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THE OPTIMAL DIVIDEND PROBLEM IN THE DUAL MODEL

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Abstract

We study de Finetti's optimal dividend problem, also known as the optimal harvesting problem, in the dual model. In this model, the firm value is affected both by continuous fluctuations and by upward directed jumps. We use a fixed point method to show that the solution of the optimal dividend problem with jumps can be obtained as the limit of a sequence of stochastic control problems for a diffusion. In each problem, the optimal dividend strategy is of barrier type, and the rate of convergence of the barrier and the corresponding value function is exponential.

Keywords: Optimal distribution of dividends; de Finetti's dividend problem; optimal harvesting; singular stochastic control; jump diffusion model

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1. Introduction

In the classical optimal dividend problem by de Finetti one seeks to maximize the expected value of the discounted dividends paid out to the share holders of a firm until the ruin time. In [2] and [15], this dividend problem was solved in the case when the underlying firm value is modelled as a linear Brownian motion using methods from singular stochastic control theory. It was shown that the optimal strategy is of barrier type, i.e. to distribute all surplus above a certain level as dividends, and then do nothing as long as the firm value is below this level.

More recent literature has to a large extent dealt with models allowing for negative jumps of the firm value; see, for example, [1], [5], [13], and [14]. The main application of such models is in the insurance industry, where the negative jumps have a natural interpretation as insurance claims. Mathematically, the inclusion of negative jumps is tractable since the process then never jumps over the barrier.

We study the optimal dividend problem by de Finetti in a model allowing for positive jumps of the underlying firm value. This is also known in the literature as the dividend problem in the dual model; cf. [3], [4], [6], and [9]. To include positive jumps is natural for example in the case of a research-based firm. The jump is then interpreted as the net present value of future income stemming from an invention. Since the firm value may jump over the barrier (we show below that a barrier strategy is optimal also in our setting), there is in general little hope of an explicit solution of the dividend problem. Instead, we connect the dividend problem with a problem of finding a fixed point of a certain functional operator. Moreover, we show that the fixed point can be obtained as the limit of a recursively defined sequence of stochastic control problems for a diffusion process, and each problem in the sequence is readily solved using

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standard methods for stochastic singular control of a diffusion process. One advantage of this fixed-point approach, in comparison with for example a study of the dividend problem based on viscosity solutions of integrodifferential equations, is that the fixed-point approach gives control of the regularity of the value function. In particular, it is straightforward to provide verification results based on Itô's formula that connect the analytical solution of a free boundary problem with the corresponding stochastic control problem.

The technique that we use to write the dividend problem for a jump process as the limit of a sequence of control problems for a diffusion process is inspired by corresponding studies in optimal stopping theory. The classical references are [10] and [12], where this technique is developed for piecewise deterministic Markov processes. For generalizations to processes involving both jumps and Brownian fluctuations, see [7], [11], and the references therein. To the best of our knowledge, this technique has not been applied to any singular stochastic control problem before. Along with the financial interest of the dividend problem in the dual model mentioned above, a key contribution of the current paper is thus to provide the technical details of this procedure for the first time in stochastic control theory. Even though the overall structure of the procedure is the same as in optimal stopping theory, we encountered a number of technical problems, for example in connection with the concavity of the value function and with the monotonicity of the sequence of barriers, which seem intimately connected with stochastic control theory.

The paper is organized as follows. In Section 2 we set up the model and formulate de Finetti's optimal dividend problem in the presence of positive jumps, and we prove a verification result. In Section 3 we introduce a related stochastic control problem for a diffusion process, and we show that it can be solved using a free boundary approach. Next, in Section 4 we use this control problem as a building block in the recursive construction of a sequence of control problems, and the corresponding solutions are shown to converge to the dividend problem formulated in Section 2. In Section 5 we show that the rate of convergence is exponential both for the value functions and for the corresponding barriers. Finally, in Section 6 we provide a sensitivity analysis of the solution with respect to the different parameters of the model.

Remark. After finishing a first version of the current paper, we were informed about the article [8]. In that paper the authors proved the optimality of a barrier strategy in de Finetti's problem for spectrally positive Lévy processes using fluctuation theory, and the optimal barrier is characterized in terms of a scale function. The current paper offers an alternative approach of determining the optimal barrier and the value function under the additional hypothesis of a finite activity of jumps. This assumption is crucial below in the definition of the functional operator J, and it appears not easily disposed of. However, our approach does seem flexible enough to extend in another direction, that is, to include for example models where the drift, volatility, jump rate, and jump size are level dependent.

2. The optimal dividend problem and a verification result

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting a Poisson random measure $N(\mathrm{d}t, \mathrm{d}y)$ on $[0, \infty) \times \mathbb{R}_+$ and a Brownian motion $W = \{W_t,\ t \geq 0\}$ such that N and W are independent. We assume that the mean measure of N is $\lambda F(\mathrm{d}y)\,\mathrm{d}t$, where F is a probability distribution on $\mathbb{R}_+$ with finite mean $\varepsilon := \int_0^\infty yF(\mathrm{d}y)$ and the jump intensity $\lambda > 0$ is a constant.

Let $D = \{D_t,\ t \geq 0\}$ be a nonnegative, right-continuous, and nondecreasing process adapted to the filtration generated by N and W. Below D_t represents the cumulative dividends paid out up to time t. In particular, if $D_0 > 0$ then a dividend payment of size D_0 is distributed at time 0. Let $X^D = \{X_t^D,\ t \geq 0\}$ be the risk process of a firm after dividends are distributed according to the strategy D. We assume that X^D satisfies

$$X_t^D = X_0^D + \mu t + \sigma W_t + \int_0^t \int_0^\infty y N(\mathrm{d}t, \mathrm{d}y) - D_t, \tag{1}$$

where the drift μ and the volatility σ are constants. Note that in the absence of dividend payments, i.e. $D \equiv 0$, the firm value evolves between jump times according to a linear Brownian motion with drift μ and volatility σ. Also, note that each negative jump of the firm value corresponds to a lump sum dividend payment. For a given dividend policy D, the ruin time γ^D of the firm is defined by

$$\gamma^D = \inf\{t \geq 0 \mid X_t^D \leq 0\}.$$

We only consider dividend strategies D such that

$$X_0^D \geq 0 \quad \text{and} \quad D_t - D_{t-} \leq X_{t-}^D + \int_0^\infty y N(\mathrm{d}t, \mathrm{d}y) \quad \text{for } t \in (0, \gamma^D]. \tag{2}$$

The class of such strategies is denoted Θ.

Remark. Condition (2) asserts that a lump sum dividend payment never results in a negative value of X^D. Note that we allow for lump sum dividend payments to occur at the same time as a positive jump in X^D.

Our objective is to solve the stochastic control problem

$$V(x) = \sup_{D \in \Theta} \mathbb{E}_{x - D_0}\bigg[D_0 + \int_0^{\gamma^D} \mathrm{e}^{-rt}\,\mathrm{d}D_t\bigg], \tag{3}$$

where

$$\int_0^{\gamma^D} \mathrm{e}^{-rt}\,\mathrm{d}D_t := \int_{(0,\gamma^D]} \mathrm{e}^{-rt}\,\mathrm{d}D_t,$$

r is a constant positive interest rate and $x - D_0 = X_0^D$ denotes the initial firm value immediately after dividends at time 0 have been deducted. Accordingly, the parameter x denotes the initial firm value before the dividends at time 0 have been deducted. Note that $V(x) \geq x$ since the strategy

$$D_t = x \tag{4}$$

of deducting all money as dividends immediately is admissible. Next, for a given $b > 0$, define the barrier strategy D^b to be the minimal dividend strategy D such that $X_t^D \leq b$ for all $t \geq 0$. More explicitly, if $X_t = x + \mu t + \sigma W_t + \int_0^t \int_0^\infty y N(\mathrm{d}t, \mathrm{d}y)$ is the firm value in the absence of dividend payments and

$$S_t = \sup\{X_s : s \in [0, t]\}$$

is its supremum process, then $D_t^b = (S_t - b)^+$ and $X_t^{D^b} = X_t - (S_t - b)^+$.

