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HEAVY-TAILED ASYMPTOTICS OF STATIONARY PROBABILITY VECTORS OF MARKOV CHAINS OF GI/G/1 TYPE

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Abstract

In this paper, we provide a novel approach to studying the heavy-tailed asymptotics of the stationary probability vector of a Markov chain of GI/G/1 type, whose transition matrix is constructed from two matrix sequences referred to as a boundary matrix sequence and a repeating matrix sequence, respectively. We first provide a necessary and sufficient condition under which the stationary probability vector is heavy tailed. Then we derive the long-tailed asymptotics of the R-measure in terms of the RG-factorization of the repeating matrix sequence, and a Wiener–Hopf equation for the boundary matrix sequence. Based on this, we are able to provide a detailed analysis of the subexponential asymptotics of the stationary probability vector.

Keywords: Markov chain of GI/G/1 type; R-measure; RG-factorization; heavy tail; long tail; subexponentiality; regular variation

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1. Introduction

This paper is devoted to the study of the heavy-tailed asymptotics of the stationary probability vector of a Markov chain of GI/G/1 type, whose transition probability matrix in block-partitioned notation can be written as

$$P = \begin{pmatrix} D_0 & D_1 & D_2 & D_3 & D_4 & \cdots \\ D_{-1} & A_0 & A_1 & A_2 & A_3 & \cdots \\ D_{-2} & A_{-1} & A_0 & A_1 & A_2 & \cdots \\ D_{-3} & A_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ D_{-4} & A_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the sizes of the matrices A_l for $-\infty < l < \infty$, D_0, D_i for $i \geq 1$, and D_{-j} for $j \geq 1$ are $m\times m$, $m_0\times m_0$, $m_0\times m$, and $m\times m_0$, respectively. $\{A_k\}$ and $\{D_k\}$ are referred to as the repeating matrix sequence and the boundary matrix sequence, respectively. In this paper, a matrix is called finite if all its entries are finite. Throughout the paper, we assume that (i) the Markov chain of GI/G/1 type is irreducible, aperiodic, and positive recurrent; (ii) $\sum_{k=1}^{\infty} kD_k$ and $\sum_{k=-\infty}^{\infty} |k|A_k$ are both finite; (iii) $A = \sum_{k=-\infty}^{\infty} A_k$ is irreducible and stochastic; and (iv) $\phi_{A-} < 1$,

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where ϕ_{A-} is the radius of convergence of the matrix function $\sum_{k=1}^{\infty} z^{-k} A_{-k}$. (The radius of convergence of a matrix function is defined as the minimal radius of convergence of the entry functions. This condition is often satisfied and is assumed in order for the transformation method to work.) The stationary probability vector π of the Markov chain of GI/G/1 type is partitioned accordingly into vectors $\pi = (\pi_0, \pi_1, \pi_2, \dots)$.

Markov chains of GI/G/1 type have been investigated by a number of researchers, among whom are Asmussen [2], Grassmann and Heyman [26], Asmussen and Møller [4], Zhao *et al.* [51], Zhao [50], Zhao *et al.* [52], and Li and Zhao [33], [35]. Two important examples of Markov chains of GI/G/1 type are Markov chains of GI/M/1 type and M/G/1 type. We refer the reader to Neuts [37], [38] and Li and Zhao [32], [34] for details.

Since the mid 1960s, considerable attention has been paid to studying heavy-tailed distributions and their applications. These have been well documented, for example in books by Feller [22], Seneta [45], Bingham *et al.* [8], Resnick [41], and Embrechts *et al.* [20], and in the survey papers of Embrechts [15], Resnick [42], Goldie and Klüppelberg [25], and Sigman [46]. Important properties, for example on convolution tails, integrated tails, and closeness of tails, have been obtained. Two important heavy-tailed classes, the classes of subexponential distributions and regular variations, have been discussed extensively. Readers may refer to Teugels [47], Goldie [24], Embrechts *et al.* [19], Embrechts and Goldie [16], [17], Embrechts and Omey [18], Cline [13], [14], Klüppelberg [29], [30], and Bingham *et al.* [8] for details. Two important references on discrete heavy-tailed random variables are Chover *et al.* [12] and Embrechts *et al.* [20]. In this paper, some of the existing properties of heavy-tailed asymptotics will be generalized to matrix form for sequences of nonnegative matrices, which are always useful and often necessary in studying the heavy-tailed asymptotics of performance measures of block-structured stochastic models.

Since the publication of the research on Ethernet network data by Leland *et al.* [31] in 1993, it has been well known that queues fed by long-range-dependent traffic may produce heavy-tailed performance measures such as busy periods, waiting times, and queue lengths, while queues with heavy-tailed distributions can result in self-similar or long-range-dependent traffic. Readers may refer to Norros [39], Erramilli *et al.* [21], Resnick [42], Adler *et al.* [1], and Park and Willinger [40] for more information. The subexponential asymptotics of queueing and risk processes was discussed in Asmussen *et al.* [5], Asmussen [3], and Asmussen *et al.* [7]. The subexponential asymptotics of random walks was studied in Borovkov and Korshunov [9] for homogeneous and partially homogeneous Markov chains, in Jelenković and Lazar [28] and Foss and Zachary [23] for modulated random walks, and in Zachary [49], who provided a novel probabilistic proof of Veraverbeke's theorem.

For a regularly varying tail of the busy period, Meyer and Teugels [36] and Zwart [53] studied the M/G/1 queue and the GI/G/1 queue, respectively. Boxma and Dumas [11] discussed a fluid queue. There is considerable volume in the literature on the subexponential asymptotics of stationary waiting times in queues with subexponential service times. For example, see Boxma and Cohen [10], Whitt [48], and references therein.

For the subexponential asymptotics of stationary queue lengths or, more generally, the heavy-tailed asymptotics of stationary probability vectors of positive-recurrent Markov chains, available results are fewer. In terms of stochastic comparison, Resnick and Samorodnitsky [43] analyzed the heavy-tailed asymptotic behavior of the stationary queue length of a G/M/1 queue when the arrival process is long-range dependent. Based on a property of the generating functions of regularly varying sequences, Roughan *et al.* [44] derived the power law asymptotics of the stationary queue length of an M/G/1 queue with power law service times. Using the distributional version of Little's law, Asmussen *et al.* [6] studied the subexponential asymptotics of the stationary queue length of a GI/G/1 queue with subexponential service times. Using the Mellin transform, Jacquet [27] provided results on polynomial tails for the stationary queue length of a single-server queue when the arrival process contains a finite or infinite number of on–off input sources. For a Markov chain of GI/G/1 type with subexponential increments and tail-equivalent repeating and boundary matrix sequences, Asmussen and Møller [4] discussed the subexponential asymptotics of the stationary level process.

The purpose of this paper is to present a novel approach to analyzing the heavy-tailed asymptotics of the stationary probability vector of a Markov chain of GI/G/1 type. The use of the R-measure, the RG-factorization of the repeating matrix sequence, and a Wiener–Hopf equation for the boundary matrix sequence are the keys to this approach. We illustrate that the RG-factorization of the repeating matrix sequence and a Wiener–Hopf equation for the boundary matrix sequence play a similarly important role to that played by the Wiener–Hopf factorization in studying stationary waiting times. Our main contributions are threefold. First, some useful properties of heavy-tailed sequences of nonnegative scalars are extended to matrix form for sequences of nonnegative matrices. Second, we provide a necessary and sufficient condition under which the stationary probability vector is heavy tailed. Third, the long-tailed asymptotics of the R-measure is derived in terms of the RG-factorization of the repeating matrix sequence and a Wiener–Hopf equation for the boundary matrix sequence. Based on this, we are able to provide a detailed analysis of the subexponential asymptotics of the stationary probability vector. The results obtained in this paper are much stronger than those in [4], due to the following facts. First, in [4] the boundary matrix sequence is very special, in that the subexponential tail of the stationary probability vector is independent of this boundary matrix sequence. In this paper, a much wider class of boundary matrix sequences is considered and we illustrate that the subexponential tail of the stationary probability vector strongly depends on the boundary matrix sequence. Second, in a unified matrix-structured form, subexponential expressions with respect to various boundary matrix sequences are provided explicitly in this paper. Third, the matrix-form expressions provided here are more convenient for computations of the subexponential tail of the stationary probability vector of a block-structured stochastic model.

The rest of the paper is organized as follows. Basic definitions and preliminary properties of heavy-tailed sequences of nonnegative matrices are given in Section 2. A necessary and sufficient condition under which the stationary probability vector is heavy tailed is provided in Section 3. The long-tailed asymptotics of the R-measure is derived in Section 4 and the subexponential asymptotics of the stationary probability vector is analyzed in Section 5. Final remarks are also made in Section 5.

2. Preliminaries

In this section, we provide definitions and preliminary properties for heavy tails, long tails, and subexponentiality for sequences of nonnegative matrices. These preliminaries will be used in subsequent sections.

For a sequence of nonnegative scalars $\{g_n\}$ with $\sum_{n=0}^{\infty} g_n < \infty$, we define two associative functions by $g_{\le x} = \sum_{0\le k\le x} g_k$ and $g_{>x} = \sum_{k>x} g_k$ for an arbitrary real number $x \ge 0$. Specifically, for an integer $n \ge 0$, $g_{\le n} = \sum_{k=0}^{n} g_k$ and $g_{>n} = \sum_{k=n+1}^{\infty} g_k$. For convenience, we also write $g_{>n}$ as $g_{\ge n+1}$.

For the real function $g_{\le x}$ associated with the sequence $\{g_n\}$, the tail of $g_{\le x}$ is defined and expressed as $\overline{g_{\le x}} = g_{<\infty} - g_{\le x} = g_{>x}$ for $x \ge 0$. Specifically, for an integer $n \ge 0$,

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$\overline{g_{\le n}} = g_{\ge n+1}$. It is clear that if $\{g_n\}$ is a probability sequence, then $g_{\le x}$ is its distribution function and $\overline{g_{\le x}}$ is the tail of this function.

In terms of the Riemann–Stieltjes integral, the convolution of two functions $F(x)$ and $G(x)$ is defined as

$$F(x) * G(x) = \int_0^x F(x-y)\,\mathrm{d}G(y). \tag{2.1}$$

We denote by $\lfloor x \rfloor$ the maximum integer part of x. For two sequences $\{c_n\}$ and $\{d_n\}$, it follows from (2.1) that

$$c_{\le x} * d_{\le x} = \int_0^x F(x-y)\,\mathrm{d}G(y) = \sum_{k=0}^{\lfloor x \rfloor}\Bigg(\sum_{i=0}^{\lfloor x \rfloor - k} c_i\Bigg)d_k = \sum_{k=0}^{\lfloor x \rfloor} c_{\le \lfloor x \rfloor - k} d_k.$$

Specifically, for an integer $n \ge 0$,

$$c_{\le n} * d_{\le n} = \sum_{k=0}^{n} c_{\le n-k} d_k, \tag{2.2}$$

which is referred to as the convolution associated with the two sequences $\{c_n\}$ and $\{d_n\}$. Furthermore, for a sequence $\{c_n\}$ we define $c_{\le n}^{r*} = c_{\le n}^{(r-1)*} * c_{\le n}$ for $r \ge 2$, with $c_{\le n}^{1*} = c_{\le n}$.

It should be noted that, in this paper, the usual convolution of two sequences $\{c_n\}$ and $\{d_n\}$ is denoted by $c_n \circledast d_n$ and defined as

$$c_n \circledast d_n = \sum_{k=0}^{n} c_{n-k} d_k. \tag{2.3}$$

We further define $c_n^{r\circledast} = c_n^{(r-1)\circledast} \circledast c_n$ for $r \ge 2$, with $c_n^{1\circledast} = c_n$.

