(\mathcal{U}, M \times R)$.

Step 3: For two coverings

$$\Phi_1 = \{(U_i, x_i) | \ i \in I\},$$
$$\Phi_2 = \{(V_j, x_j) | \ j \in J\},$$

we can construct two $\tilde{\Phi}_1, \tilde{\Phi}_2$ as follows;

$$\tilde{\Phi}_1 = \{(U_i \cap V_j, x_i |_{U_i \cap V_j} \mid (i,j) \in I \times J\}$$
$$\tilde{\Phi}_2 = \{(U_i \cap V_j, x_j |_{U_i \cap V_j} \mid (i,j) \in I \times J\}$$

Then both $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ satisfy the above conditions.

Let $\theta_1, \theta_2, \tilde{\theta}_1, \tilde{\theta}_2$ be the cohomology elements with respect to $\Phi_1, \Phi_2, \tilde{\Phi}_1, \tilde{\Phi}_2$ respectively. Then they are all the same by Step 1 and Step 2. This proves our Lemma. ∎

Note that the above Lemmas show that the obstruction θ depends only on the affine connection D and the base manifold M.

We are now ready to prove the global solvability of the equation $dG = \operatorname{tr} \omega$.

3. **Global solvability of** $dG = \operatorname{tr} \omega$.

THEOREM 2. *Any collection of local solutions, $\{G_\alpha\}$, gives a globally well defined solution dV an affine volume form if, and only if, the obstruction $\theta = 0$ in $\check{H}^1(M, R)$.*

PROOF. ($\Longleftarrow$)

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If $\theta = 0$ in $\check{H}^1(M, R)$, $[\{c_{\alpha\beta}\}] = 0$ in $\check{H}^1(\Phi, M \times R)$. That is, $\{c_{\alpha\beta}\} = \delta\{c_\alpha\}$ for some 0-cochain $\{c_\alpha\}$,i.e.,

$$c_{\alpha\beta} = c_\beta - c_\alpha.$$

Now, define $dV \equiv \exp(G_\alpha - c_\alpha)\, dx^1 \wedge \cdots \wedge dx^n$ on each $(U_\alpha, x_\alpha) \in \Phi$.

Then $d(G_\alpha - c_\alpha) = \operatorname{tr} \omega$ on each U_α, and on every intersection $U_\alpha \cap U_\beta \neq \emptyset$,

$$\begin{aligned}\exp(G_\alpha - c_\alpha) dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n &= \exp(G_\alpha - c_\alpha)|A_{\alpha\beta}|\, dx_\beta^1 \wedge \cdots \wedge dx_\beta^n \\ &= \exp(G_\alpha - c_\alpha + \log|A_{\alpha\beta}|)\, dx_\beta^1 \wedge \cdots \wedge dx_\beta^n \\ &= \exp(G_\beta - c_\beta)\, dx_\beta^1 \wedge \cdots \wedge dx_\beta^n,\end{aligned}$$

because $c_{\alpha\beta} = c_\beta - c_\alpha = G_\beta - G_\alpha - \log|A_{\alpha\beta}|$.

Therefore dV is a well defined n-form which satisfies the equation $(*)$.

$(\Longrightarrow)$

Conversely, let dV be a affine volume form. Then we may put $dV = \exp(G_\alpha)\, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$ on each (U_α, x_α), and we know that $G_\beta = G_\alpha + \log|A_{\alpha\beta}|$ on the intersection $U_\alpha \cap U_\beta$.

Hence we have $c_{\alpha\beta} = 0$ for all α, β.

Therefore $\theta = 0$ in $\check{H}^1(M, R)$. ∎

From the Theorem 1 and Theorem 2 we obtain the complete main result.

MAIN THEOREM. *An affine connection D admits an affine volume form dV if, and only if,* $\operatorname{tr} \Omega = 0$ *and* $\theta = 0$.

COROLLARY 1. *On a orientable smooth manifold M with* $\check{H}^1(M) = 0$, *any affine connection D admits an affine volume form if, and only if,* $\operatorname{tr} \Omega = 0$.

PROOF. trivial. ∎

COROLLARY 2. *An affine connection D with* $\operatorname{tr} \Omega \neq 0$ *or* $\theta \neq 0$ *can not be a Riemannian connection, i.e., any metric can not induce D as a Riemannian connection.*

PROOF. If D is induced from a metric, it must give a volume form. ∎

The obstruction θ is very far from being trivial since there are many affinely flat manifolds which can not have a volume like form. For example let $a \in D^* = \{z \in C^* \mid |z| < 1\}$ and let $\mathbb{Z}$ act on C^* by $n(z) = a^n z$. Set $T_a^2 = C^*/Z$ together the induced affine structure from the plane. Then T_a^2 is a affinely flat manifold.

Let $U_1 = \{z \in C^* \mid |a| + \epsilon < |z| < 1 - \epsilon\}$ and $U_2 = \{z \in C^* \mid |a| \leq |z| < |a| + 2\epsilon \text{ or } 1 - 2\epsilon < |z| \leq 1\}$. Then it is easy to see that $<\theta, \alpha> = 0$ and $<\theta, \beta> = -2\log|a|$ since the Jacobian determinant of $z \longrightarrow az$ as a real linear map is $|a|^2$. We thank the refree for suggesting the above example to us.

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Department of Mathematics
College of Natural Sciences
Seoul National University
Seoul 151-748
Korea