A common approach in the literature on the dividend problem is to somehow construct a candidate solution, and then to appeal to a verification result which shows that the candidate solution actually coincides with the value function. Since we have not been able to find in the literature a rigorous verification result that applies in the current setting, we include a detailed result (Theorem 1). Let $\mathcal{A}$ be the differential operator

$$\mathcal{A}v = \frac{\sigma^2}{2} v'' + \mu v'.$$

Theorem 1. (Verification result for the dividend problem with jumps.) *Assume that*

$$v\colon [0,\infty) \to [0,\infty)$$

is twice continuously differentiable with

(i) $1 \le v' \le C$ *for some constant* C,

(ii) $\mathcal{A}v - rv + \lambda \int_0^\infty (v(x+y) - v(x)) F(\mathrm{d}y) \le 0$.

Then $V \le v$.

If there exists a point $b > 0$ *such that* v, *in addition to (i) and (ii), also satisfies*

(iii) $v(0) = 0$,

(iv) $\mathcal{A}v - rv + \lambda \int_0^\infty (v(x+y) - v(x)) F(\mathrm{d}y) = 0$ *for* $x \in (0, b]$,

(v) $v(x) = v(b) + x - b$ *for* $x \in (b, \infty)$,

then $V = v$, *and the barrier strategy* D^b *is optimal.*

Proof. Assume that (i) and (ii) hold. Let $D \in \Theta$ be a given dividend strategy, and let D^{c} be its continuous part. Itô's formula for semimartingales yields

$$\begin{aligned}
&\mathrm{e}^{-r(t\wedge\gamma^D)} v(X^D_{t\wedge\gamma^D}) \\
&\quad = v(X_0^D) - \int_0^{t\wedge\gamma^D} \mathrm{e}^{-rt} v'(X^D_{s-})\,\mathrm{d}D^{\mathrm{c}}_s + \int_0^{t\wedge\gamma^D} \sigma \mathrm{e}^{-rs} v'(X^D_{s-})\,\mathrm{d}W_s \\
&\qquad + \int_0^{t\wedge\gamma^D} \mathrm{e}^{-rs} (\mathcal{A} - r) v(X^D_{s-})\,\mathrm{d}s \\
&\qquad + \sum_{s \le t\wedge\gamma^D} \mathrm{e}^{-rs} \bigg(v\bigg(X^D_{s-} + \int_0^\infty y N(\mathrm{d}s, \mathrm{d}y) - \Delta D_s \bigg) - v(X^D_{s-}) \bigg) \\
&\quad = v(X_0^D) - \int_0^{t\wedge\gamma^D} \mathrm{e}^{-rt} v'(X^D_{s-})\,\mathrm{d}D^{\mathrm{c}}_s + \int_0^{t\wedge\gamma^D} \sigma \mathrm{e}^{-rs} v'(X^D_{s-})\,\mathrm{d}W_s \\
&\qquad + \int_0^{t\wedge\gamma^D} \mathrm{e}^{-rs} \bigg((\mathcal{A} - r) v(X^D_{s-}) + \lambda \int_0^\infty (v(X^D_{s-} + y) - v(X^D_{s-})) F(\mathrm{d}y) \bigg)\,\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\gamma^D} \mathrm{e}^{-rs} \int_0^\infty (v(X^D_{s-} + y) - v(X^D_{s-})) \tilde{N}(\mathrm{d}s, \mathrm{d}y) \\
&\qquad + \sum_{s \le t\wedge\gamma^D} \mathrm{e}^{-rs} \bigg(v\bigg(X^D_{s-} + \int_0^\infty y N(\mathrm{d}s, \mathrm{d}y) - \Delta D_s \bigg) - v\bigg(X^D_{s-} + \int_0^\infty y N(\mathrm{d}s, \mathrm{d}y) \bigg) \bigg),
\end{aligned}$$

where

$$\tilde{N}(\mathrm{d}s, \mathrm{d}y) := N(\mathrm{d}s, \mathrm{d}y) - \lambda F(\mathrm{d}y)\,\mathrm{d}s$$

is the compensated Poisson random measure. Since v' is bounded, the integral with respect to Brownian motion is a martingale. Similarly, the integral with respect to $\tilde{N}$ is also a martingale,

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so taking expected values and using (i) and (ii) gives

$$
\begin{aligned}
&\mathbb{E}_{x-D_0}[\mathrm{e}^{-r(t\wedge\gamma^D)}v(X^D_{t\wedge\gamma^D})]\\
&\quad= v(x-D_0) - \mathbb{E}_{x-D_0}\bigg[\int_0^{t\wedge\gamma^D} \mathrm{e}^{-rs}v'(X^D_{s-})\,\mathrm{d}D^{\mathrm{c}}_s\bigg]\\
&\qquad+ \mathbb{E}_{x-D_0}\bigg[\int_0^{t\wedge\gamma^D} \mathrm{e}^{-rs}\bigg((\mathcal{A}-r)v(X^D_{s-}) + \lambda\int_0^\infty (v(X^D_{s-}+y) - v(X^D_{s-}))F(\mathrm{d}y)\bigg)\,\mathrm{d}s\bigg]\\
&\qquad+ \mathbb{E}_{x-D_0}\bigg[\sum_{s\le t\wedge\gamma^D} \mathrm{e}^{-rs}\bigg(v\bigg(X^D_{s-} + \int_0^\infty yN(\mathrm{d}s,\mathrm{d}y) - \Delta D_s\bigg)\\
&\qquad\qquad\qquad - v\bigg(X^D_{s-} + \int_0^\infty yN(\mathrm{d}s,\mathrm{d}y)\bigg)\bigg)\bigg]\\
&\quad\le v(x-D_0) - \mathbb{E}_{x-D_0}\bigg[\int_0^{t\wedge\gamma^D} \mathrm{e}^{-rs}\,\mathrm{d}D_s\bigg].
\end{aligned}
$$

Next, using $v' \ge 1$ and $v \ge 0$, and letting $t \to \infty$, we have

$$
v(x) \ge v(x-D_0) + D_0 \ge \mathbb{E}_{x-D_0}\bigg[D_0 + \int_0^{\gamma^D} \mathrm{e}^{-rs}\,\mathrm{d}D_s\bigg]
$$

by monotone convergence. Since the dividend strategy D was arbitrary, this implies that

$$
V(x) = \sup_{D\in\Theta} \mathbb{E}_{x-D_0}\bigg[D_0 + \int_0^{\gamma^D} \mathrm{e}^{-rs}\,\mathrm{d}D_s\bigg] \le v(x).
$$

Now assume that (i)–(v) holds for some $b > 0$, and choose the dividend strategy $D = D^b$ as the barrier strategy that pushes the controlled process X^D down below the level b. As above,

$$
\begin{aligned}
&\mathbb{E}_{x-D_0}[\mathrm{e}^{-r(t\wedge\gamma^D)}v(X^D_{t\wedge\gamma^D})]\\
&\quad= v(x-D_0) - \mathbb{E}_{x-D_0}\bigg[\int_0^{t\wedge\gamma^D} \mathrm{e}^{-rs}v'(X^D_{s-})\,\mathrm{d}D^{\mathrm{c}}_s\bigg]\\
&\qquad+ \mathbb{E}_{x-D_0}\bigg[\int_0^{t\wedge\gamma^D} \mathrm{e}^{-rs}\bigg((\mathcal{A}-r)v(X^D_{s-}) + \lambda\int_0^\infty (v(X^D_{s-}+y) - v(X^D_{s-}))F(\mathrm{d}y)\bigg)\,\mathrm{d}s\bigg]\\
&\qquad+ \mathbb{E}_{x-D_0}\bigg[\sum_{s\le t\wedge\gamma^D} \mathrm{e}^{-rs}\bigg(v\bigg(X^D_{s-} + \int_0^\infty yN(\mathrm{d}s,\mathrm{d}y) - \Delta D_s\bigg)\\
&\qquad\qquad\qquad - v\bigg(X^D_{s-} + \int_0^\infty yN(\mathrm{d}s,\mathrm{d}y)\bigg)\bigg)\bigg]\\
&\quad= v(x-D_0) - \mathbb{E}_{x-D_0}\bigg[\int_0^{t\wedge\gamma^D} \mathrm{e}^{-rs}\,\mathrm{d}D_s\bigg],
\end{aligned}
$$

where the last equality follows from the fact that $v' = 1$ on the support of the measure $\mathrm{d}D$ and (iv). By bounded convergence and monotone convergence, letting $t \to \infty$ gives

$$
v(x-D_0) = \mathbb{E}_{x-D_0}\bigg[\int_0^{\gamma^D} \mathrm{e}^{-rs}\,\mathrm{d}D_s\bigg].
$$

Since $D_0 = (x - b)^+$ and $v'(x) = 1$ for $x \geq b$, $v(x) = \mathbb{E}_{x-D_0}[D_0 + \int_0^{\gamma^D} \mathrm{e}^{-rs}\,\mathrm{d}D_s]$, which completes the proof of the second claim.