It is worthwhile to note the relationship between the usual convolution and the convolution associated with the sequences, using (2.2) and (2.3):

$$c_{\le n} * d_{\le n} = c_{\le n} \circledast d_n = \sum_{k=0}^{n} c_k \circledast d_k \tag{2.4}$$

and

$$c_n \circledast d_n = c_{\le n} * d_{\le n} - c_{\le n-1} * d_{\le n-1}.$$

Also, it is clear from (2.4) that

$$\overline{c_{\le n} * d_{\le n}} = \sum_{k=n+1}^{\infty} c_k \circledast d_k. \tag{2.5}$$

Note that the two convolutions can be extended to sequences $\{c_n, n = 0, \pm 1, \pm 2, \dots\}$ and $\{d_n, n = 0, \pm 1, \pm 2, \dots\}$ by writing $c_{\le n} * d_{\le n} = \sum_{i+j=n} c_{\le i} d_j$ and $c_n \circledast d_n = \sum_{i+j=n} c_i d_j$, respectively.

For a sequence $\{c_n, n \ge 0\}$, if we set $c_{-n} = 0$ for all $n \ge 1$, then

$$c_{\le n}^{2*} = \sum_{k=0}^{n} c_{\le k} c_{n-k} = \sum_{k=0}^{\infty} c_{\le k} c_{n-k}.$$

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Specifically, if $\{c_n, n \geq 0\}$ is a probability sequence, simple computations lead to

$$\overline{c^{2*}_{\leq n}} = 1 - c^{2*}_{\leq n} = \sum_{k=0}^{\infty} c_k(1 - c_{\leq n-k}) = \sum_{k=0}^{\infty} c_k \overline{c_{\leq n-k}}.$$

Following Subsections 1.3 and 1.4 of [20], we provide the following definitions for a sequence of nonnegative scalars to be heavy tailed, long tailed, or subexponential.

Definition 2.1. (i) A sequence $\{c_n\}$ of nonnegative scalars with $\sum_{n=0}^{\infty} c_n < \infty$ is called heavy tailed if, for all $\varepsilon > 0$,

$$\sum_{n=0}^{\infty} c_n \exp\{\varepsilon n\} = \infty.$$

Otherwise, $\{c_n\}$ is called light tailed. Denote by $\mathcal{H}$ the class of heavy-tailed sequences.

(ii) A sequence $\{c_n\}$ of nonnegative scalars with $\sum_{n=0}^{\infty} c_n < \infty$ is called long tailed if $\overline{c_{\leq n}} > 0$ for all $n > N$, where N is a sufficiently large positive integer, and if

$$\lim_{n\to\infty} \frac{\overline{c_{\leq n+m}}}{\overline{c_{\leq n}}} = 1 \quad \text{for any integer } m \geq 0.$$

Denote by $\mathcal{L}$ the class of long-tailed sequences.

(iii) A probability sequence $\{c_n\}$ is called subexponential if

$$\lim_{n\to\infty} \frac{\overline{c^{2*}_{\leq n}}}{\overline{c_{\leq n}}} = 2.$$

Denote by $\mathcal{S}$ the class of subexponential sequences.

Let $\{c_k\}$ be a sequence of nonnegative scalars with $\sum_{k=0}^{\infty} c_k = c < \infty$. Then $\overline{c^{2*}_{\leq n}} = c^2 - c^{2*}_{\leq n}$, and

$$\lim_{n\to\infty} \frac{\overline{c^{2*}_{\leq n}}}{\overline{c_{\leq n}}} = 2c \tag{2.6}$$

if and only if $\{c_k/c\}$ is subexponential. According to [47], properties of a subexponential sequence also hold for a sequence of nonnegative scalars satisfying (2.6). Therefore, in this paper a sequence of nonnegative scalars satisfying (2.6) is also called subexponential.

To characterize the subexponential asymptotics of the stationary probability vector of a Markov chain of GI/G/1 type, we need to introduce a particular class $\mathcal{S}^* \subset \mathcal{S}$. For a sequence $\{c_k\}$ of nonnegative scalars with $\mu_c = \sum_{k=0}^{\infty} k c_k < \infty$, we define $c_k^{(I)} = (1/\mu_c) \sum_{l=0}^{k} \overline{c_{\leq l}}$. Clearly, $\{c_k^{(I)}\}$ is a probability sequence. Following Klüppelberg [29], the integral tail of the sequence $\{c_k\}$ is defined as $\overline{c^{(I)}_{\leq k}}$, for $k \geq 1$. Klüppelberg [29] illustrated that, for $\{c_k\} \in \mathcal{S}$, it is possible that $\{c_k^{(I)}\} \notin \mathcal{S}$, and provided a useful sufficient condition under which $\{c_k^{(I)}\} \in \mathcal{S}$, which is restated in Proposition 2.1(i), below.

Definition 2.2. A sequence $\{c_k\}$ of nonnegative scalars is in $\mathcal{S}^*$ if $\mu_c < \infty$ and

$$\lim_{k\to\infty} \frac{\overline{c_{\leq k}} \circledast \overline{c_{\leq k}}}{\overline{c_{\leq k}}} = 2\mu_c.$$

Proposition 2.1. (i) *If* $\{c_k\} \in \mathcal{S}^*$ *then* $\{c_k^{(I)}\} \in \mathcal{S}$.

(ii) *If* $\{p_k\}$, $\{q_k\} \in S$ *then* $\{p_k \circledast q_k\} \in S$ *if and only if* $\{\lambda p_k + (1-\lambda) q_k\} \in S$ *for all* $\lambda \in (0, 1)$.

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Proof. From part (c) of Theorem 5.1 of [25], this proposition follows by noting that (i) the sequence $\{p_k\} \in \mathcal{S}$ if and only if the function $p_{\le k} \in \mathcal{S}$, and (ii) the sequence $\{p_k \circledast q_k\} \in \mathcal{S}$ if and only if the function of convolution $p_{\le k} * q_{\le k} \in \mathcal{S}$, since $p_{\le k} * q_{\le k} = \sum_{l=0}^{k} p_l \circledast q_l$.

Definition 2.3. (i) (*Tail equivalence*.) Two sequences $\{c_k\}$ and $\{d_k\}$ of nonnegative scalars are called tail equivalent, denoted by $\overline{c_{\le k}} \backsim \xi \overline{d_{\le k}}$, if $\lim_{k\to\infty} \overline{c_{\le k}}/\overline{d_{\le k}} = \xi$ for some $\xi \in (0, \infty)$.

(ii) (*Tail lightness*.) A sequence $\{c_k\}$ of nonnegative scalars is tail lighter than a sequence $\{d_k\}$ of nonnegative scalars (or $\{d_k\}$ is tail heavier than $\{c_k\}$), denoted by $\overline{c_{\le k}} = o(\overline{d_{\le k}})$, if $\lim_{k\to\infty} \overline{c_{\le k}}/\overline{d_{\le k}} = 0$.

Remark 2.1. It is easy to check that $\mathcal{H}$ and $\mathcal{L}$ are both closed with respect to tail equivalence. Also, Teugels [47] proved that $\mathcal{S}$ is closed with respect to tail equivalence, while Goldie and Klüppelberg [25, p. 445] illustrated that $\mathcal{S}^*$ is closed with respect to tail equivalence.

In what follows, we extend the above notion for sequences of nonnegative scalars to one for sequences of nonnegative matrices. In an abuse of our notation, which should not cause any confusion, we will use the same symbols $\mathcal{H}$, $\mathcal{L}$, and $\mathcal{S}$ and $\mathcal{S}^*$ for the classes of heavy-tailed, long-tailed, and subexponential matrix sequences, respectively.

Definition 2.4. We assume that the nonnegative matrices B_n, $n \ge 1$, have the same size, and that $\sum_{n=0}^{\infty} B_n$ is finite.

(i) The sequence $\{B_n\}$ of nonnegative matrices is called heavy tailed if there exists at least one entry sequence of $\{B_n\}$ that is heavy tailed. Otherwise, $\{B_n\}$ is called light tailed. Denote by $\mathcal{H}$ the class of the heavy-tailed matrix sequences of all sizes.

(ii) The sequence $\{B_n\}$ of nonnegative matrices is called long tailed or subexponential, respectively, if there exists at least one entry sequence of $\{B_n\}$ that is long tailed or subexponential and all the other entry sequences are either long tailed or subexponential or tail lighter than some long-tailed or subexponential entry sequence of $\{B_n\}$. Denote by $\mathcal{L}$ and $\mathcal{S}$ the classes of long-tailed and subexponential matrix sequences of all sizes, respectively.

(iii) The sequence $\{B_n\}$ of nonnegative matrices is in $\mathcal{S}^*$ if there exists at least one entry sequence of $\{B_n\}$ that is in $\mathcal{S}^*$ and all the other entry sequences are either in $\mathcal{S}^*$ or are tail lighter than some entry sequence of $\{B_n\}$ in $\mathcal{S}^*$.

In the remainder of this paper, we denote by $b(i, j)$ the (i, j)th entry of the matrix B. For a sequence $\{B_k\}$ of matrices, $B_{\le k}$ and $\overline{B_{\le k}}$ are defined elementwise as $B_{\le k} = (b_{\le k}(i, j))$ and $\overline{B_{\le k}} = (\overline{b_{\le k}}(i, j))$, respectively.

We denote by Ω the class of heavy-tailed matrix sequences with the property that, for each sequence $\{B_k\}$ in Ω, there exists a heavy-tailed scalar sequence $\{\beta_k\}$ and a finite, nonzero, nonnegative matrix W such that $\lim_{k\to\infty} \overline{B_{\le k}}/\overline{\beta_{\le k}} = W$. The sequence $\{\beta_k\}$ of nonnegative scalars and the matrix W are called a uniformly dominant sequence of the matrix sequence $\{B_k\}$ and the associated ratio matrix, respectively.

Proposition 2.2. *A heavy-tailed matrix sequence $\{B_k\}$ is in Ω if and only if there exists at least one pair (i_0, j_0) such that the sequence $\{b_k(i_0, j_0)\}$ is heavy tailed and the limit $\lim_{k\to\infty} \overline{b_{\le k}}(i, j)/\overline{b_{\le k}}(i_0, j_0)$ is either* 0 *or a positive number for all i and j.*

Proof. For sufficiency, if there exists at least one pair (i_0, j_0) such that the sequence $\{b_k(i_0, j_0)\}$ is heavy tailed and the limit $\lim_{k\to\infty} \overline{b_{\le k}}(i, j)/\overline{b_{\le k}}(i_0, j_0)$ is either 0 or a positive

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number for all i and j, then the matrix $W = \lim_{k\to\infty} \overline{B_{\leq k}}/\overline{b_{\leq k}}(i_0, j_0)$ is finite, nonzero, and nonnegative. We then take $\beta_k = b_k(i_0, j_0)$ for $k \geq 1$, which implies that $\{B_k\} \in \Omega$.

By necessity, if $\{B_k\} \in \Omega$ then there exist a heavy-tailed scalar sequence $\{\beta_k\}$ and a finite, nonzero, nonnegative matrix W such that $\lim_{k\to\infty} \overline{B_{\leq k}}/\overline{\beta_{\leq k}} = W$. We assume that the (i_0, j_0)th entry $w(i_0, j_0)$ of the matrix W is not 0. Then, for all i and j,

$$\lim_{k\to\infty} \frac{\overline{b_{\leq k}}(i, j)}{\overline{b_{\leq k}}(i_0, j_0)} = \lim_{k\to\infty} \frac{\overline{b_{\leq k}}(i, j)/\overline{\beta_{\leq k}}}{\overline{b_{\leq k}}(i_0, j_0)/\overline{\beta_{\leq k}}} = \frac{w(i, j)}{w(i_0, j_0)},$$

which is either 0 or a positive number. Since $\overline{b_{\leq k}}(i_0, j_0) \backsim w(i_0, j_0)\overline{\beta_{\leq k}}$, $\{\beta_k\}$ is heavy tailed, and $w(i_0, j_0) > 0$, it is obvious that $\{b_k(i_0, j_0)\}$ is heavy tailed. This completes the proof.

