On two linear vector spaces associated with a vector in an L_n

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Let v_λ be a vector (*i.e.* a vector field) in an affinely connected space L_n, ∇_κ the symbol of covariant differentiation, and r the rank of the matrix $\| \nabla_\kappa v_\lambda \|$, then there exist two sets of $n - r$ independent vectors $\underset{x}{i}^\kappa$ and $\underset{x}{j}^\kappa$ $(x = n - r + 1, \ldots, n)$ which satisfy respectively the equations

$$(1) \qquad i^\kappa \nabla_\kappa v_\lambda = 0,$$

$$(1') \qquad j^\lambda \nabla_\kappa v_\lambda = 0.$$

We denote by E_{n-r} and $\bar{E}_{n-r}$ the local linear vector spaces of $n - r$ dimensions spanned by $\underset{x}{i}^\kappa$ and $\underset{x}{j}^\kappa$ and defined at every point of L_n. Evidently any vector in E_{n-r} is a solution of (1) and any vector in $\bar{E}_{n-r}$ is a solution of (1′).

For the vector $'v_\lambda = \sigma v_\lambda$ where σ is a scalar (*i.e.* a scalar function) we have the corresponding $'E_{n-r'}$, $'\bar{E}_{n-r'}$ defined by

$$(2) \qquad i^\kappa \nabla_\kappa {'v_\lambda} = 0,$$

$$(2') \qquad j^\lambda \nabla_\kappa {'v_\lambda} = 0,$$

r' being the rank of $\| \nabla_\kappa {'v_\lambda} \|$. The purpose of this note is to show that the nature of the relation between the two pairs of local linear vector spaces E_{n-r} and $'E_{n-r'}$, $\bar{E}_{n-r}$ and $'\bar{E}_{n-r'}$, is completely characterised by the ranks r, r_1, r_2, r_3 of the matrices

$$(3) \quad M = \| \nabla_\kappa v_\lambda \|, \; M_1 = \left\| \begin{matrix} \nabla_\kappa v_\lambda \\ \nabla_\kappa \log \sigma \end{matrix} \right\|, \; M_2 = \| \nabla_\kappa v_\lambda, v_\lambda \|, \; M_3 = \left\| \begin{matrix} \nabla_\kappa v_\lambda, & -v_\lambda \\ \nabla_\kappa \log \sigma, & 1 \end{matrix} \right\|.$$

The matrices M, M_2 and the determinant of M_3 have appeared in Eisenhart's investigation on the transversals of parallelism of a given vector, where he considered[1] the *aggregate* of the vectors i^κ in the spaces $'E_{n-r'}$ for all possible scalars σ.

[1] Eisenhart, *Non-Riemannian Geometry* (New York, 1927), 38-43.

We shall now investigate how the nature of the relation between $'E_{n-r'}$ and E_{n-r} at any point is dependent on the values of the r's at that point. Let us first consider the vectors common to $'E_{n-r'}$ and E_{n-r}. Equation (2) when written out is

$$i^\kappa \nabla_\kappa v_\lambda = - v_\lambda (i^\kappa \nabla_\kappa \log \sigma). \tag{4}$$

Putting $i^\kappa = \overset{x}{c}\, \underset{x}{i}{}^\kappa$, where $\overset{x}{c}$ are $n-r$ parameters, we get

$$\overset{x}{c}\, \underset{x}{i}{}^\kappa \nabla_\kappa \log \sigma = 0. \tag{5}$$

Comparison of (5) with (1) shows that (5) is identically satisfied or gives a linear homogeneous relation between the $\overset{x}{c}$ according as $r_1 = r$ or $r_1 \neq r$. In the former case every vector in E_{n-r} is also a vector in $'E_{n-r'}$; in the latter case $'E_{n-r'}$ has an E_{n-r-1} in common with E_{n-r}. Hence *$'E_{n-r'}$ contains E_{n-r} or has an E_{n-r-1} in common with E_{n-r} according as $r_1 = r$ or $r_1 \neq r$.*