Corollary 1. *If $\mu + \varepsilon\lambda \leq 0$ then $V(x) = x$, so D defined in (4) is an optimal dividend strategy.*

Proof. The function $v(x) = x$ certainly satisfies (i). Moreover,

$$\mathcal{A}v - rv + \lambda \int_0^\infty (v(x+y) - v(x))F(\mathrm{d}y) = \mu + \lambda\varepsilon - rx \leq 0,$$

so the first part of Theorem 1 yields $V \leq x$. Since we always have $V \geq x$, the result follows.

Throughout the remainder of this paper, we assume that

$$\mu + \varepsilon\lambda > 0, \tag{5}$$

so that the expected value of the firm is increasing in time if no dividends are deducted.

Corollary 2. *The value function V satisfies $V(x) \leq x + (\mu + \lambda\varepsilon)/r$.*

Proof. The function $v(x) := x + (\mu + \lambda\varepsilon)/r$ satisfies

$$\mathcal{A}v - rv + \lambda \int_0^\infty (v(x+y) - v(x))F(\mathrm{d}y) = -rx \leq 0,$$

so the result follows from the first part of Theorem 1.

In the presence of positive jumps, the construction of an explicit candidate solution seems feasible only in the special cases of hyperexponentially distributed jumps; see [4] and [6] (the authors of [9] claim that they include a general positive jump structure, but a closer inspection of their candidate function reveals that it does not satisfy the conditions needed for a verification argument). The recent preprint [8] transforms the problem of finding an explicit solution to a problem of finding an explicit representation of the scale function. In Sections 3 and 4 we instead construct a candidate function as the limit of a sequence of value functions for an inductively defined sequence of stochastic control problems written in terms of a diffusion process.

3. The building block: a stochastic control problem without jumps

In this section we study a stochastic control problem for an underlying process without jumps. This control problem is the basic building block in Section 4 when showing that the value function V in (3) for a problem with jumps can be written as the limit of a sequence of value functions in problems with no jumps.

Let $D = \{D_t,\ t \geq 0\}$ be a dividend strategy consisting of a nonnegative, right-continuous, nondecreasing process adapted to the filtration generated by a Brownian motion W, and let

$$\mathrm{d}Y_t^D = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t - \mathrm{d}D_t,$$

where the constants μ and σ are the same as in (1). For a given dividend policy D, let

$$\tau^D = \inf\{t \geq 0 \mid Y_t^D \leq 0\}$$

be the ruin time of the process Y^D. We denote by Π the set of dividend strategies D such that $Y_0^D \geq 0$ and $D_t - D_{t-} \leq Y_t^D$ for all $0 < t \leq \tau^D$.

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Next we introduce the functional operator J whose action on a test function $f\colon (0,\infty)\to(0,\infty)$ is defined by

$$Jf(x)=\sup_{D\in\Pi}\mathbb{E}_{x-D_0}\bigg[D_0+\int_0^{\tau^D}\mathrm{e}^{-(r+\lambda)t}\,\mathrm{d}D_t+\lambda\int_0^{\tau^D}\mathrm{e}^{-(r+\lambda)t}Sf(Y_t^D)\,\mathrm{d}t\bigg] \tag{6}$$

for $x\geq 0$, where

$$Sf(y):=\int_0^\infty f(y+z)F(\mathrm{d}z)$$

is a weighted translation of f. We will only consider functions f belonging to the class

$$\mathbb{F}:=\bigg\{f\colon (0,\infty)\to(0,\infty),\ f\text{ is concave},\ x\leq f(x)\leq x+\frac{\mu+\lambda\varepsilon}{r}\bigg\}.$$

As before, for a given $b>0$, we define the barrier strategy D^b to be the minimal dividend strategy D such that $Y_t^D\leq b$ for all $t\geq 0$. More explicitly, if $\mathrm{d}Y_t=\mu\,\mathrm{d}t+\sigma\,\mathrm{d}W_t$ and

$$S_t=\sup\{Y_s\colon s\in[0,t]\}$$

is its supremum process, then $D_t^b=(S_t-b)^+$ and $Y_t^{D^b}=Y_t-(S_t-b)^+$ (the process D^b is then the local time of the process $\mathrm{d}Y_t=\mu\,\mathrm{d}t+\sigma\,\mathrm{d}W_t$ reflected at the point b).

We begin our analysis of the control problem (6) by providing a verification result. To formulate it, let the differential operator $\mathcal{L}$ be defined by

$$\mathcal{L}u:=\frac{\sigma^2}{2}u''+\mu u'-(\lambda+r)u.$$

Theorem 2. (Verification result for the control problem without jumps.) *Let $f\in\mathbb{F}$, and assume that $v\colon[0,\infty)\to[0,\infty)$ is twice continuously differentiable with*

(i) $1\leq v'\leq C$ *for some constant C,*

(ii) $\mathcal{L}v+\lambda Sf\leq 0$.

Then $v\geq Jf$.

If there exists a point $b>0$ such that v, in addition to (i) and (ii), also satisfies

(iii) $v(0)=0$,

(iv) $\mathcal{L}v+\lambda Sf=0$ *for $x\in(0,b]$,*

(v) $v(x)=v(b)+x-b$ *for $x\in(b,\infty)$,*

then $v=Jf$, and the barrier strategy D^b is optimal in (6).

Remark. Note that if $v=f$ then $\mathcal{L}v+\lambda Sf=\mathcal{A}v-rv+\lambda\int_0^\infty(v(x+y)-v(x))F(\mathrm{d}y)$. Consequently, in view of Theorems 1 and 2, the optimal dividend problem (3) in the dual model is closely related to a fixed-point problem for the operator J in (6).

Proof of Theorem 2. First assume that (i) and (ii) hold. For a given dividend strategy $D \in \Pi$, an application of Itô's formula yields

$$\begin{aligned}
&\mathrm{e}^{-(r+\lambda)(t\wedge\tau^D)} v(Y^D_{t\wedge\tau^D}) \\
&\quad = v(Y^D_0) + \int_0^{t\wedge\tau^D} \mathrm{e}^{-(r+\lambda)s} \mathcal{L}v(Y^D_{s-})\,\mathrm{d}s + \int_0^{t\wedge\tau^D} \mathrm{e}^{-(r+\lambda)s} \sigma v'(Y^D_{s-})\,\mathrm{d}W_s \\
&\qquad - \int_0^{t\wedge\tau^D} \mathrm{e}^{-(r+\lambda)s} v'(Y^D_{s-})\,\mathrm{d}D^{\mathrm{c}}_s + \sum_{s\le t\wedge\tau^D} \mathrm{e}^{-(r+\lambda)s}(v(Y_{s-} - \Delta D_s) - v(Y_{s-})), \qquad (7)
\end{aligned}$$

where D^{c} denotes the continuous part of D. Since v' is bounded, the process

$$\left\{ \int_0^{t\wedge\tau^D} \mathrm{e}^{-(r+\lambda)s} \sigma v'(Y^D_{s-})\,\mathrm{d}W_s \right\}_{t\ge 0}$$

is a martingale. Consequently, taking the expected value in (7) yields

$$\begin{aligned}
&\mathbb{E}_{x-D_0}[\mathrm{e}^{-(r+\lambda)(t\wedge\tau^D)} v(Y^D_{t\wedge\tau^D})] \\
&\quad = v(x - D_0) + \mathbb{E}_{x-D_0}\left[\int_0^{t\wedge\tau^D} \mathrm{e}^{-(r+\lambda)s} \mathcal{L}v(Y^D_{s-})\,\mathrm{d}s\right] \\
&\qquad - \mathbb{E}_{x-D_0}\left[\int_0^{t\wedge\tau^D} \mathrm{e}^{-(r+\lambda)s} v'(Y^D_{s-})\,\mathrm{d}D^{\mathrm{c}}_s\right] \\
&\qquad + \mathbb{E}_{x-D_0}\left[\sum_{t_k\le t\wedge\tau^D} \mathrm{e}^{-(r+\lambda)t_k}(v(Y_{t_k-} - \Delta D_{t_k}) - v(Y_{t_k-}))\right], \qquad (8)
\end{aligned}$$

so using $\mathcal{L}v + \lambda Sf \le 0$, $v \ge 0$, and $v' \ge 1$ gives

$$v(x - D_0) \ge \mathbb{E}_{x-D_0}\left[\int_0^{t\wedge\tau^D} \mathrm{e}^{-(r+\lambda)s}\,\mathrm{d}D_s + \lambda \int_0^{t\wedge\tau^D} \mathrm{e}^{-(r+\lambda)s} Sf(Y^D_s)\,\mathrm{d}s\right].$$