The following proposition provides a way of using a sequence of nonnegative scalars to characterize the tail of a sequence of nonnegative matrices. The proof follows from Definition 2.4 and Remark 2.1.

Proposition 2.3. *Given a heavy-tailed matrix sequence $\{B_k\} \in \Omega$ with a uniformly dominant sequence $\{\beta_k\}$, and the associated ratio matrix W,*

(i) *$\{B_k\}$ is long tailed if and only if $\{\beta_k\}$ is long tailed;*

(ii) *$\{B_k\}$ is subexponential if and only if $\{\beta_k\}$ is subexponential; and*

(iii) *$\{B_k\} \in \mathcal{S}^*$ if and only if $\{\beta_k\} \in \mathcal{S}^*$.*

Now we provide some basic properties for heavy-tailed matrix sequences. For simplicity, we assume that all the nonnegative matrices involved are square matrices of common size m.

Proposition 2.4. *For two sequences $\{B_k\}$ and $\{C_k\}$ of nonnegative matrices, $\{B_k\}$ is heavy tailed if*

(i) *there exists a nonnegative, invertible matrix W such that $B_k \geq WC_k$ for all $k > N$, where N is a sufficiently large positive integer; and*

(ii) *$\{C_k\}$ is heavy tailed.*

Proof. If $\{C_k\}$ is heavy tailed, then there exists at least one pair (i_0, j_0) for which the (i_0, j_0)th entry sequence $\{c_k(i_0, j_0)\}$ is heavy tailed. Since W is invertible, each column of W is nonzero. For the i_0th column of W, we assume that the (i_1, i_0)th entry $w(i_1, i_0) > 0$. We then find that

$$\sum_{l=1}^{m} w(i_1, l)c_k(l, j_0) \geq w(i_1, i_0)c_k(i_0, j_0).$$

Since $B_k \geq WC_k$, we have $b_k(i_1, j_0) \geq w(i_1, i_0)c_k(i_0, j_0)$. Notice that $w(i_1, i_0) > 0$ and $\{c_k(i_0, j_0)\}$ is heavy tailed. It follows from Definition 2.1(i) that $\{b_k(i_1, j_0)\}$ is heavy tailed. Therefore, $\{B_k\}$ is heavy tailed according to Definition 2.4(i).

Proposition 2.5. *For two sequences $\{B_k\}$ and $\{C_k\}$ of nonnegative matrices, suppose that there exist a nonnegative, invertible matrix V and an invertible matrix $W \geq V$ such that*

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$VC_k \leq B_k \leq WC_k$ for all $k > N$, where N is a sufficiently large positive integer; that $\{C_k\} \in \Omega$; and that $\{B_k\} \in \mathcal{L}$. Then

(i) *$\{B_k\} \in \mathcal{S}$ if $\{C_k\} \in \mathcal{S}$, and*

(ii) *$\{B_k\} \in \mathcal{S}^*$ if $\{C_k\} \in \mathcal{S}^*$.*

Proof. Using Theorem 2.1 of [29], a similar argument to the proof of Proposition 2.4 will lead to the stated results.

For two sequences $\{B_k\}$ and $\{C_k\}$ of matrices, $B_{\leq k} * C_{\leq k}$ is defined elementwise as

$$B_{\leq k} * C_{\leq k} = \left(\sum_r b_{\leq k}(i, r) * c_{\leq k}(r, j) \right).$$

The following two propositions characterize the tail behavior of convolutions of sequences of nonnegative matrices.

Proposition 2.6. *If*

(i) *$\{p_k\} \in \mathcal{S}$, $\{q_k\}$ is any probability sequence, and $\overline{q_{\leq k}} = o(\overline{p_{\leq k}})$; and*

(ii) *$\overline{B_{\leq k}} \backsim W\overline{p_{\leq k}}$ and $\overline{C_{\leq k}} \backsim V\overline{q_{\leq k}}$,*

then $\overline{B_{\leq k} * C_{\leq k}} \backsim WV\overline{p_{\leq k}}$.

Proof. It is easy to check that

$$\overline{B_{\leq k} * C_{\leq k}} = \left(\sum_{r=1}^{m} \overline{b_{\leq k}(i, r) * c_{\leq k}(r, j)} \right).$$

Since $\overline{B_{\leq k}} \backsim W\overline{p_{\leq k}}$ and $\overline{C_{\leq k}} \backsim V\overline{q_{\leq k}}$, we obtain

$$\overline{b_{\leq k}(i, r)} \backsim w(i, r)\overline{p_{\leq k}} \quad \text{and} \quad \overline{c_{\leq k}(r, j)} \backsim v(r, j)\overline{c_{\leq k}(r, j)}.$$

If $w(i, r) = 0$ or $v(r, j) = 0$, then we take $\overline{b_{\leq k}(i, r) * c_{\leq k}(r, j)} \backsim 0$. If $w(i, r) \neq 0$ and $v(r, j) \neq 0$, then

$$\overline{b_{\leq k}(i, r) * c_{\leq k}(r, j)} = w(i, r)v(r, j) \cdot \overline{\frac{b_{\leq k}(i, r)}{w(i, r)} * \frac{c_{\leq k}(r, j)}{v(r, j)}}.$$

Since

$$\overline{\frac{b_{\leq k}(i, r)}{w(i, r)}} \backsim \overline{p_{\leq k}}, \qquad \overline{\frac{c_{\leq k}(r, j)}{v(r, j)}} \backsim \overline{q_{\leq k}},$$

$\{p_k\} \in \mathcal{S}$, and $\overline{q_{\leq k}} = o(\overline{p_{\leq k}})$, it follows from Proposition 2.7 of [46] that $\overline{p_{\leq k} * q_{\leq k}} \backsim \overline{p_{\leq k}}$ and, so, $\overline{b_{\leq k}(i, r) * c_{\leq k}(r, j)} = w(i, r)v(r, j)\overline{p_{\leq k}}$. Therefore, we obtain

$$\overline{B_{\leq k} * C_{\leq k}} \backsim \left(\sum_{r=1}^{m} w(i, r)v(r, j)\overline{p_{\leq k}} \right) = WV\overline{p_{\leq k}},$$

which completes the proof.

Proposition 2.7. *If $\{p_k\} \in \mathcal{S}$ and two sequences $\{C_k^{(1)}\}$ and $\{C_k^{(2)}\}$ of nonnegative matrices satisfy $\overline{C_{\le k}^{(l)}} \backsim H_l \overline{p_{\le k}}$ for $l = 1, 2$, where H_1 and H_2 are two finite, nonzero, nonnegative matrices, then $\overline{C_{\le k}^{(1)} * C_{\le k}^{(2)}} \backsim (H_1 ee^\top + ee^\top H_2)\overline{p_{\le k}}$, where e is a column vector of 1s and '$\top$' denotes the transpose of a matrix.*

Proof. The condition that $\overline{C_{\le k}^{(l)}} \backsim H_l \overline{p_{\le k}}$ for $l = 1, 2$ implies that

$$\overline{c_{\le k}^{(1)}}(i, r) \backsim h_1(i, r)\overline{p_{\le k}}, \qquad \overline{c_{\le k}^{(2)}}(r, j) \backsim h_2(r, j)\overline{p_{\le k}}.$$

Noting that

$$\overline{C_{\le k}^{(1)} * C_{\le k}^{(2)}} = \left(\sum_{r=1}^{m} \overline{c_{\le k}^{(1)}(i, r) * c_{\le k}^{(2)}(r, j)} \right),$$

using Theorem 5.1 of [25] leads to

$$\overline{c_{\le k}^{(1)}(i, r) * c_{\le k}^{(2)}(r, j)} = [h_1(i, r) + h_2(r, j)]\overline{p_{\le k}}.$$

Simple computations then lead to $\overline{C_{\le k}^{(1)} * C_{\le k}^{(2)}} \backsim (H_1 ee^\top + ee^\top H_2)\overline{p_{\le k}}$, which completes the proof.

3. A condition on the heavy tail of $\{\pi_k\}$

In this section, we provide an expression for the stationary probability vector in terms of the R-measure. Using this expression, we obtain a necessary and sufficient condition under which $\{\pi_k\}$ is heavy tailed.

For the transition probability matrix of GI/G/1 type, the R- and G-measures have been extensively studied, for example in [26] and [50]. Note that the R- and G-measures have decompositions in terms of $\{R_{0,k}\}$ and $\{R_k\}$, and $\{G_{k,0}\}$ and $\{G_k\}$, respectively. The sequences $\{R_k\}$ and $\{G_k\}$ can be expressed by means of a matrix sequence $\{\Phi_i, -\infty < i < \infty\}$. Let $L_0 = \{(0, j) : 1 \le j \le m_0\}$ and $L_i = \{(i, j) : 1 \le j \le m\}$. Also let $L_{\le i} = \bigcup_{k=0}^{i} L_k$ and write $L_{\ge i}$ for the complement of $L_{\le (i-1)}$. To define the matrices Φ_i, $-\infty < i < \infty$, we write

$$P = \begin{array}{c} \\ L_{\le n} \\ L_{\ge (n+1)} \end{array} \begin{array}{c} \begin{array}{cc} L_{\le n} & L_{\ge (n+1)} \end{array} \\ \begin{pmatrix} Q_0 & U \\ V & Q_1 \end{pmatrix} \end{array}$$

for $n \ge 1$. Let $P^{[n]} = Q_0 + U\widehat{Q}_1 V$, where $\widehat{Q}_1 = \sum_{k=0}^{\infty} Q_1^k$. We denote by $P_{i,j}^{[n]}$ the (i, j)th block entry of $P^{[n]}$ corresponding to the levels of P. Grassmann and Heyman [26] showed that the matrices $P_{n-i,n}^{[n]}$, for $0 \le i \le n-1$, and $P_{n,n-j}^{[n]}$, for $0 \le j \le n-1$, are all independent of $n \ge 1$. Therefore, for $n \ge 1$, $0 \le i \le n-1$, and $0 \le j \le n-1$, we define

$$\Phi_i = P_{n-i,n}^{[n]}, \qquad \Phi_{-j} = P_{n,n-j}^{[n]}.$$

It is shown in Section 3 of [26] that

$$R_i = \Phi_i (I - \Phi_0)^{-1}, \qquad i \ge 1,$$

and

$$G_i = (I - \Phi_0)^{-1}\Phi_{-i}, \qquad i \ge 1, \tag{3.1}$$

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where I denotes the identity matrix. Furthermore, $\{R_k\}$ and $\{G_k\}$ are the minimal nonnegative solutions to the Wiener–Hopf equations

$$R_i(I - \Phi_0) = A_i + \sum_{l=1}^{\infty} R_{i+l}(I - \Phi_0)G_l, \qquad i \geq 1, \tag{3.2}$$

and

$$(I - \Phi_0)G_j = A_{-j} + \sum_{l=1}^{\infty} R_l(I - \Phi_0)G_{j+l}, \qquad j \geq 1. \tag{3.3}$$

Similarly, after we correct two typographical errors, the Wiener–Hopf equations [50, Equations (18) and (19)] for $\{R_{0,j}\}$ and $\{G_{i,0}\}$ are given by

$$R_{0,k}(I - \Phi_0) = D_k + \sum_{i=1}^{\infty} R_{0,k+i}(I - \Phi_0)G_i, \qquad k \geq 1, \tag{3.4}$$

and

$$(I - \Phi_0)G_{k,0} = D_{-k} + \sum_{i=1}^{\infty} R_i(I - \Phi_0)G_{i+k,0}, \qquad k \geq 1.$$

Let

$$\Psi_0 = D_0 + \sum_{i=1}^{\infty} R_{0,i}(I - \Phi_0)G_{i,0}.$$

This is the censored matrix of P to level 0, and it is positive recurrent since the Markov chain is irreducible and positive recurrent.