We shall now consider those vectors of $'E_{n-r'}$ not contained in E_{n-r}. If (4) has a solution for i^κ not lying in E_{n-r}, then $i^\kappa \nabla_\kappa \log \sigma$ does not vanish, as is seen from (4). If we put

$$u^\kappa = i^\kappa / (i^\lambda \nabla_\lambda \log \sigma), \tag{6}$$

equation (4) becomes

$$u^\kappa \nabla_\kappa v_\lambda = - v_\lambda. \tag{7}$$

Therefore *in order that such an i^κ may exist it is necessary that $r_2 = r$.* We now suppose that $r_2 = r$. Then the solution for u^κ of (7) contains $n - r$ parameters[1]. In fact if $\underset{0}{u}{}^\kappa$ is a particular solution, the general solution is

$$u^\kappa = \underset{0}{u}{}^\kappa + \overset{x}{a}\, \underset{x}{i}{}^\kappa, \tag{8}$$

where $\overset{x}{a}$ are $n - r$ parameters. Comparing (6) and (8) we have

$$i^\kappa / (i^\lambda \nabla_\lambda \log \sigma) = \underset{0}{u}{}^\kappa + \overset{x}{a}\, \underset{x}{i}{}^\kappa, \tag{9}$$

from which it follows that

$$\underset{0}{u}{}^\kappa \nabla_\kappa \log \sigma + \overset{x}{a}\, \underset{x}{i}{}^\kappa \nabla_\kappa \log \sigma = 1. \tag{10}$$

This is the necessary and sufficient condition for the existence of an i^κ corresponding to a solution u^κ of (7).

[1] Bocher, *Introduction to Higher Algebra* (New York, 1907), 43-46.

If $r_1 = r\,(= r_2)$, then $\underset{x}{i}^{\kappa}\, \nabla_\kappa \log \sigma = 0$ and equation (10) becomes

$$\text{(11)} \qquad \underset{0}{u}^{\kappa}\, \nabla_\kappa \log \sigma = 1.$$

Remembering that $\underset{0}{u}^{\kappa}$ is a solution of (7) we see that (11) is satisfied or not according as $r_3 = r_1$ or $r_3 \neq r_1$. In the former case, *i.e.* $r = r_1 = r_2 = r_3$, no restriction is imposed on $\overset{x}{a}$ and therefore $r' = r - 1$ and $'E_{n-r'}$ is spanned by $\underset{0}{u}^{\kappa}$ and E_{n-r}; in the latter case, *i.e.* $r = r_1 = r_2 \neq r_3$, there is no i^κ in $'E_{n-r'}$ outside E_{n-r} and therefore $r' = r$ and $'E_{n-r'}$ coincides with E_{n-r}.

If $r_1 \neq r\,(= r_2)$, let $\underset{0}{\overset{x}{a}}$ be a set of particular solutions for $\overset{x}{a}$ of (10). Then if $\overset{x}{c}$ is any set of parameters satisfying (5), the general solutions of (10) and (4) are respectively

$$\overset{x}{a} = \underset{0}{\overset{x}{a}} + \overset{x}{c},$$

$$\text{(12)} \qquad i^{\kappa} = \underset{0}{u}^{\kappa} + \underset{0}{\overset{x}{a}}\, \underset{x}{i}^{\kappa} + \overset{x}{c}\, \underset{x}{i}^{\kappa},$$

after omitting a scalar factor. Remembering that when the $\overset{x}{c}$ satisfy (5) $\overset{x}{c}\, \underset{x}{i}^{\kappa}$ span the common E_{n-r-1} of $'E_{n-r'}$ and E_{n-r}, we see from (12) that $r' = r$ and $'E_{n-r'}$ is spanned by the vector $\underset{0}{u}^{\kappa} + \underset{0}{\overset{x}{a}}\, \underset{x}{i}^{\kappa}$ and E_{n-r-1}.