Letting $t \to \infty$ we find by monotone convergence that

$$v(x - D_0) \ge \mathbb{E}_{x-D_0}\left[\int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)s}\,\mathrm{d}D_s + \lambda \int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)s} Sf(Y^D_s)\,\mathrm{d}s\right].$$

Since $v' \ge 1$ and the dividend strategy $D \in \Pi$ is arbitrary, it follows that

$$\begin{aligned}
Jf(x) &= \sup_{D\in\Pi} \mathbb{E}_{x-D_0}\left[D_0 + \int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)t}\,\mathrm{d}D_t + \lambda \int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)t} Sf(Y^D_t)\,\mathrm{d}t\right] \\
&\le v(x).
\end{aligned}$$

To prove the second statement, assume that (i)–(v) hold for some $b > 0$. Note that the strategy $D = D^b$ is continuous (although we may have $D^b_0 > 0$), so (8) yields

$$\begin{aligned}
v(x - D_0) = {}&\mathbb{E}_{x-D_0}\left[\int_0^{t\wedge\tau^D} \mathrm{e}^{-(r+\lambda)s} v'(Y^D_s)\,\mathrm{d}D_s + \lambda \int_0^{t\wedge\tau^D} \mathrm{e}^{-(r+\lambda)s} Sf(Y^D_s)\,\mathrm{d}s\right] \\
&+ \mathbb{E}_{x-D_0}[\mathrm{e}^{-(r+\lambda)(t\wedge\tau^D)} v(Y^D_{t\wedge\tau^D})]
\end{aligned}$$

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since $\mathcal{L}v(x)+\lambda Sf(x)=0$ for $x\le b$. By (iii) and bounded convergence, the last term vanishes as $t\to\infty$, so, by monotone convergence,

$$v(x-D_0)=\mathbb{E}_{x-D_0}\bigg[\int_0^{\tau^D}\mathrm{e}^{-(r+\lambda)s}v'(Y_s^D)\,\mathrm{d}D_s+\lambda\int_0^{\tau^D}\mathrm{e}^{-(r+\lambda)s}Sf(Y_s^D)\,\mathrm{d}s\bigg].$$

Since $v'(x)=1$ for $x\ge b$ and since the support of the measure $\mathrm{d}D$ is contained in $\{s\ge 0\colon Y_s^D\ge b\}$, we have

$$v(x)-D_0=v(x-D_0)=\mathbb{E}_{x-D_0}\bigg[\int_0^{\tau^D}\mathrm{e}^{-(r+\lambda)s}\,\mathrm{d}D_s+\lambda\int_0^{\tau^D}\mathrm{e}^{-(r+\lambda)s}Sf(Y_s^D)\,\mathrm{d}s\bigg].$$

Consequently,

$$Jf(x)=\sup_{D\in\Pi}\mathbb{E}_{x-D_0}\bigg[D_0+\int_0^{\tau^D}\mathrm{e}^{-(r+\lambda)t}\,\mathrm{d}D_t+\lambda\int_0^{\tau^D}\mathrm{e}^{-(r+\lambda)t}Sf(Y_t^D)\,\mathrm{d}t\bigg]\ge v(x),$$

which completes the proof.

In view of the above verification result, we now construct a function U satisfying properties (i)–(v). The existence of such a function is guaranteed by Theorem 3 together with Propositions 1 and 2 below; see Theorem 4.

Theorem 3. *Assume that $f\in\mathbb{F}$. Then there exists a unique solution (U,b) of the free boundary problem*

$$\mathcal{L}U(x)+\lambda Sf(x)=0,\qquad 0<x<b,\tag{9a}$$

$$U(0)=0,\tag{9b}$$

$$U'(b)=1,\tag{9c}$$

$$U''(b)=0\tag{9d}$$

such that $b>0$.

Proof. Let $\gamma_1<0$ and $\gamma_2>0$ be the solutions of the quadratic equation

$$\gamma^2+\frac{2\mu}{\sigma^2}\gamma-\frac{2(\lambda+r)}{\sigma^2}=0,$$

so that $\psi=\mathrm{e}^{\gamma_2 x}$ and $\varphi=\mathrm{e}^{\gamma_1 x}$ are the increasing and decreasing, respectively, fundamental solutions to the homogeneous equation $\mathcal{L}U=0$.

Let, for a given $b>0$,

$$\begin{aligned}U(x)&:=U^b(x)\\&:=C\varphi(x)\int_0^x\frac{Sf(y)}{\varphi(y)}\,\mathrm{d}y-C\psi(x)\int_0^x\frac{Sf(y)}{\psi(y)}\,\mathrm{d}y\\&\quad+\frac{\psi(x)-\varphi(x)}{\psi'(b)-\varphi'(b)}\bigg(1+C\psi'(b)\int_0^b\frac{Sf(y)}{\psi(y)}\,\mathrm{d}y-C\varphi'(b)\int_0^b\frac{Sf(y)}{\varphi(y)}\,\mathrm{d}y\bigg)\end{aligned}$$

for $x\in(0,b)$, where $C=2\lambda/\sigma^2(\gamma_2-\gamma_1)$. Then $U(0)=0$ and

$$\begin{aligned}U'(x)&=C\varphi'(x)\int_0^x\frac{Sf(y)}{\varphi(y)}\,\mathrm{d}y-C\psi'(x)\int_0^x\frac{Sf(y)}{\psi(y)}\,\mathrm{d}y\\&\quad+\frac{\psi'(x)-\varphi'(x)}{\psi'(b)-\varphi'(b)}\bigg(1+C\psi'(b)\int_0^b\frac{Sf(y)}{\psi(y)}\,\mathrm{d}y-C\varphi'(b)\int_0^b\frac{Sf(y)}{\varphi(y)}\,\mathrm{d}y\bigg),\end{aligned}$$

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so inserting $x = b$ gives $U'(b) = 1$. Moreover,

$$U''(x) = C\varphi''(x)\int_0^x \frac{Sf(y)}{\varphi(y)}\,\mathrm{d}y - C\psi''(x)\int_0^x \frac{Sf(y)}{\psi(y)}\,\mathrm{d}y - C(\gamma_2-\gamma_1)Sf(x) + \frac{\psi''(x)-\varphi''(x)}{\psi'(b)-\varphi'(b)}\bigg(1 + C\psi'(b)\int_0^b \frac{Sf(y)}{\psi(y)}\,\mathrm{d}y - C\varphi'(b)\int_0^b \frac{Sf(y)}{\varphi(y)}\,\mathrm{d}y\bigg), \quad (10)$$

and it is straightforward to check that $\mathcal{L}U(x) + \lambda Sf(x) = 0$ for $x < b$. Consequently, the pair (U, b) satisfies the first three equations in (9). Moreover, for a given b, U is the unique such solution. To find a unique b so that the fourth equation also holds, we show that the function $h(b) := (U^b)''(b)$ has a unique positive zero. Note that inserting $x = b$ in (10) yields

$$h(b) = C(\gamma_2-\gamma_1)\frac{\gamma_1\gamma_2\varphi(b)\psi(b)}{\gamma_2\psi(b)-\gamma_1\varphi(b)}\int_0^b \frac{\varphi(y)-\psi(y)}{\varphi(y)\psi(y)}Sf(y)\,\mathrm{d}y - C(\gamma_2-\gamma_1)Sf(b) + \frac{\gamma_2^2\psi(b)-\gamma_1^2\varphi(b)}{\gamma_2\psi(b)-\gamma_1\varphi(b)}.$$

Define a function $l\colon [0,\infty) \to \mathbb{R}$ by

$$l(b) := C(\gamma_2-\gamma_1)\int_0^b \frac{\varphi(y)-\psi(y)}{\varphi(y)\psi(y)}Sf(y)\,\mathrm{d}y - C(\gamma_2-\gamma_1)\bigg(\frac{1}{\gamma_1\varphi(b)} - \frac{1}{\gamma_2\psi(b)}\bigg)Sf(b) + \frac{\gamma_2}{\gamma_1\varphi(b)} - \frac{\gamma_1}{\gamma_2\psi(b)}, \quad (11)$$

so that

$$l(b) = h(b)\frac{\gamma_2\psi(b)-\gamma_1\varphi(b)}{\gamma_2\gamma_1\varphi(b)\psi(b)}.$$

Since $(\gamma_2\psi(b)-\gamma_1\varphi(b))/\gamma_2\gamma_1\varphi(b)\psi(b) < 0$ for all b, it suffices to show that there exists a unique zero of l. Note that

$$l(0) = \frac{2(\gamma_1-\gamma_2)}{\sigma^2\gamma_1\gamma_2}(\mu + \lambda Sf(0)) > 0,$$

where the inequality follows from $\lambda Sf(0) + \mu \geq \lambda\varepsilon + \mu > 0$.