Define the generating functions for the matrix sequences $\{A_k\}$, $\{R_k\}$, and $\{G_k\}$ as

$$A^*(z) = \sum_{k=-\infty}^{\infty} z^k A_k, \qquad R^*(z) = \sum_{k=1}^{\infty} z^k R_k, \qquad G^*(z) = \sum_{k=1}^{\infty} z^{-k} G_k.$$

The following RG-factorization will be used in our study (refer to Zhao [50] or Zhao *et al.* [52] for a proof using (3.2) and (3.3)):

$$I - A^*(z) = [I - R^*(z)](I - \Phi_0)[I - G^*(z)]. \tag{3.5}$$

Let ϕ_{A+}, ϕ_D, ϕ_R, and ϕ_{R_0} be the radii of convergence of the matrix functions $\sum_{k=1}^{\infty} z^k A_k$, $\sum_{k=1}^{\infty} z^k D_k$, $R^*(z)$, and $\sum_{k=1}^{\infty} R_{0,k} z^k$, respectively. By using the RG-factorization, Theorem 1 and Lemma 3 of [33] provide important relations between the radii of convergence:

$$\phi_R = \phi_{A+}, \qquad \phi_{R_0} = \phi_D. \tag{3.6}$$

If the Markov chain of GI/G/1 type is positive recurrent, then [26, Equation (28)] shows that the stationary probability vector $\{\pi_k\}$ is given by

$$\begin{aligned} \pi_0 &= x_0, \\ \pi_k &= \pi_0 R_{0,k} + \sum_{i=1}^{k-1} \pi_i R_{k-i}, \qquad k \geq 1, \end{aligned} \tag{3.7}$$

where $x_0 = c\widehat{x}_0$ and $R_0 = \sum_{k=1}^{\infty} R_{0,k}$. Here, $\widehat{x}_0$ is the stationary probability vector of the censored matrix Ψ_0 of P to level 0, $c = (\widehat{x}_0 R_0 (I - R)^{-1} e)^{-1}$, and $R = \sum_{k=1}^{\infty} R_k$. Let $\Pi^*(z) = \sum_{k=1}^{\infty} z^k \pi_k$. It follows from (3.7) that

$$\Pi^*(z)[I - R^*(z)] = x_0 R_0^*(z).$$

Since the Markov chain is positive recurrent, Corollary 30 of [52] shows that all solutions to the equation $\det(I - R^*(z)) = 0$, if any exist, reside outside the unit circle $|z| > 1$. This means that $I - R^*(z)$ is always invertible for all $|z| \leq 1$. Therefore,

$$\Pi^*(z) = x_0 R_0^*(z)[I - R^*(z)]^{-1},$$

which implies that

$$\pi_k = x_0 R_{0,k} \circledast \sum_{n=0}^{\infty} R_k^{n\circledast} \tag{3.8}$$

and, thus,

$$\overline{\pi_{\leq k}} = \sum_{l=k+1}^{\infty} x_0 R_{0,l} \circledast \sum_{n=0}^{\infty} R_l^{n\circledast}.$$

The following lemma provides an expression for the tail of the stationary probability vector $\{\pi_k\}$, and later plays a key role in our study.

Lemma 3.1. *For all* $k \geq 1$,

$$\overline{\pi_{\leq k}} = x_0 R_{0,\leq k} * \overline{\sum_{n=0}^{\infty} R_{\leq k}^{n\circledast}},$$

where $R_{0,\leq k} = \sum_{l=1}^{k} R_{0,l}$, $R_{\leq k}^{n\circledast} = \sum_{l=1}^{k} R_l^{n\circledast}$, *and*

$$R_{0,\leq k} * \overline{\sum_{n=0}^{\infty} R_{\leq k}^{n\circledast}} = R_0(I - R)^{-1} - R_{0,\leq k} * \sum_{n=0}^{\infty} R_{\leq k}^{n\circledast}.$$

Proof. Noting that

$$\sum_{l=0}^{k} x_0 R_{0,l} \circledast \sum_{n=0}^{\infty} R_l^{n\circledast} + \sum_{l=k+1}^{\infty} x_0 R_{0,l} \circledast \sum_{n=0}^{\infty} R_l^{n\circledast} = R_0(I - R)^{-1},$$

it follows from (3.8) and (2.5) that

$$\begin{aligned} \overline{\pi_{\leq k}} &= x_0 R_0 (I - R)^{-1} - \sum_{l=1}^{k} x_0 R_{0,l} \circledast \sum_{n=0}^{\infty} R_l^{n\circledast} \\ &= R_0(I - R)^{-1} - R_{0,\leq k} * \sum_{n=0}^{\infty} R_{\leq k}^{n\circledast} = x_0 R_{0,\leq k} * \overline{\sum_{n=0}^{\infty} R_{\leq k}^{n\circledast}}. \end{aligned}$$

This completes the proof.

The following lemma provides a useful tail heaviness property for the matrix sequences $\{A_k\}$ and $\{D_k\}$. The proof is obvious, from the definition of heavy tails.

Lemma 3.2. (i) *If* $\phi_{A+} = 1$ *then the matrix sequence* $\{A_k\}$ *is heavy tailed.*

(ii) *If* $\phi_D = 1$ *then the matrix sequence* $\{D_k\}$ *is heavy tailed.*

Under the assumptions in Lemma 3.2, the matrix sequences $\{R_k\}$ and $\{R_{0,k}\}$ are heavy tailed since $\phi_R = \phi_{A+}$ and $\phi_{R_0} = \phi_D$.

In this work, we carry out the tail analysis of Markov chains of GI/G/1 type under the assumption that $\min\{\phi_{A+}, \phi_D\} = 1$, which is an extension of [33], where it was assumed that $\min\{\phi_{A+}, \phi_D\} > 1$. The following theorem shows that satisfaction of our condition is necessary and sufficient for the stationary probability vector $\{\pi_k\}$ to be heavy tailed.

Theorem 3.1. *If the Markov chain of GI/G/*1 *type is positive recurrent, then the stationary probability vector* $\{\pi_k\}$ *is heavy tailed if and only if* $\min\{\phi_{A+}, \phi_D\} = 1$.

Proof. We first prove the necessity of the condition. Suppose that $\min\{\phi_{A+}, \phi_D\} > 1$. Then both $\phi_{A+} > 1$ and $\phi_D > 1$. Using (3.6) and

$$\Pi^*(z) = x_0 R_0^*(z)[I - R^*(z)]^{-1}$$

gives $\phi_\pi = \min\{\phi_{A+}, \phi_D, \eta\} > 1$, where ϕ_π is the radius of convergence of the vector function $\Pi^*(z)$ and $\eta > 1$ is the minimal positive solution, if one exists, to the equation $\det(I - R^*(z)) = 0$ (with the convention that $\eta = \infty$ if there does not exist such a solution). Hence, $\{\pi_k\}$ is light tailed, but this is in contradiction with the assumption that $\{\pi_k\}$ is heavy tailed.

We now prove the sufficiency of the condition. Note that when $\phi_{A+} \geq 1$ and $\phi_D \geq 1$, the assumption that $\min\{\phi_{A+}, \phi_D\} = 1$ implies that $\phi_{A+} = 1$ or $\phi_D = 1$.

Case I: $\phi_D = 1$. In this case, since

$$\Pi^*(z) = x_0 R_0^*(z)[I - R^*(z)]^{-1} \geq x_0 R_0^*(z),$$

we have $\pi_k \geq x_0 R_{0,k}$ for $k \geq 1$. Note that, since the Markov chain of GI/G/1 type is irreducible and positive recurrent, the censored chain Ψ_0 to level 0 is also irreducible and positive recurrent, which implies that $x_0 > 0$. Under the assumption that $\phi_D = 1$, $\{R_{0,k}\}$ is heavy tailed. Thus, there always exists at least one pair (i, j) such that the sequence $\{r_{0,k}(i, j)\}$ is heavy tailed, where $r_{0,k}(i, j)$ is the (i, j)th entry of the matrix $R_{0,k}$ for each $k \geq 1$. It is clear that $\pi_k \geq x_0 R_{0,k}$ implies that

$$\pi_k \geq (\underbrace{0, \dots, 0}_{j-1 \text{ zeros}}, x_0(i) r_{0,k}(i, j), \underbrace{0, \dots, 0}_{m-j \text{ zeros}}),$$

where $x_0(i)$ is the ith entry of the positive row vector x_0. Therefore, $\{\pi_k\}$ is heavy tailed.

Case II: $\phi_D > 1$ *and* $\phi_{A+} = 1$. In this case, since

$$\Pi^*(z) = x_0 R_0^*(z)[I - R^*(z)]^{-1} \geq x_0 R_0^*(z) R^*(z),$$

we have

$$\pi_k \geq x_0 R_{0,k} \circledast R_k \quad \text{for all } k \geq 1. \tag{3.9}$$

Under the assumption that $\phi_{A+} = 1$, $\{R_k\}$ is heavy tailed. Thus, there exists at least one pair (i_0, j_0) such that $\{r_k(i_0, j_0)\}$ is heavy tailed. The assumption that the Markov chain is irreducible and positive recurrent leads to $x_0 > 0$, and using Theorem 6 of [33] leads to $x_0 R_0 > 0$. (Recall that $R_0 = \sum_{k=1}^{\infty} R_{0,k}$.) Therefore, there always exist an $l_0 \geq 1$ and an i^* such that the (i^*, i_0)th element $r_{0,l_0}(i^*, i_0)$ of R_{0,l_0} is positive. Thus, it follows from (3.9) that, for $k \geq N$,

$$\pi_k \geq (\underbrace{0, \dots, 0}_{i_0-1 \text{ zeros}}, x_0(i^*) r_{0,l_0}(i^*, i_0) r_{k-l_0}(i_0, j_0), \underbrace{0, \dots, 0}_{m-i_0 \text{ zeros}}). \tag{3.10}$$

Since $x_0(i^*) r_{0,l_0}(i^*, i_0) > 0$ and $\{r_{k-l_0}(i_0, j_0)\}$ is heavy tailed, (3.10) implies that $\{\pi_k\}$ is heavy tailed.

4. The long-tailed asymptotics of the R-measure

In this section, we provide the long-tailed asymptotics of the R-measure. In particular, we show that the matrix sequence $\{R_{0,k}\}$ is long tailed if the matrix sequence $\{D_k\}$ is long tailed, and that the matrix sequence $\{R_k\}$ is long tailed if the matrix sequence $\{A_k\}$ is long tailed. These results are the key to deriving the subexponential asymptotics of the stationary probability vector in the next section. Throughout the rest of the paper, we assume that a heavy-tailed matrix sequence $\{A_k\}$ or $\{D_k\}$ is an element of Ω.

We first discuss the long-tailed asymptotics of the matrix sequence $\{R_{0,k}\}$. To do so, we must provide an expression for the matrix $R_{0,k}$, $k \geq 1$. The following lemma is useful for this purpose.