Hence the nature of the $'E_{n-r'}$ of $'v = \sigma v^\kappa$ is completely characterised by the numbers r, r_1, r_2, r_3.

Proceeding in an analogous manner we can start with equations (1′) and (2′) and classify the nature of the $'\bar{E}_{n-r'}$ according to r, r_1, r_2, r_3. We shall not enter into detail but write down the corresponding equations which appear in the discussion. They are

$$\text{(4′)} \qquad j^{\lambda}\, \nabla_\kappa v_\lambda = -\,(j^\lambda v_\lambda)\, \nabla_\kappa \log \sigma,$$

$$\text{(5′)} \qquad \overset{x}{c}\, \underset{x}{j}^{\lambda}\, v_\lambda = 0,$$

$$\text{(6′)} \qquad u^{\lambda} = j^{\lambda}/(j^{\mu} v_\mu),$$

$$\text{(7′)} \qquad u^{\lambda}\, \nabla_\kappa v_\lambda = -\,\nabla_\kappa \log \sigma,$$

$$\text{(8′)} \qquad u^{\lambda} = \underset{0}{u}^{\lambda} + \overset{x}{a}\, \underset{x}{j}^{\lambda},$$

$$\text{(9′)} \qquad j^{\lambda}/(j^{\mu} v_\mu) = \underset{0}{u}^{\lambda} + \overset{x}{a}\, \underset{x}{j}^{\lambda},$$

(10′) $\underset{0}{u}^{\lambda} v_{\lambda} + \overset{x}{a} \underset{x}{j}^{\lambda} v_{\lambda} = 1,$

(11′) $\underset{0}{u}^{\lambda} v_{\lambda} = 1,$

(12′) $j^{\lambda} = \underset{0}{u}^{\lambda} + \underset{0}{\overset{x}{a}} \underset{x}{j}^{\lambda} + \overset{x}{c} \underset{x}{j}^{\lambda}.$

The four matrices (3) appear in the order M, M_2, M_1, M_3 instead of M, M_1, M_2, M_3.

Summing up these results we have

THEOREM. *The nature of the* $'E_{n-r'}$ *and* $'\bar{E}_{n-r'}$ *of* $'v^{\kappa} = \sigma v^{\kappa}$ *is completely characterised by the ranks* r, r_1, r_2, r_3 *of the matrices* (3). *More precisely,*

(i) *if* $r = r_1 \neq r_2$, *then* $r' = r$ *and* $'E_{n-r'}$ *coincides with* E_{n-r} *while* $'\bar{E}_{n-r'}$ *has an* $\bar{E}_{n-r-1}$ *in common with* $\bar{E}_{n-r}$;

(ii) *if* $r \neq r_1, r_2$, *then* $r' = r + 1$ *and* $'E_{n-r'}$ *and* $'\bar{E}_{n-r'}$ *are contained in* E_{n-r} *and* $\bar{E}_{n-r}$ *respectively*;

(iii) *if* $r = r_1 = r_2 = r_3$, *then* $r' = r - 1$ *and* $'E_{n-r'}$ *and* $'\bar{E}_{n-r'}$ *contain* E_{n-r} *and* $\bar{E}_{n-r}$ *respectively*;

(iv) *if* $r = r_1 = r_2 \neq r_3$, *then* $r' = r$ *and* $'E_{n-r'}$ *and* $'\bar{E}_{n-r'}$ *coincide with* E_{n-r} *and* $\bar{E}_{n-r}$ *respectively*;

(v) *if* $r = r_2 \neq r_1$, *then* $r' = r$ *and* $'E_{n-r'}$ *has an* E_{n-r-1} *in common with* E_{n-r} *while* $'\bar{E}_{n-r'}$ *coincides with* $\bar{E}_{n-r}$.

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