Moreover,

$$\begin{aligned} l'_+(b) &= \frac{\gamma_1}{\psi(b)} - \frac{\gamma_2}{\varphi(b)} - C(\gamma_2-\gamma_1)(Sf)'_+(b)\bigg(\frac{1}{\gamma_1\varphi(b)} - \frac{1}{\gamma_2\psi(b)}\bigg) \\ &= \bigg(\frac{\gamma_1}{\psi(b)} - \frac{\gamma_2}{\varphi(b)}\bigg)\bigg(1 - \frac{\lambda}{\lambda+r}(Sf)'_+(b)\bigg), \end{aligned}$$

where the second equality follows from $\gamma_1\gamma_2 = -2(\lambda+r)/\sigma^2$, and where l'_+ and $(Sf)'_+$ denote the right derivatives of l and Sf, respectively. Hence $l'_+(b)$ behaves like $-\gamma_2 r/(\lambda+r)\varphi(b)$ for large b, so $l(\infty) < 0$. Thus, by continuity of l, there exists b^* such that $l(b^*) = 0$.

To prove the uniqueness of b^*, note that, since Sf is concave, l is nondecreasing on $(0, \hat{b})$ and nonincreasing on $(\hat{b}, \infty)$, where

$$\hat{b} := \inf\bigg\{b \in (0,\infty)\colon (Sf)'_+(b) \leq \frac{\lambda+r}{\lambda}\bigg\} \in [0,\infty) \quad (12)$$

(note that if $\hat{b} = 0$ then l is nonincreasing on $(0,\infty)$). Together with $l(0) > 0$ and $l(\infty) < 0$, this proves uniqueness of the zero of l.

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We let (U, b) be the unique solution of the free boundary problem (9), and we extend the domain of definition of U by defining

$$U(x) = U(b) + x - b \quad \text{for } x > b. \tag{13}$$

Note that U is C^2 on $(0, \infty)$ by construction.

Proposition 1. *The function U satisfies $\mathcal{L}U(x) + \lambda Sf(x) \leq 0$ for all $x \in (0, \infty)$.*

Proof. Define

$$H(x) := \mathcal{L}U(x) + \lambda Sf(x) = \frac{\sigma^2}{2}U'' + \mu U'(x) - (\lambda + r)U(x) + \lambda Sf(x).$$

To see that $H(x) \leq 0$, first note that $H(x) = 0$ for $x \leq b$ by definition, so it suffices to show that $H'(x) \leq 0$ for $x \geq b$. For the unique $b > 0$ satisfying $l(b) = 0$, we have $b > \hat{b}$, where $\hat{b}$ is defined in (12). Consequently, by the concavity of Sf, $(Sf)'(x) \leq (\lambda + r)/\lambda$ for all $x \geq b$. Thus,

$$H'(x) = -(\lambda + r) + \lambda Sf'(x) \leq 0$$

for $x > b$, so $H(x) \leq 0$ for $x \geq b$.

Proposition 2. *The function U satisfies $U'(x) \geq 1$ for $x \in (0, \infty)$, and $U'(x) > 1$ for $0 < x < b$.*

Proof. First note that $U'(x) = 1$ for $x \geq b$ by definition. Recall that, for $x < b$,

$$\begin{aligned} U'(x) &= \varphi'(x) \int_0^x \frac{CSf(y)}{\varphi(y)}\,\mathrm{d}y - \psi'(x) \int_0^x \frac{CSf(y)}{\psi(y)}\,\mathrm{d}y \\ &\quad + (\psi'(x) - \varphi'(x)) \frac{1 + \psi'(b) \int_0^b (CSf(y)/\psi(y))\,\mathrm{d}y - \varphi'(b) \int_0^b (CSf(y)/\varphi(y))\,\mathrm{d}y}{\psi'(b) - \varphi'(b)}. \end{aligned}$$

Define $k \colon [0, \infty) \to \mathbb{R}$ by

$$k(x) := \frac{1 + \psi'(x) \int_0^x (CSf(y)/\psi(y))\,\mathrm{d}y - \varphi'(x) \int_0^x (CSf(y)/\varphi(y))\,\mathrm{d}y}{\psi'(x) - \varphi'(x)},$$

so that

$$U'(x) = \varphi'(x) \int_0^x \frac{CSf(y)}{\varphi(y)}\,\mathrm{d}y - \psi'(x) \int_0^x \frac{CSf(y)}{\psi(y)}\,\mathrm{d}y + (\psi'(x) - \varphi'(x))k(b).$$

Straightforward calculations show that

$$k'(x) = \frac{-\varphi'(x)\psi'(x)l(x)}{(\psi'(x) - \varphi'(x))^2},$$

where l is defined in (11). Since $l(x) > 0$ for $x < b$, k is strictly increasing on $(0, b)$. Consequently,

$$U'(x) > \varphi'(x) \int_0^x \frac{CSf(y)}{\varphi(y)}\,\mathrm{d}y - \psi'(x) \int_0^x \frac{CSf(y)}{\psi(y)}\,\mathrm{d}y + (\psi'(x) - \varphi'(x))k(x) = 1$$

for $x < b$, which completes the proof.

Theorem 4. *Let $f \in \mathbb{F}$, and let (U, b) be the solution to the free boundary problem (9), with U extended linearly above b as in (13). Then $U = Jf$, and the supremum in (6) is attained for the dividend strategy D^b.*

Proof. The lower bound $U' \geq 1$ in (i) of Theorem 2 follows from Proposition 2, and the upper bound $U' \leq C$ holds since U' has a finite limit at 0 (since U solves an ordinary differential equation with nondegenerate coefficients) and satisfies $U'(x) = 1$ for $x \geq b$. Condition (ii) holds by Proposition 1. Finally, (iii), (iv), and (v) are fulfilled by construction. Consequently, Theorem 2 applies, which yields the result.

We end this section by providing a condition under which $Jf \in \mathbb{F}$.

Theorem 5. *Assume that $f \in \mathbb{F} \cap C^2([0, \infty))$, where $f \in \mathbb{F}$ is extended to $[0, \infty)$ by $f(0) := f(0+)$. Furthermore, assume that*

$$\mu f'(0) + \lambda S f(0) \geq 0. \tag{14}$$

Then $U = Jf \in \mathbb{F} \cap C^2([0, \infty))$.

Remark. The assumption that f is C^2 can easily be removed using an approximation argument, but, for simplicity, we include it since we only need the result below for functions f in C^2. Condition (14) is trivially satisfied in the case $\mu \geq 0$. We do not know whether Jf is also concave without (14).

Proof of Theorem 5. Clearly, $Jf(x) \geq x$. Moreover, using $f(x) \leq x + (\mu + \lambda\varepsilon)/r$, it is straightforward to check that

$$\begin{aligned}\mathcal{L}\left(x + \frac{\mu + \lambda\varepsilon}{r}\right) + \lambda S f &\leq \mu - (\lambda + r)\left(x + \frac{\mu + \lambda\varepsilon}{r}\right) + \lambda\left(x + \frac{\mu + \lambda\varepsilon}{r} + \varepsilon\right) \\ &= -rx \\ &\leq 0.\end{aligned}$$

Thus, applying the first part of Theorem 2 gives also the upper bound

$$Jf(x) \leq x + \frac{\mu + \lambda\varepsilon}{r}.$$

To prove the concavity of $U = Jf$, let $u(x) = U''(x)$. Differentiating the differential equation in (9) twice gives

$$\frac{\sigma^2}{2} u_{xx} + \mu u_x - (\lambda + r)u + \lambda(Sf)'' = 0.$$

By the definition of b, we have $u(b) = U''(b) = 0$. Therefore, by the maximum principle, it suffices to show that $u(0) \leq 0$. Using $U'(0) \geq 0$, $U(0) = 0$, and $f \geq 0$ in (9) shows that $U''(0) \leq 0$ provided the drift μ is nonnegative.