Lemma 4.1. *Let* $B_i = \Phi_{-i}(I - \Phi_0)^{-1}$, $i \geq 1$, *and*

$$\Lambda = \begin{pmatrix} I & & & & \\ -B_1 & I & & & \\ -B_2 & -B_1 & I & & \\ -B_3 & -B_2 & -B_1 & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$\Lambda^{-1} = \begin{pmatrix} I & & & & \\ X_1 & I & & & \\ X_2 & X_1 & I & & \\ X_3 & X_2 & X_1 & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$X_l = \sum_{i=1}^{\infty} \sum_{\substack{n_1+n_2+\cdots+n_i=l \\ n_j \geq 1,\ 1 \leq j \leq i}} B_{n_1} B_{n_2} \cdots B_{n_i}, \qquad l \geq 1.$$

Proof. Noting that $\Lambda^{-1}\Lambda = I$, we find that $-B_k - \sum_{i=1}^{k-1} B_{k-i} X_i + X_k = 0$ for all $k \geq 1$. Let $X^*(z) = \sum_{k=1}^{\infty} z^k X_k$ and $B^*(z) = \sum_{k=1}^{\infty} z^k B_k$. Then

$$X^*(z) = [I - B^*(z)]^{-1} B^*(z) = \sum_{i=1}^{\infty} [B^*(z)]^i = \sum_{l=1}^{\infty} z^l \sum_{i=1}^{\infty} \sum_{\substack{n_1+n_2+\cdots+n_i=l \\ n_j \geq 1,\ 1 \leq j \leq i}} B_{n_1} B_{n_2} \cdots B_{n_i}, \tag{4.1}$$

and we thus have

$$X_l = \sum_{i=1}^{\infty} \sum_{\substack{n_1+n_2+\cdots+n_i=l \\ n_j \geq 1,\ 1 \leq j \leq i}} B_{n_1} B_{n_2} \cdots B_{n_i}, \qquad l \geq 1.$$

This completes the proof.

The following theorem characterizes the long-tailed asymptotics of the matrix sequence $\{R_{0,k}\}$.

Theorem 4.1. *Suppose that the Markov chain of GI/G/*1 *type is positive recurrent. If* $\{D_k\}$ *is long tailed with a uniformly dominant sequence* $\{q_k\}$ *and associated ratio matrix* V*, then*

$$\lim_{k\to\infty} \frac{\overline{R_{0,\leq k}}}{\overline{q_{\leq k}}} = V\left(I - \sum_{i=0}^{\infty} \Phi_{-i}\right)^{-1}.$$

Proof. It follows from (3.4) and (3.1) that, for all $k \geq 1$,

$$R_{0,k} - \sum_{i=1}^{\infty} R_{0,k+i}\Phi_{-i}(I - \Phi_0)^{-1} = D_k(I - \Phi_0)^{-1},$$

or

$$(R_{0,1}, R_{0,2}, R_{0,3}, R_{0,4}, \dots)\begin{pmatrix} I & & & & \\ -B_1 & I & & & \\ -B_2 & -B_1 & I & & \\ -B_3 & -B_2 & -B_1 & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} F_1^\top \\ F_2^\top \\ F_3^\top \\ F_4^\top \\ \vdots \end{pmatrix}^\top,$$

where $F_k = D_k(I - \Phi_0)^{-1}$ for $k \geq 1$. Using Lemma 4.1 gives, for all $k \geq 1$,

$$R_{0,k} = F_k + \sum_{i=1}^{\infty} F_{k+i}X_i = D_k(I - \Phi_0)^{-1} + \sum_{i=1}^{\infty} D_{k+i}(I - \Phi_0)^{-1}X_i.$$

Since $(I - \Phi_0)^{-1} \geq 0$ and $B_i = \Phi_{-i}(I - \Phi_0)^{-1} \geq 0$ for $i \geq 1$, it follows from (4.1) that

$$0 \leq \frac{\overline{D_{\leq k+i}}}{\overline{q_{\leq k}}}(I - \Phi_0)^{-1}X_i \leq \frac{\overline{D_{\leq k+i}}}{\overline{q_{\leq k}}}(I - \Phi_0)^{-1}[I - B^*(1)]^{-1}B^*(1).$$

Thus, using the dominated convergence theorem, we obtain

$$\lim_{k\to\infty} \sum_{i=1}^{\infty} \frac{\overline{D_{\leq k+i}}}{\overline{q_{\leq k}}}(I - \Phi_0)^{-1}X_i = \sum_{i=1}^{\infty} \lim_{k\to\infty} \frac{\overline{D_{\leq k+i}}}{\overline{q_{\leq k}}}(I - \Phi_0)^{-1}X_i.$$

Also note that, since $\{D_k\}$ is long tailed with a uniformly dominant sequence $\{q_k\}$ and associated ratio matrix V, we obtain

$$\begin{aligned}\lim_{k\to\infty} \frac{\overline{R_{0,\leq k}}}{\overline{q_{\leq k}}} &= \lim_{k\to\infty}\left(\frac{\overline{D_{\leq k}}}{\overline{q_{\leq k}}}(I - \Phi_0)^{-1} + \sum_{i=1}^{\infty} \frac{\overline{D_{\leq k+i}}}{\overline{q_{\leq k}}}(I - \Phi_0)^{-1}X_i\right) \\ &= V(I - \Phi_0)^{-1}\left(I + \sum_{i=1}^{\infty} X_i\right),\end{aligned}$$

where we have used the fact that $\lim_{k\to\infty} \overline{D_{\le k}}/\overline{q_{\le k}} = \lim_{k\to\infty} \overline{D_{\le k+i}}/\overline{q_{\le k}} = V$ for an arbitrary $i \ge 1$. It follows, from Lemma 4.1, that $\sum_{i=1}^{\infty} X_i = [I - B^*(1)]^{-1} B^*(1)$ and $I + \sum_{i=1}^{\infty} X_i = [I - B^*(1)]^{-1}$. Hence, we obtain

$$[I - B^*(1)]^{-1} = \left[I - \sum_{i=1}^{\infty} \Phi_{-i}(I - \Phi_0)^{-1}\right]^{-1} = (I - \Phi_0)\left(I - \sum_{i=0}^{\infty} \Phi_{-i}\right)^{-1}$$

and, therefore,

$$\lim_{k\to\infty} \frac{\overline{R_{0,\le k}}}{\overline{q_{\le k}}} = V\left(I - \sum_{i=0}^{\infty} \Phi_{-i}\right)^{-1},$$

which completes the proof.

In what follows, we study the long-tailed asymptotics of the matrix sequence $\{R_k\}$. Throughout the rest of the paper, we denote by ω the stationary probability vector of the matrix A. The following lemma, in which $sp(R)$ denotes the spectral radius of R, will be needed; readers are referred to Theorem 4 of [51] or Theorem 23 of [52] for details.

Lemma 4.2. *Suppose that the Markov chain P of GI/G/1 type is irreducible.*

(i) *If P is positive recurrent then $\omega R \lneqq \omega$ (i.e. $sp(R) < 1$) and G is stochastic.*

(ii) *If P is null recurrent then $\omega R = \omega$ (i.e. $sp(R) = 1$) and G is stochastic.*

(iii) *If P is transient then $\omega R = \omega$ (i.e. $sp(R) = 1$) and G is strictly substochastic.*

The following two lemmas are useful in determining a uniformly dominant subsequence of the matrix sequence $\{R_k\}$, and the associated ratio matrix if the matrix sequence $\{A_k\}$ is long tailed.

Lemma 4.3. *If the Markov chain of GI/G/1 type is positive recurrent and $\sum_{k=-\infty}^{\infty} |k| A_k$ is finite, then $\sum_{k=1}^{\infty} k G_k$ is finite.*

Proof. It follows from (3.5) that

$$\sum_{k=1}^{\infty} k A_{-k} - \sum_{k=1}^{\infty} k A_k = (I - R)(I - \Phi_0) \sum_{k=1}^{\infty} k G_k - \sum_{k=1}^{\infty} k R_k (I - \Phi_0)(I - G).$$

Since the Markov chain is positive recurrent, it follows from Lemma 4.2(i) that $I - R$ is invertible and $(I - G)e = 0$. It is clear that $(I - \Phi_0)^{-1}(I - R)^{-1} \gneqq 0$ and is finite. Since $\sum_{k=-\infty}^{\infty} |k| A_k$ is also finite,

$$\sum_{k=1}^{\infty} k G_k e = (I - \Phi_0)^{-1}(I - R)^{-1}\left(\sum_{k=1}^{\infty} k A_{-k} - \sum_{k=1}^{\infty} k A_k\right) e$$

is finite. Therefore, $\sum_{k=1}^{\infty} k G_k$ is finite.

When the Markov chain of GI/G/1 type is positive recurrent, the matrix $I - R$ is invertible, according to Lemma 4.2. It follows from (3.5) that $I - A = (I - R)(I - \Phi_0)(I - G)$. When A is irreducible and stochastic, the maximal eigenvalue of A is simple and is equal to 1. Hence, $\text{rank}(I - A) = m - 1$. Since the matrix $I - \Phi_0$ is invertible, we find that $\text{rank}(I - G) = m - 1$; hence, the maximal eigenvalue of G is simple and is equal to 1.

Letting $g_1 = 1$ and g_i, $2 \le i \le m$, be the m eigenvalues of the nonnegative matrix G, we have the following lemma.

Lemma 4.4. *If the Markov chain of GI/G/1 type is positive recurrent and the matrix A is irreducible and stochastic, then the adjoint matrix of $I - G$ can be expressed as*

$$\operatorname{adj}(I - G) = \kappa_G e \frac{\omega(I - R)(I - \Phi_0)}{\omega(I - R)(I - \Phi_0)e}, \tag{4.2}$$

where $\kappa_G = \prod_{k=2}^{m}(1 - g_i) \neq 0$.

Proof. Note that when the maximal eigenvalue of the matrix G is simple and equal to 1, we obtain

$$\operatorname{adj}(I - \alpha G) = \det(I - \alpha G) \cdot (I - \alpha G)^{-1} = \prod_{i=2}^{m}(1 - \alpha g_i) \cdot (1 - \alpha)(I - \alpha G)^{-1},$$

where $\alpha \in (0, 1)$. Thus, there exists an invertible matrix $T(\alpha)$ such that

$$T(\alpha)^{-1}(I - \alpha G)T(\alpha) = \begin{pmatrix} 1 - \alpha & \\ & J(\alpha) \end{pmatrix},$$

which is the canonical Jordan form of the matrix $I - \alpha G$. It follows that

$$(1 - \alpha)(I - \alpha G)^{-1} = T(\alpha) \begin{pmatrix} 1 & \\ & (1 - \alpha)J(\alpha)^{-1} \end{pmatrix} T(\alpha)^{-1}.$$

Since, in the matrix

$$T(1)^{-1}(I - G)T(1) = \begin{pmatrix} 0 & \\ & J(1) \end{pmatrix},$$

$J(1)$ is invertible due to the fact that $\operatorname{rank}(I - G) = m - 1$, we have $\lim_{\alpha \nearrow 1}(1 - \alpha)J(\alpha)^{-1} = 0$. Note that, since $\operatorname{adj}(I - \alpha G)$ is continuous for $\alpha \in (0, 1]$, we obtain

$$\begin{aligned} \operatorname{adj}(I - G) &= \lim_{\alpha \nearrow 1} \operatorname{adj}(I - \alpha G) = \lim_{\alpha \nearrow 1} \prod_{i=2}^{m}(1 - \alpha g_i) \cdot (1 - \alpha)(I - \alpha G)^{-1} \\ &= \kappa_G T(1) \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} T(1)^{-1} = \kappa_G e \frac{\omega(I - R)(I - \Phi_0)}{\omega(I - R)(I - \Phi_0)e}. \end{aligned}$$

The final equality holds because the vectors e and $\omega(I - R)(I - \Phi_0)/\omega(I - R)(I - \Phi_0)e$ are the right and left Perron–Frobenius eigenvectors of G, respectively. Note that, since $\operatorname{rank}(I - G) = m - 1$, it is clear that $\operatorname{adj}(I - G) \neq 0$, which implies that $\kappa_G \neq 0$. This completes the proof.

To study the long-tailed asymptotics of the matrix sequence $\{R_k\}$, we need to extend the results in Lemma 4 and Proposition 1 of [28] to a matrix setting. We do so in the following two lemmas. All the measures involved in the following can be signed measures.

Let $\mathfrak{B}(\mathbb{R})$ be the σ-algebra of Borel sets on $\mathbb{R} = (-\infty, \infty)$. The convolution of two measures μ_1 and μ_2 is defined as

$$(\mu_1 * \mu_2)(B) = \int_{(-\infty,\infty)} \mu_1(B - x)\mu_2(\mathrm{d}x), \qquad B \in \mathfrak{B}(\mathbb{R}),\ B - x = \{y \colon y + x \in B\}.$$

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For $B \in \mathfrak{B}(\mathbb{R})$, let $U(B)$ and $V(B)$ be two $m \times m$ matrices whose entries are finite measures, given as

$$U(B) = (u_{ij}(B))_{1\le i,j\le m} \quad \text{and} \quad V(B) = (v_{ij}(B))_{1\le i,j\le m}.$$

The convolution of the two matrices U and V of finite measures is defined as

$$(U * V)(B) = \left(\sum_{k=1}^{m}(u_{ik} * v_{kj})(B)\right)_{1\le i,j\le m}$$

and the convolution of a matrix U of finite measures and a finite scalar measure v is defined as

$$(U * v)(B) = ((u_{ij} * v)(B))_{1\le i,j\le m}.$$

Remark 4.1. It should be noted that when B is a singleton, the convolution for measures coincides with the ordinary convolution for sequences.

A distribution function $F(x)$ is called long tailed if $\lim_{x\to\infty} \overline{F}(x+y)/\overline{F}(x) = 1$ for all $y \in \mathbb{R}$, where $\overline{F}(x) = 1 - F(x)$.

Lemma 4.5. *Let U and U_- be two $m \times m$ matrices of finite measures on* $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$. *If* $\lim_{x\to\infty} U([x,\infty))/\overline{F}(x) = C$, *where $F(x)$ is a long-tailed distribution function and the matrix C is finite, and U_- has support on* $(-\infty, 0]$, *then the matrix $\Gamma = U_- * U$ satisfies*

$$\lim_{x\to\infty} \frac{\Gamma([x,\infty))}{\overline{F}(x)} = U_-((-\infty,0])C$$

and the matrix $\widetilde{\Gamma} = U * U_-$ satisfies

$$\lim_{x\to\infty} \frac{\widetilde{\Gamma}([x,\infty))}{\overline{F}(x)} = CU_-((-\infty,0]).$$

Proof. Using Lemma 4 of [28], we obtain

$$\begin{aligned}\lim_{x\to\infty} \frac{\Gamma([x,\infty))}{\overline{F}(x)} &= \lim_{x\to\infty} \frac{1}{\overline{F}(x)} \left(\sum_{k=1}^{m}(u_{-ik} * u_{kj})([x,\infty))\right)_{1\le i,j\le m} \\ &= \left(\sum_{k=1}^{m} u_{-ik}((-\infty,0])c_{kj}\right)_{1\le i,j\le m} = U_-((-\infty,0])C.\end{aligned}$$

This completes the proof.

Lemma 4.6. *Let U and U_+ be two $m \times m$ matrices of finite measures on* $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$. *Assume that* (i) *$\mu_-$ is a finite scalar measure with support on* $(-\infty, 0]$, *such that* $\mu_-((-\infty,0]) = 0$ *and* $0 < |\int_{(-\infty,0]} x\mu_-(\mathrm{d}x)| < \infty$; (ii) *$U_+$ has support on* $[0,\infty)$ *with at least one nonzero element, and all the nonzero elements of U_+ are strictly positive on* $[a,\infty)$ *for* $a > 0$; *and* (iii) $\lim_{x\to\infty} U([x,\infty))/\overline{F}(x) = C$, *where $F(x)$ is a long-tailed distribution function and the matrix C is finite. If $U = \mu_- * U_+$, then*

$$\lim_{x\to\infty} \frac{U_+([x,\infty))}{\int_{[x,\infty)} \overline{F}(y)\,\mathrm{d}y} = \frac{C}{\int_{(-\infty,0]} x\mu_-(\mathrm{d}x)}.$$

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Proof. The proof is obvious from Proposition 1 of [28].

Lemma 4.7. *Suppose that the Markov chain of GI/G/1 type is positive recurrent. If $\{A_k\}$ is long tailed with a uniformly dominant sequence $\{p_k\}$ and associated ratio matrix W, then*

$$\varliminf_{k\to\infty} \frac{\overline{R_{\le k}}}{\overline{p_{\le k}}} \ge W(I-\Phi_0)^{-1}.$$

Proof. Note that, since the Markov chain of GI/G/1 type is positive recurrent, it follows from (3.2) and (3.1) that, for all $k \ge 1$,

$$R_k = A_k(I-\Phi_0)^{-1} + \sum_{l=1}^{\infty} R_{k+l}\Phi_{-l}(I-\Phi_0)^{-1} \ge A_k(I-\Phi_0)^{-1},$$

since $(I-\Phi_0)^{-1} \ge 0$ and $R_k \ge 0$ and $\Phi_{-k} \ge 0$ for $k \ge 1$. Hence, for all $k \ge 1$,

$$\overline{R_{\le k}} = \sum_{l=k+1}^{\infty} R_l \ge \sum_{l=k+1}^{\infty} A_k(I-\Phi_0)^{-1} = \overline{A_{\le k}}(I-\Phi_0)^{-1}.$$

Since $\{A_k\}$ is long tailed, it is clear that

$$\varliminf_{k\to\infty} \frac{\overline{R_{\le k}}}{\overline{p_{\le k}}} \ge \varliminf_{k\to\infty} \frac{\overline{A_{\le k}}}{\overline{p_{\le k}}}(I-\Phi_0)^{-1} = \lim_{k\to\infty} \frac{\overline{A_{\le k}}}{\overline{p_{\le k}}}(I-\Phi_0)^{-1} = W(I-\Phi_0)^{-1}.$$

Now we are able to prove the following theorem, which characterizes the long-tailed asymptotics of the matrix sequence $\{R_k\}$.

Theorem 4.2. *Suppose that the Markov chain of GI/G/1 type is positive recurrent, $\phi_{A-} < 1$, and $\sum_{k=-\infty}^{\infty} |k| A_k$ is finite. If $\{A_k\}$ is long tailed with a uniformly dominant probability sequence $\{p_k\}$ and associated ratio matrix W, then*

$$\lim_{k\to\infty} \frac{\overline{R_{\le k}}}{\overline{\overline{p_{\le k}}}} = \frac{We\omega(I-R)}{\omega(I-R)(I-\Phi_0)\sum_{j=1}^{\infty} jG_j e}, \tag{4.3}$$

where $\overline{\overline{p_{\le k}}} = \sum_{n=k+1}^{\infty} \overline{p_{\le n}}$.

Proof. It follows from (3.5) that

$$[R^*(z)-I]\det(I-G^*(z)) = [A^*(z)-I]\cdot \operatorname{adj}(I-G^*(z))\cdot(I-\Phi_0)^{-1} \tag{4.4}$$

when the matrix $I-G^*(z)$ is invertible. To evaluate the asymptotics of the coefficient matrix sequence in the generating function $R^*(z)-I$, we first analyze the asymptotics of the coefficient matrix sequence in the generating function $[A^*(z)-I]\cdot\operatorname{adj}(I-G^*(z))$, using Lemma 4.5. Since $A^*(z)-I$ and $\operatorname{adj}(I-G^*(z))$ are analytic for $\phi_{A-} < |z| < \phi_{A+} = 1$ (where $\phi_{A+} = 1$ results from the fact that $\{A_k\}$ is long tailed), we can write $A^*(z)-I = \sum_{k=-\infty}^{\infty} z^k \widehat{A}_k$, where

$$\widehat{A}_k = \begin{cases} A_k & \text{if } k \ne 0, \\ A_0 - I & \text{if } k = 0. \end{cases}$$

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Since $G^*(z) = \sum_{k=1}^{\infty} z^{-k} G_k$, by the definition of the adjoint matrix we can write

$$\operatorname{adj}(I - G^*(z)) = \sum_{k=-\infty}^{0} z^k S_k$$

and define $S_k = 0$ for all $k \geq 1$. Let

$$\sum_{k=-\infty}^{\infty} z^k Q_k = [A^*(z) - I] \cdot \operatorname{adj}(I - G^*(z)).$$

Then the equality

$$\sum_{k=-\infty}^{\infty} z^k Q_k = \sum_{k=-\infty}^{\infty} z^k \widehat{A}_k \cdot \sum_{k=-\infty}^{\infty} z^k S_k,$$

with $\sum_{k=1}^{\infty} z^k S_k = 0$, implies that

$$Q_k = \sum_{i+j=k} \widehat{A}_i S_j = \widehat{A}_k \circledast S_k, \tag{4.5}$$

with $S_j = 0$ for $j \geq 1$. Therefore, we obtain $\overline{Q_{\leq k}} = \overline{A_{\leq k} * S_{\leq k}}$ for $k \geq 1$. If, for a matrix sequence $\{C_k\}$, we define the matrix of measures by $\mu_C(B) = \sum_{k \in B} C_k$, then it follows from Remark 4.1, (4.5), and Lemma 4.5 that

$$\lim_{k \to \infty} \frac{\overline{Q_{\leq k}}}{\overline{p_{\leq k}}} = W \cdot \sum_{k=-\infty}^{0} S_k = W \cdot \operatorname{adj}(I - G). \tag{4.6}$$

We now evaluate the asymptotics of the coefficient matrix sequence in the generating function $R^*(z) - I$, using (4.4), (4.6), and Lemma 4.6. Let $\det(I - G^*(z)) = \sum_{k=-\infty}^{0} z^k g_k$ and define $\mu_-(B) = \sum_{k \in B} g_k$. Since the Markov chain of GI/G/1 type is positive recurrent, applying Lemma 4.2(i) leads to $\sum_{k=-\infty}^{0} g_k = \det(I - G^*(1)) = \det(I - G) = 0$. It is clear that $\sum_{k=-\infty}^{0} k g_k = (\mathrm{d}/\mathrm{d}z)\{\det(I - G^*(z))\}|_{z=1}$. To compute $\sum_{k=-\infty}^{0} k g_k$, taking the derivative (elementwise) of the equation $\det(I - G^*(z)) \cdot I = \operatorname{adj}(I - G^*(z)) \cdot [I - G^*(z)]$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}z}\{\det(I - G^*(z))\}|_{z=1} \cdot I = \operatorname{adj}(I - G) \cdot \sum_{k=1}^{\infty} k G_k + \frac{\mathrm{d}}{\mathrm{d}z}\{\operatorname{adj}(I - G^*(z))\}|_{z=1} \cdot (I - G). \tag{4.7}$$

After multiplying (4.7) by ω and e and using the facts that $\omega e = 1$ and $(I - G)e = 0$, it follows from Lemma 4.4 that

$$\begin{aligned}\sum_{k=-\infty}^{0} k g_k &= \frac{\mathrm{d}}{\mathrm{d}z}\{\det(I - G^*(z))\}|_{z=1} = \omega \cdot \operatorname{adj}(I - G) \sum_{k=1}^{\infty} k G_k \cdot e \\ &= \kappa_G \frac{\omega(I - R)(I - \Phi_0)}{\omega(I - R)(I - \Phi_0)e} \cdot \sum_{k=1}^{\infty} k G_k e. \end{aligned} \tag{4.8}$$