The case $\mu < 0$ is more involved, and we deal with it as follows. First define the affine function $\tilde{f} \colon [0, \infty) \to [0, \infty)$ by

$$\tilde{f}(x) = \frac{\lambda S f(0)}{-\mu} x + S f(0) - \frac{\lambda \varepsilon S f(0)}{-\mu}.$$

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We claim that $S\tilde{f} \geq Sf$. Indeed, this follows since the function

$$h(x) := S\tilde{f}(x) - Sf(x) = \frac{\lambda Sf(0)}{-\mu}x + Sf(0) - Sf(x)$$

satisfies $h(0) = 0$,

$$h'(0) = \frac{\lambda Sf(0)}{-\mu} - (Sf)'(0) \geq \frac{\lambda Sf(0)}{-\mu} - f'(0) \geq 0$$

(by (14)), and $h''(x) = -(Sf)''(x) \geq 0$.

Define

$$\tilde{U}(x) := \frac{\lambda Sf(0)}{-\mu}x. \tag{15}$$

Then $\tilde{U}$ satisfies

$$\mathcal{L}\tilde{U} + \lambda Sf \leq \mathcal{L}\tilde{U} + \lambda S\tilde{f} = -r\frac{\lambda(Sf)(0)}{-\mu}x \leq 0.$$

Applying the first part of Theorem 2 gives $U \leq \tilde{U}$. It then follows from (15) that $U'(0) \leq \lambda Sf(0)/(-\mu)$. The differential equation in (9) thus yields

$$\frac{\sigma^2}{2}U''(0) = -\mu U'(0) - \lambda Sf(0) \leq 0,$$

which completes the proof.

4. An iterative procedure to determine V

In this section we define a sequence $\{v_n\}_{n=0}^{\infty}$ of functions $v_n : [0, \infty) \to [0, \infty)$ by $v_0(x) = x$ and

$$v_{n+1}(x) = Jv_n(x) \quad \text{for } n \geq 0.$$

Proposition 3. *Each function v_n belongs to $\mathbb{F}$, so the sequence $\{v_n\}_{n=0}^{\infty}$ is well defined. Moreover, the sequence is increasing in n.*

Proof. First note that $v_0 \in \mathbb{F} \cap C^2([0, \infty))$ and v_0 satisfies (14) by (5). Moreover, $v_1 \geq v_0$ since $Jf(x) \geq x$ for any $f \in \mathbb{F}$.

Now assume that $v_n \in \mathbb{F} \cap C^2([0, \infty))$, $v_{n+1} := Jv_n \geq v_n$, and that v_n satisfies (14) for some $n \geq 0$. Then $v_{n+1} \in \mathbb{F} \cap C^2([0, \infty))$ by Theorem 5. Moreover,

$$\mu v_{n+1}'(0) + \lambda S v_{n+1}(0) \geq \mu v_{n+1}'(0) + \lambda S v_n(0) = -\frac{\sigma^2}{2}v_{n+1}''(0) \geq 0,$$

where we have used (9) and the concavity of v_{n+1}. Consequently, v_{n+1} also solves (14). Moreover, $v_{n+1} \geq v_n$ clearly implies that $v_{n+2} \geq v_{n+1}$ since J preserves order. The result thus follows by induction.

Since $\{v_n\}_{n=0}^{\infty}$ is an increasing sequence of concave functions with a pointwise bound $x + (\mu + \lambda\varepsilon)/r$, the sequence has the limit

$$v_\infty(x) := \sup_{n \geq 0} v_n(x), \tag{16}$$

which is also concave, and the limit satisfies the same pointwise bound. Consequently, the limit v_∞ belongs to $\mathbb{F}$. We show below that the limit v_∞ coincides with V defined in (3). Consequently, V is determined as the limit of a sequence of standard stochastic control problems (where the underlying process contains no jumps).

Lemma 1. *The function v_∞ is a fixed point of the operator J.*

Proof. For $x \geq 0$, we have

$$\begin{aligned}
v_\infty(x) &= \sup_{n\geq 0} v_n(x) \\
&= \sup_{n\geq 0}\sup_{D\in\Pi} \mathbb{E}_x\bigg[\int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)t}\,\mathrm{d}D_t + \lambda\int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)t} S v_n(Y_t^D)\,\mathrm{d}t\bigg] \\
&= \sup_{D\in\Pi}\sup_{n\geq 0} \mathbb{E}_x\bigg[\int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)t}\,\mathrm{d}D_t + \lambda\int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)t} S v_n(Y_t^D)\,\mathrm{d}t\bigg] \\
&= \sup_{D\in\Pi} \mathbb{E}_x\bigg[\int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)t}\,\mathrm{d}D_t + \lambda\int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)t} S\Big(\sup_{n\geq 0} v_n\Big)(Y_t^D)\,\mathrm{d}t\bigg] \\
&= J v_\infty(x),
\end{aligned}$$

where the second last equality follows by applying the monotone convergence theorem twice.

Remark. In fact, the function v_∞ is the smallest fixed point of J that is larger than x. Indeed, let v^* be any function satisfying $v^*(x) = Jv^*(x)$ and $v^* \geq x = v_0$. Assuming that $v_n \leq v^*$, we find that $v_{n+1} = Jv_n \leq Jv^* = v^*$ since J preserves order, so $v_n \leq v^*$ holds by induction for all $n \geq 0$. Consequently, $v_\infty \leq v^*$.

Corollary 3. *The function v_∞ belongs to $\mathbb{F} \cap C^2([0,\infty))$ and satisfies $v_\infty(0) = 0$. Moreover, if b is the unique solution of the boundary equation $l(b) = 0$, where in the definition of l given in (11) we let $f = v_\infty$, then v_∞ satisfies $v_\infty' \geq 1$ and*

$$\begin{aligned}
\mathcal{L}v_\infty(x) + \lambda S v_\infty(x) &= 0 \quad \textit{for } x \leq b, \\
\mathcal{L}v_\infty(x) + \lambda S v_\infty(x) &\leq 0 \quad \textit{everywhere}, \\
v_\infty(x) &= v_\infty(b) + x - b \quad \textit{for } x \geq b.
\end{aligned}$$

Proof. The claims follow from the fact that $v_\infty = Jv_\infty$ together with Theorems 3 and 4 and Propositions 1 and 2.

Theorem 6. *Let V be the value function in (3), and let v_∞ be the limit of v_n as in (16). Moreover, let b be the corresponding barrier defined as in Corollary 3. Then $V \equiv v_\infty$, and D^b is an optimal dividend strategy in (3).*

Proof. This is a direct consequence of Theorem 1 and Corollary 3.

Remark. It can be shown that the functional operator J in (6), acting on the space of continuous functions bounded below and above by x and $x + (\mu + \lambda\varepsilon)/r$, respectively, and equipped with the metric defined by $d(f_1, f_2) = \sup_x |f_1(x) - f_2(x)|$, is a contraction. Consequently, by the Banach fixed-point theorem, there exists a unique fixed point (which then by uniqueness has to coincide with $V = v_\infty$ above). Moreover, any choice of v_0 would produce a sequence converging exponentially fast to $V = v_\infty$. For example, choosing $v_0 = x + (\mu + \lambda\varepsilon)/r$ would give rise to a decreasing sequence v_n. However, in this case we do not know whether the corresponding v_n is concave, and, in particular, whether the optimal dividend strategy is automatically a barrier strategy.

Another reason to choose $v_0 = x$ is that it can be shown that v_n then has a natural interpretation as the value function of a dividend problem with time horizon $\gamma^D \wedge T_n$, where T_n is the nth jump time of the process $\int_0^t \int_0^\infty y N(\mathrm{d}t, \mathrm{d}y)$; cf. [12, Theorem 1]. In fact, this can be seen by noting that v_n satisfies a variational inequality involving v_{n-1} and then appealing to an appropriate verification argument. However, we do not provide details since this fact is not used in the analysis.