Note that, since $\sum_{k=-\infty}^{\infty} |k| A_k$ is finite, Lemma 4.3 illustrates that $\sum_{k=1}^{\infty} kG_k$ is also finite. Thus, $\sum_{k=-\infty}^{0} kg_k$ is finite and nonzero according to Lemma 4.4, and $\sum_{k=1}^{\infty} kG_k e \geq Ge = e$. Therefore, $0 < |\sum_{k=-\infty}^{0} kg_k| < \infty$. Since $\{A_k\}$ is long tailed, Lemma 4.7 implies that

$$\varliminf_{k\to\infty} \frac{\overline{R_{\leq k}}}{\overline{p_{\leq k}}} \geq W(I-\Phi_0)^{-1}.$$

Noting the facts that $W \gneqq 0$ and $(I-\Phi_0)^{-1} \gneqq 0$, it is clear that $W(I-\Phi_0)^{-1} \gneqq 0$, since the matrix $I-\Phi_0$ is invertible. Hence, there exists at least one pair (i_0, j_0) such that the (i_0, j_0)th element of the matrix $W(I-\Phi_0)^{-1}$ is positive. Therefore, Lemma 4.7 implies that $\overline{r_{\leq k}}(i_0, j_0) > 0$ for all $k \geq N$, where N is a sufficiently large positive integer. Similarly, for each positive element of the matrix $W(I-\Phi_0)^{-1}$, written as the (i^*, j^*)th element, we have $\overline{r_{\leq k}}(i^*, j^*) > 0$ for all $k \geq N$. Define $U_+(B) = \sum_{k\in B} R_k$ and $U(B) = \sum_{k\in B} Q_k (I-\Phi_0)^{-1}$. It follows from (4.4), (4.6), and Lemma 4.6 that

$$\lim_{k\to\infty} \frac{\overline{R_{\leq k}}}{\overline{\overline{p_{\leq k}}}} = \frac{W \cdot \mathrm{adj}(I-G)(I-\Phi_0)^{-1}}{\sum_{k=-\infty}^{0} kg_k}. \tag{4.9}$$

Substituting (4.2) and (4.8) into (4.9) leads to the expression in (4.3). This completes the proof.

Remark 4.2. It is easy to see from Theorem 4.2 that the equilibrium excess probability sequence $\{p_k^{(I)}\}$ of the sequence $\{p_k\}$ is a uniformly dominant sequence of the matrix sequence $\{R_k\}$. Compare this with the fact that the sequence $\{q_k\}$ is a uniformly dominant sequence of the matrix sequence $\{R_{0,k}\}$. Here, $\{p_k\}$ and $\{q_k\}$ are uniformly dominant sequences of the matrix sequences $\{A_k\}$ and $\{D_k\}$, respectively. This means that $\{R_k\}$ is tail heavier than $\{R_{0,k}\}$ if $\lim_{k\to\infty} \overline{p_{\leq k}}/\overline{q_{\leq k}} = c$, where c is a positive constant, since

$$\lim_{k\to\infty} \frac{\overline{q_{\leq k}}}{\overline{\overline{p_{\leq k}^{(I)}}}} = 0$$

according to Lemma 3.1 of [46].

Since $\overline{\pi_{\leq k}} = x_0 \overline{R_{0,\leq k} * \sum_{n=0}^{\infty} R_{\leq k}^{n\circledast}}$ for $k \geq 1$, according to Lemma 3.1, the tail of the vector sequence $\{\pi_k\}$ can be expressed as a tail of convolution of the two matrix sequences $\{R_{0,k}\}$ and $\{\sum_{n=0}^{\infty} R_k^{n\circledast}\}$. It is well known that the convolution of two long-tailed matrix sequences may not be long tailed; therefore, it is possible that the vector sequence $\{\pi_k\}$ is not long tailed, even though the two matrix sequences $\{A_k\}$ and$\{D_k\}$ are. In the remainder of the paper, the two matrix sequences $\{A_k\}$ and$\{D_k\}$ will be restricted to either the subexponential class or the class of light-tailed sequences.

5. The subexponential asymptotics of $\{\pi_k\}$

In this section, under the condition $\min\{\phi_{A+}, \phi_D\} = 1$, we consider the following three cases:

(a) $\{A_k\} \in \mathcal{S}^*$ and $\{D_k\}$ is light tailed,

(b) $\{A_k\}$ is light tailed and $\{D_k\} \in \mathcal{S}$, and

(c) $\{A_k\} \in \mathcal{S}^*$ and $\{D_k\} \in \mathcal{S}$.

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In these three cases, we can characterize the subexponential asymptotics of the stationary probability vector $\{\pi_k\}$. We can also explicitly express a uniformly dominant subsequence of $\{\pi_k\}$, and the associated ratio vector.

According to Lemma 3.1, it is crucial to characterize the subexponential asymptotics of the matrix sequence $\{\sum_{n=0}^{\infty} R_k^{n\circledast}\}$. To do this, we need Lemma 4.3 of [5], which we now restate.

Lemma 5.1. (Asmussen *et al*. [5].) *Let $H(x)$ be a matrix of nonnegative functions such that $H = H(\infty) - H(0)$ is strictly substochastic (and, therefore, the spectral radius of H is strictly less than 1). If there exist a probability distribution $F(x) \in \mathcal{S}$ and a finite matrix L such that $\lim_{x\to\infty} \overline{H}(x)/\overline{F}(x) = L$, then*

$$\lim_{k\to\infty} \frac{\sum_{n=0}^{\infty} \overline{H^{n\circledast}}(x)}{\overline{F}(x)} = (I-H)^{-1}L(I-H)^{-1}.$$

The following lemma characterizes the subexponential asymptotics of $\{\sum_{n=0}^{\infty} R_k^{n\circledast}\}$.

Lemma 5.2. *Suppose that the Markov chain of GI/G/1 type is positive recurrent, $\phi_{A-} < 1$, and $\sum_{k=-\infty}^{\infty} |k| A_k$ is finite. If $\{A_k\} \in \mathcal{S}^*$ with a uniformly dominant probability sequence $\{p_k\}$ and associated ratio matrix W, then $\{\sum_{n=0}^{\infty} R_k^{n\circledast}\} \in \mathcal{S}$ and*

$$\lim_{k\to\infty} \frac{\overline{\sum_{n=0}^{\infty} R_{\le k}^{n\circledast}}}{\overline{\overline{p_{\le k}}}} = \frac{(I-R)^{-1}We\omega}{\omega(I-R)(I-\Phi_0)\sum_{j=1}^{\infty} jG_j e}.$$

Proof. Since $\{A_k\} \in \mathcal{S}^* \subset \mathcal{L}$ and $p_k^{(I)} = (1/\mu_p)\overline{p_{\le k}}$ for $k \ge 1$, where $\mu_p = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} \overline{p_{\le k}} < \infty$ according to the assumption that $\sum_{k=-\infty}^{\infty} |k| A_k$ is finite, it follows from Theorem 4.2 that

$$\lim_{k\to\infty} \frac{\overline{R_{\le k}}}{\overline{\overline{p_{\le k}^{(I)}}}} = \lim_{k\to\infty} \frac{\overline{R_{\le k}}}{(1/\mu_p)\overline{\overline{p_{\le k}}}} = \mu_p L, \tag{5.1}$$

where

$$L = \frac{We\omega(I-R)}{\omega(I-R)(I-\Phi_0)\sum_{j=1}^{\infty} jG_j e}.$$

Let $\widetilde{G}_k = \Delta^{-1} R_k^{\top} \Delta$ for $k \ge 1$, where $\Delta = \operatorname{diag}(\omega_1, \omega_2, \dots, \omega_m)$. Then Lemma 4.2(i) gives $\omega R \lneq \omega$ and, so, $\widetilde{G} = \sum_{k=1}^{\infty} \widetilde{G}_k$ is strictly substochastic, since

$$\widetilde{G}e = \sum_{k=1}^{\infty} \widetilde{G}_k e = \sum_{k=1}^{\infty} \Delta^{-1} R_k^{\top} \Delta e = \Delta^{-1}(\omega R)^{\top} \lneq \Delta^{-1}\omega^{\top} = e.$$

It follows from (5.1) that $\lim_{k\to\infty} \overline{\widetilde{G}_{\le k}}/\overline{p_{\le k}^{(I)}} = \mu_p \Delta^{-1} L^{\top} \Delta$, and it follows from Proposition 2.1(i) that $\{p_k^{(I)}\}$ is a probability sequence in $\mathcal{S}$. Therefore, Lemma 5.1 implies that

$$\lim_{k\to\infty} \frac{\sum_{n=0}^{\infty} \overline{\widetilde{G}_{\le k}^{n\circledast}}}{\overline{p_{\le k}^{(I)}}} = (I-\widetilde{G})^{-1} \cdot \mu_p \Delta^{-1} L^{\top} \Delta \cdot (I-\widetilde{G})^{-1}. \tag{5.2}$$

Since the Markov chain of GI/G/1 type is positive recurrent, $I-R$ is invertible. It is easy to see that $(I-R)^{-1}L(I-R)^{-1} \ne 0$, since $L \ne 0$. Therefore, it follows from (5.2) that

$$\lim_{k\to\infty} \frac{\sum_{n=0}^{\infty} \overline{R_{\le k}^{n\circledast}}}{\overline{p_{\le k}^{(I)}}} = \mu_p (I-R)^{-1} L (I-R)^{-1} = \frac{\mu_p (I-R)^{-1} We\omega}{\omega(I-R)(I-\Phi_0)\sum_{j=1}^{\infty} jG_j e}. \tag{5.3}$$

Noting that

$$\overline{\sum_{n=0}^{\infty} R_{\leq k}^{n\circledast}} = \sum_{n=0}^{\infty} \overline{R_{\leq k}^{n\circledast}},$$

it follows from (5.3) that

$$\lim_{k\to\infty} \frac{\overline{\sum_{n=0}^{\infty} R_{\leq k}^{n\circledast}}}{\overline{\overline{p_{\leq k}}}} = \lim_{k\to\infty} \frac{(1/\mu_p)\overline{\sum_{n=0}^{\infty} R_{\leq k}^{n\circledast}}}{\overline{p_{\leq k}^{(I)}}} = \frac{(I-R)^{-1}We\omega}{\omega(I-R)(I-\Phi_0)\sum_{j=1}^{\infty} jG_j e}.$$

Therefore, $\{\sum_{n=0}^{\infty} R_k^{n\circledast}\} \in \mathcal{S}$ according to Proposition 2.3. This completes the proof.

Remark 5.1. As seen from this proof, Lemma 5.2 still holds under the weaker conditions that $\{A_k\} \in \mathcal{L}$ and $\{\overline{p_{\leq k}}\} \in \mathcal{S}$. The proof of this lemma can be made shorter if results of [23] are used.

The following corollary is a direct result of Theorem 4.1, using Proposition 2.3.

Corollary 5.1. *Suppose that the Markov chain of GI/G/1 type is positive recurrent. If* $\{D_k\} \in \mathcal{S}$ *with a uniformly dominant sequence* $\{q_k\}$ *and associated ratio matrix* V*, then* $\{R_{0,k}\} \in \mathcal{S}$ *and*

$$\lim_{k\to\infty} \frac{\overline{R_{0,\leq k}}}{\overline{q_{\leq k}}} = V\left(I - \sum_{i=0}^{\infty} \Phi_{-i}\right)^{-1}.$$

The following lemma characterizes the light-tailed asymptotics of the matrix sequence $\{\sum_{n=0}^{\infty} R_k^{n\circledast}\}$ when the matrix sequence $\{A_k\}$ is light tailed. Its proof can be obtained by computations similar to those in Subsection 4.1 of [33].