5. Rate of convergence

In this section we provide some further properties of the value function V and the optimal dividend barrier b. In particular, we study the rate of convergence of (v_n, b_n) to (V, b). As noted above, the Banach fixed-point theorem shows that v_n converges exponentially fast to V. Rather than proving that J is a contraction and then applying the fixed-point theorem, we first give a direct proof of this fact.

Theorem 7. (Rate of convergence of v_n to V.) *The inequality*

$$0 \le v_{n+1}(x) - v_n(x) \le \frac{\mu+\lambda\varepsilon}{\lambda+r}\left(\frac{\lambda}{\lambda+r}\right)^n \tag{17}$$

holds for all x and $n \ge 0$. Consequently,

$$v_n(x) \le V(x) \le v_n(x) + \frac{\mu+\lambda\varepsilon}{r}\left(\frac{\lambda}{\lambda+r}\right)^n, \tag{18}$$

so the sequence $\{v_n\}_{n\ge 0}$ converges uniformly to V, and the rate of convergence is exponential.

Proof. The first inequality in (17) is proved in Proposition 3. For the second inequality, we use an induction argument. First note that the function $\tilde{v}(x) := x + (\mu+\lambda\varepsilon)/(\lambda+r)$ satisfies

$$\mathcal{L}\tilde{v} + \lambda Sx \le 0,$$

so applying Theorem 2 yields $v_1 = Jx \le \tilde{v}$. Consequently, $v_1 - v_0 \le (\mu+\lambda\varepsilon)/(\lambda+r)$, so (17) holds for $n = 0$. Next, assume that (17) holds for some $n \ge 0$. Then we have

$$\begin{aligned}
v_{n+2}(x) &= \sup_{D\in\Pi} \mathbb{E}_x\left[\int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)t}\,\mathrm{d}D_t + \lambda\int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)t} Sv_{n+1}(Y_t^D)\,\mathrm{d}t\right] \\
&\le \sup_{D\in\Pi} \mathbb{E}_x\left[\int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)t}\,\mathrm{d}D_t + \lambda\int_0^{\tau^D} \mathrm{e}^{-(r+\lambda)t} Sv_{n}(Y_t^D)\,\mathrm{d}t\right] \\
&\quad + \lambda\int_0^{\infty} \mathrm{e}^{-(r+\lambda)t}\frac{\mu+\lambda\varepsilon}{\lambda+r}\left(\frac{\lambda}{\lambda+r}\right)^n \mathrm{d}t \\
&= v_{n+1}(x) + \frac{\mu+\lambda\varepsilon}{\lambda+r}\left(\frac{\lambda}{\lambda+r}\right)^{n+1},
\end{aligned}$$

which completes the proof of (17).

Finally, note that (18) is a consequence of (17). Indeed, using (17), we have

$$\begin{aligned}
v_\infty(x) - v_n(x) &= \sum_{k=n}^{\infty} v_{k+1}(x) - v_k(x) \\
&\le \sum_{k=n}^{\infty} \frac{\mu+\lambda\varepsilon}{\lambda+r}\left(\frac{\lambda}{\lambda+r}\right)^k \\
&= \frac{\mu+\lambda\varepsilon}{r}\left(\frac{\lambda}{\lambda+r}\right)^n.
\end{aligned}$$

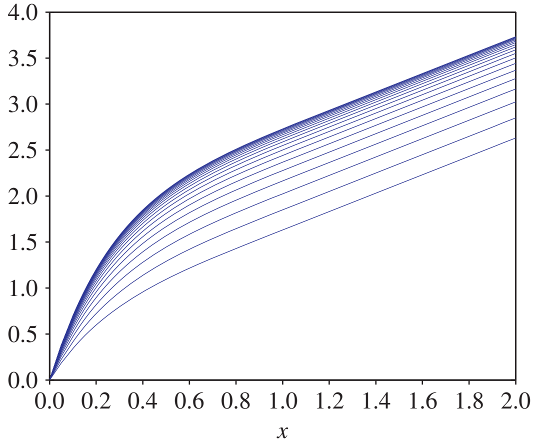

FIGURE 1: The convergence of the value functions for a constant jump size $\varepsilon = 0.2$, $\lambda = 0.5$, $r = 0.1$, $\sigma = 0.4$, and $\mu = 0.2$.

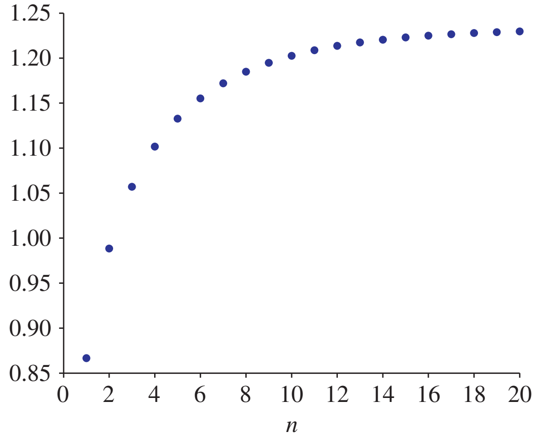

FIGURE 2: The convergence of the corresponding optimal boundaries.

For an illustration of the convergence of the value functions, see Figure 1. We next show that b_n increases monotonically in n and that the limit is b, where b is defined as in Corollary 3. The convergence of the boundaries is illustrated in Figure 2.

Theorem 8. *The sequence* $\{b_n\}_{n=1}^{\infty}$ *is nondecreasing, and* $\lim_{n\to\infty} b_n = b$*. Moreover, we have* $b \leq (\mu + \lambda\varepsilon)/r$*.*

Proof. We first treat the monotonicity of b_n. Since $b_n = \inf\{x \in (0,\infty) \colon v_n'(x) = 1\}$, it suffices to show that the functions $v_{n+1}(x) - v_n(x)$ are nondecreasing. We do this using an inductive argument.

Note that, since $v_0(x) = x$ and v_1 is concave with $v_1' \geq 1$, the function $v_1 - v_0$ is certainly nondecreasing. Now assume that $v_n(x) - v_{n-1}(x)$ is nondecreasing for some $n \geq 1$. Let l_{n+1} and l_n be defined as in (11) with $f = v_n$ and $f = v_{n-1}$, respectively. Note that, in this notation, $l_n(b_n) = 0$. Since $v_n(x) - v_{n-1}(x)$ is nondecreasing, $Sv_n(x) - Sv_{n-1}(x)$ is also nondecreasing,

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so

$$Sv_n(y) - Sv_{n-1}(y) \le Sv_n(x) - Sv_{n-1}(x)$$

for $0 < y \le x$. Consequently,

$$\begin{aligned}\frac{l_{n+1}(x) - l_n(x)}{C(\gamma_2 - \gamma_1)} &= \left(\frac{1}{\gamma_2 \psi(x)} - \frac{1}{\gamma_1 \varphi(x)}\right)(Sv_n - Sv_{n-1})(x) \\ &\quad - \int_0^x \left(\frac{1}{\varphi(y)} - \frac{1}{\psi(y)}\right)(Sv_n - Sv_{n-1})(y)\,\mathrm{d}y \\ &\ge \left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right)(Sv_n - Sv_{n-1})(x) \\ &\ge 0\end{aligned}$$

for all $x > 0$. It follows that $b_{n+1} \ge b_n$.

Now, define $g(x) := v'_{n+1}(x) - v'_n(x)$. Since $v_{n+1}(0) = v_n(0) = 0$ and $v_{n+1} \ge v_n$, we have $g(0) \ge 0$. Moreover, $b_{n+1} \ge b_n$, so we also have $g(x) \ge 0$ for all $x \ge b_n$. Since g satisfies

$$\mathcal{L}g = \lambda(Sv'_{n-1} - Sv'_n)$$

on $(0, b_n)$, and since $Sv'_{n-1} - Sv'_n \le 0$ by the induction hypothesis, it follows from the maximum principle that $g \ge 0$ on $(0, b_n)$ also. Consequently, $v_{n+1} - v_n$ is nondecreasing, which completes the proof of the monotonicity of b_n.

Let l_∞ be defined as in (11) with $f = v_\infty$. Recall from above that $l_n(x)$ is increasing in n. Since $v_n \nearrow v_\infty$ as $n \to \infty$, we have $Sv_n \nearrow Sv_\infty$ by monotone convergence. Another application of the monotone convergence theorem shows that $l_n(x) \nearrow l_\infty(x)$ for all $x > 0$. Since $b = \inf\{y > 0 \colon l(y) \le 0\}$, this implies that $b_n \nearrow b$.

Finally, to show the upper bound of b, note that, for $x = b$, we have $U_{xx} = 0$ and $U_x = 1$. Moreover, $SU(b) = U(b) + \varepsilon$, so it follows that

$$0 = \mathcal{L}U(b) + \lambda(SU)(b) = \mu - (\lambda + r)U(b) + \lambda(U(b) + \varepsilon) = \mu + \lambda\varepsilon - rU(b).$$

Since $U(b) \ge b$, the result follows.

Theorem 9. (Rate of convergence of b_n to b.) *The inequality*

$$0 \le b_{n+1} - b_n \le \frac{\mu + \lambda\varepsilon}{r}\left(\frac{\lambda}{\lambda + r}\right)^n \tag{19}$$

holds for all $n \ge 1$. Consequently,

$$b_n \le b \le b_n + \frac{(\lambda + r)(\mu + \lambda\varepsilon)}{r^2}\left(\frac{\lambda}{\lambda + r}\right)^n, \tag{20}$$

so the rate of convergence of the sequence $\{b_n\}_{n\ge 0}$ to b is exponential.

Proof. Define l_n, $n \ge 1$, as in (11) but with $f = v_{n-1}$. Recall that $l_n(b_n) = l_{n+1}(b_{n+1}) = 0$, $b_n \le b_{n+1}$, and $l_{n+1}(b_n) \ge 0$. For $x \in [b_n, b_{n+1}]$, we have

$$\begin{aligned}l'_{n+1}(x) &= \left(\frac{\gamma_1}{\psi(x)} - \frac{\gamma_2}{\varphi(x)}\right)\left(1 - \frac{\lambda}{\lambda + r}(Sv_n)'(x)\right) \\ &= \left(\frac{\gamma_1}{\psi(x)} - \frac{\gamma_2}{\varphi(x)}\right)\frac{r}{\lambda + r} \\ &\le \left(\frac{\gamma_1}{\psi(b_n)} - \frac{\gamma_2}{\varphi(b_n)}\right)\frac{r}{\lambda + r},\end{aligned} \tag{21}$$

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where the second equality is obtained from $(Sv_n)'(x) = 1$ for $x \geq b_n$ and the inequality holds since

$$l''(x) = \frac{\gamma_1\gamma_2(1/\varphi(x) - 1/\psi(x))r}{\lambda + r} \leq 0.$$

On the other hand, by the definition of l_n and l_{n+1} we have

$$\begin{aligned} 0 &\leq l_{n+1}(b_n) - l_n(b_n) \\ &\leq C(\gamma_2 - \gamma_1)\left(\frac{1}{\gamma_2\psi(b_n)} - \frac{1}{\gamma_1\varphi(b_n)}\right)(Sv_n - Sv_{n-1})(b_n) \\ &\leq C(\gamma_2 - \gamma_1)\left(\frac{1}{\gamma_2\psi(b_n)} - \frac{1}{\gamma_1\varphi(b_n)}\right)\frac{\mu + \lambda\varepsilon}{\lambda + r}\left(\frac{\lambda}{\lambda + r}\right)^{n-1}, \end{aligned} \tag{22}$$

where the last inequality follows from (17). Inequalities (21) and (22) imply that

$$b_{n+1} - b_n \leq \frac{\mu + \lambda\varepsilon}{r}\left(\frac{\lambda}{\lambda + r}\right)^n,$$

which completes the proof of (19).

Finally, the estimate (20) follows from (19) since

$$b - b_n = \sum_{k=n}^{\infty} b_{k+1} - b_k \leq \sum_{k=n}^{\infty} \frac{\mu + \lambda\varepsilon}{r}\left(\frac{\lambda}{\lambda + r}\right)^k = \frac{(\lambda + r)(\mu + \lambda\varepsilon)}{r^2}\left(\frac{\lambda}{\lambda + r}\right)^n.$$

6. Parameter dependencies

In this section we study parameter dependencies. We first show that the value function V depends monotonically on the drift, the jump intensity, the discount rate, the jump size, and the volatility.

Theorem 10. *The value function V is increasing in the drift μ and in the jump intensity λ, and it is decreasing in the discount rate r and in the volatility σ. Moreover, V is increasing in the jump size in the sense that if $F_1(x) \geq F_2(x)$ for all $x \in \mathbb{R}_+$ then the corresponding value functions satisfy $V_1 \leq V_2$.*

Proof. Assume that $0 < \lambda_1 < \lambda_2$, $\mu_1 < \mu_2$, $F_1(x) \geq F_2(x)$ for all $x \in \mathbb{R}_+$, $0 < r_2 < r_1$, and $0 < \sigma_2 < \sigma_1$. Denote the corresponding differential operators by $\mathcal{A}_1$ and $\mathcal{A}_2$, and the corresponding weighted translation operators by S_1 and S_2, respectively. Then

$$(\mathcal{A}_1 - r_1)V_2 + \lambda_1 S_1 V_2 \leq (\mathcal{A}_2 - r_2)V_2 + \lambda_2 S_2 V_2 \leq 0$$

since V_2 is nonnegative, increasing, and concave. Using the first part of Theorem 1 gives $V_2 \geq V_1$.

The dependencies of the optimal barrier on μ, λ, and σ seem more involved. In fact, numerical experiments suggest a nonmonotone dependence on μ and λ; cf. Figures 3 and 5. In Figures 3–5 the value of the constant parameters are $\lambda = 0.5$, $r = 0.1$, $\sigma = 0.4$, $\mu = 0.2$, and $\varepsilon = 0.2$.

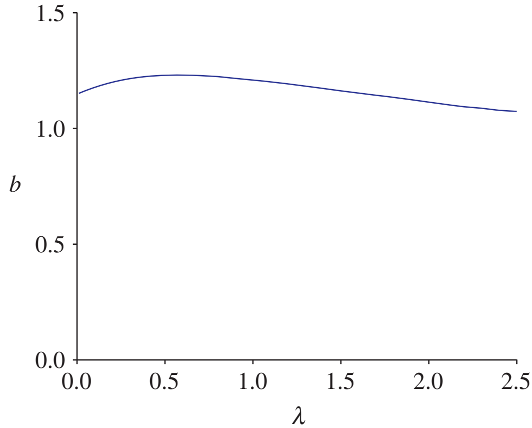

FIGURE 3: The dependence of b on λ.

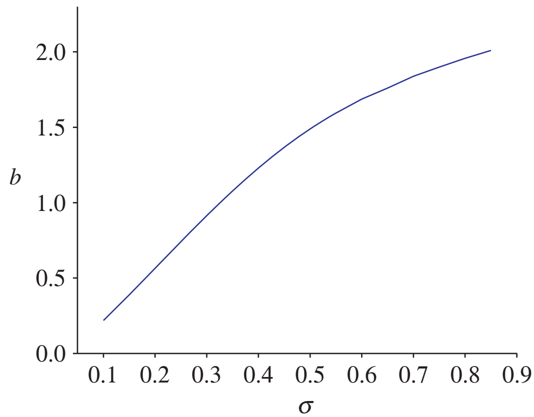

FIGURE 4: The dependence of b on σ.

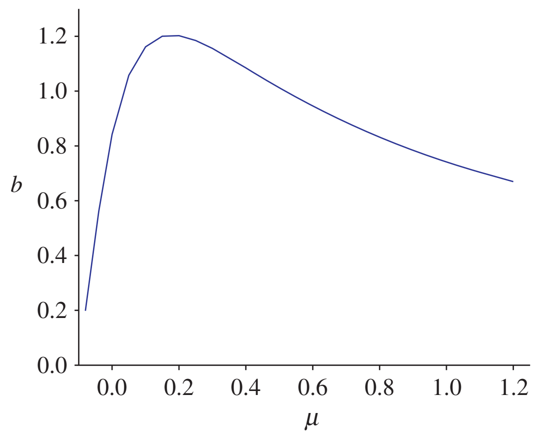

FIGURE 5: The dependence of b on μ.

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