Lemma 5.3. *Suppose that the Markov chain of GI/G/1 type is positive recurrent and* $\phi_{A+} > 1$*. If there exists a minimal positive solution* $\eta \in (1, \phi_{A+})$ *to the equation*

$$\det(I - A^*(z)) = 0,$$

then

$$\lim_{k\to\infty} \frac{\overline{\sum_{n=0}^{\infty} R_{\leq k}^{n\circledast}}}{\eta^{-(k+1)}} = \frac{(I-\Phi_0)[I-G^*(\eta)]v(\eta)u(\eta)}{(\eta-1)u(\eta)\sum_{k=1}^{\infty} k\eta^{k-1}R_k v(\eta)},$$

where $u(\eta)$ *and* $v(\eta)$ *are the left and right Perron–Frobenius vectors of* $A^*(\eta)$*, satisfying* $u(\eta)e = u(\eta)v(\eta) = 1$*.*

Remark 5.2. Suppose that the Markov chain of GI/G/1 type is positive recurrent. If $\phi_{A+} > 1$ then the matrix sequence $\{\sum_{n=0}^{\infty} R_k^{n\circledast}\}$ is light tailed. For simplicity, in the remainder of the paper we only consider the light-tailed case in which there exist a uniformly dominant subsequence $\{b_k\}$, and associated ratio matrix B, such that

$$\lim_{k\to\infty} \frac{\overline{\sum_{n=0}^{\infty} R_{\leq k}^{n\circledast}}}{\overline{b_{\leq k}}} = B.$$

If $\eta \geq \phi_{A+}$ in Lemma 5.3, then the expressions for both b_k and B might be more complicated – such examples were provided in [33].

Similarly, if $\phi_D > 1$ then $\{R_{0,k}\}$ is light tailed. In what follows, we only consider the case in which there exist a uniformly dominant subsequence $\{d_k\}$, and associated ratio matrix D, such that

$$\lim_{k\to\infty} \frac{\overline{R_{0,\leq k}}}{\overline{d_{\leq k}}} = D.$$

In general, it is not easy to give explicit expressions for both d_k and D – such examples were also discussed in [33].

The following theorem characterizes the subexponential asymptotics of $\{\pi_k\}$ for the cases in which (a) $\{A_k\} \in \mathcal{S}^*$ and $\{D_k\}$ is light tailed, and (b) $\{A_k\}$ is light tailed and $\{D_k\} \in \mathcal{S}$.

Theorem 5.1. *Suppose that the Markov chain of GI/G/1 type is positive recurrent, and that* $\sum_{k=-\infty}^{\infty} |k| A_k$ *is finite.*

(a) *If* $\phi_D > 1$, *the light-tailed sequence* $\{R_{0,k}\}$ *has a uniformly dominant subsequence* $\{d_k\}$ *with associated ratio matrix* D, *and* $\{A_k\} \in \mathcal{S}^*$ *has a uniformly dominant subsequence* $\{p_k\}$ *with associated ratio matrix* W, *then* $\{\pi_k\} \in \mathcal{S}$ *and*

$$\lim_{k\to\infty} \frac{\overline{\pi_{\leq k}}}{\overline{\overline{p_{\leq k}}}} = \frac{x_0 D (I-R)^{-1} W e \omega}{\omega (I-R)(I-\Phi_0) \sum_{j=1}^{\infty} j G_j e}.$$

(b) *If* $\phi_{A+} > 1$, *the light-tailed sequence* $\{\sum_{n=0}^{\infty} R_k^{n\circledast}\}$ *has a uniformly dominant subsequence* $\{b_k\}$ *with associated ratio matrix* B, *and* $\{D_k\} \in \mathcal{S}$ *has a uniformly dominant subsequence* $\{q_k\}$ *with associated ratio matrix* V, *then* $\{\pi_k\} \in \mathcal{S}$ *and*

$$\lim_{k\to\infty} \frac{\overline{\pi_{\leq k}}}{\overline{q_{\leq k}}} = x_0 V \left(I - \sum_{i=0}^{\infty} \Phi_{-i} \right)^{-1} B.$$

Proof. We need only prove part (a), as part (b) can be proved similarly.

If $\{A_k\} \in \mathcal{S}^*$ has a uniformly dominant sequence $\{p_k\}$ with associated ratio matrix W, then

$$\lim_{k\to\infty} \frac{\overline{\sum_{n=0}^{\infty} R_{\leq k}^{n\circledast}}}{\overline{\overline{p_{\leq k}}}} = \frac{(I-R)^{-1} W e \omega}{\omega (I-R)(I-\Phi_0) \sum_{j=1}^{\infty} j G_j e},$$

according to Lemma 5.2. If $\phi_D > 1$ then $\{R_{0,k}\}$ is light tailed. When $\{R_{0,k}\}$ has a uniformly dominant sequence $\{d_k\}$ and associated ratio matrix D, it is clear that $\overline{d_{\leq k}} = o(\overline{p_{\leq k}^{(I)}})$, since $p_k^{(I)} \in \mathcal{S}$ due to the fact that $\{A_k\} \in \mathcal{S}^*$. Therefore, it follows from Lemma 3.1 and Proposition 2.6 that

$$\lim_{k\to\infty} \frac{\overline{\pi_{\leq k}}}{\overline{\overline{p_{\leq k}}}} = \frac{x_0 D (I-R)^{-1} W e \omega}{\omega (I-R)(I-\Phi_0) \sum_{j=1}^{\infty} j G_j e}.$$

Hence, $\{\pi_k\} \in \mathcal{S}$. This completes the proof.

Remark 5.3. Although, for some stochastic models, it may not be easy to provide an explicit expression for a uniformly dominant subsequence of the light-tailed matrix sequence $\{\sum_{n=0}^{\infty} R_k^{n\circledast}\}$ or $\{R_{0,k}\}$, and the associated ratio matrix, Theorem 5.1 provides a useful relation between the subexponential asymptotics of $\{\pi_k\}$ and the repeating and boundary matrix sequences in the transition matrix of GI/G/1 type.

The following theorem characterizes the subexponential asymptotics of $\{\pi_k\}$ in case (c); that is, when $\{A_k\} \in \mathcal{S}^*$ and $\{D_k\} \in \mathcal{S}$.

Theorem 5.2. *Suppose that the Markov chain of GI/G/*1 *type is positive recurrent, and that* $\sum_{k=-\infty}^{\infty} |k| A_k$ *is finite. Assume that* $\{A_k\} \in \mathcal{S}^*$ *has a uniformly dominant subsequence* $\{p_k\}$ *with associated ratio matrix* W*, that* $\{D_k\} \in \mathcal{S}$ *has a uniformly dominant subsequence* $\{q_k\}$ *with associated ratio matrix* V*, and that the limit* $\lim_{k\to\infty} \overline{\overline{p_{\le k}}}/\overline{q_{\le k}}$ *is either* 0*, a positive number, or* ∞*. Then* $\{\pi_k\} \in \mathcal{S}$*. Furthermore,*

(i) *if* $\overline{q_{\le k}} = o(\overline{\overline{p_{\le k}}})$ *then*

$$\lim_{k\to\infty} \frac{\overline{\pi_{\le k}}}{\overline{\overline{p_{\le k}}}} = x_0 V \left(I - \sum_{i=0}^{\infty} \Phi_{-i} \right)^{-1} \frac{(I-R)^{-1} W e \omega}{\omega (I-R)(I-\Phi_0) \sum_{j=1}^{\infty} j G_j e},$$

(ii) *if* $\overline{\overline{p_{\le k}}} = o(\overline{q_{\le k}})$ *then*

$$\lim_{k\to\infty} \frac{\overline{\pi_{\le k}}}{\overline{q_{\le k}}} = x_0 V \left(I - \sum_{i=0}^{\infty} \Phi_{-i} \right)^{-1} \frac{(I-R)^{-1} W e \omega}{\omega (I-R)(I-\Phi_0) \sum_{j=1}^{\infty} j G_j e}, \quad \textit{and}$$

(iii) *if* $\overline{\overline{p_{\le k}}} \backsim \xi \overline{q_{\le k}}$ *for a constant* $\xi > 0$ *then*

$$\lim_{k\to\infty} \frac{\overline{\pi_{\le k}}}{\overline{q_{\le k}}} = x_0 \left[V \left(I - \sum_{i=0}^{\infty} \Phi_{-i} \right)^{-1} e_1 e^{\top} + \frac{\xi e_1 e^{\top} (I-R)^{-1} W e \omega}{\omega (I-R)(I-\Phi_0) \sum_{j=1}^{\infty} j G_j e} \right],$$

where e_1 *is the column vector of dimension* m_0 *with all entries equal to* 1*.*

Proof. According to Lemmas 3.1 and 5.2 and Corollary 5.1, parts (i) and (ii) can easily be proved using Proposition 2.6, while part (iii) follows directly from Proposition 2.7.

If $\{p_k\} \in \mathcal{S}^*$, $\{q_k\} \in \mathcal{S}$, and there does not exist a limit $\lim_{k\to\infty} \overline{\overline{p_{\le k}}}/\overline{q_{\le k}}$, then it is possible that $\{\pi_k\} \notin \mathcal{S}$ – such an example was provided in Theorem 6.2 of [25]. The following corollary follows from Theorem 5.1(c) of [25].

Theorem 5.3. *Suppose that the Markov chain of GI/G/*1 *type is positive recurrent, and that* $\sum_{k=-\infty}^{\infty} |k| A_k$ *is finite. If* $\{A_k\} \in \mathcal{S}^*$ *has a uniformly dominant subsequence* $\{p_k\}$ *and associated ratio matrix* W*,* $\{D_k\} \in \mathcal{S}$ *has a uniformly dominant subsequence* $\{q_k\}$ *and associated ratio matrix* V*, and* $\{\lambda q_k + (1-\lambda) p_k^{(I)}\} \in \mathcal{S}$ *for all* $\lambda \in (0,1)$*, then* $\{\pi_k\} \in \mathcal{S}$*.*

Proof. It follows from Lemma 3.1 that $\overline{\pi_{\le k}} = x_0 \overline{R_{0,\le k} * \sum_{n=0}^{\infty} R_{\le k}^{n\circledast}}$ and, hence, that $\overline{\pi_{\le k}}(j) = \sum_{i=1}^{m_0} \sum_{l=1}^{m} x_0(i) \overline{r_{0,\le k}(i,l) * \psi_{\le k}(l,j)}$, where $\psi_{\le k}(l,j)$ is the (l,j)th entry of the matrix $\sum_{n=0}^{\infty} R_{\le k}^{n\circledast}$. It follows from Corollary 5.1 and Lemma 5.2 that

$$\overline{r_{0,\le k}(i,l)} \backsim \lambda(i,l) \overline{q_{\le k}} \quad \text{and} \quad \overline{\psi_{\le k}(l,j)} \backsim \mu(l,j) \overline{\overline{p_{\le k}}},$$

where $\lambda(i,l)$ and $\mu(l,j)$ are the (i,l)th and (l,j)th entries of the matrices

$$V \left(I - \sum_{i=0}^{\infty} \Phi_{-i} \right)^{-1} \quad \text{and} \quad \frac{\mu_p W e \omega (I-R)}{\omega (I-R)(I-\Phi_0) \sum_{j=1}^{\infty} j G_j e},$$

respectively. If $\lambda(i, l) = 0$ or $\mu(l, j) = 0$, we take $\overline{r_{0,\le k}(i, l) * \psi_{\le k}(l, j)} \backsim 0$, while, if $\lambda(i, l) \neq 0$ and $\mu(l, j) \neq 0$, we obtain

$$\overline{r_{0,\le k}(i, l) * \psi_{\le k}(l, j)} = \lambda(i, l)\mu(l, j)\overline{\frac{r_{0,\le k}(i, l)}{\lambda(i, l)} * \frac{\psi_{\le k}(l, j)}{\mu(l, j)}}.$$

Since

$$\overline{\frac{r_{0,\le k}(i, l)}{\lambda(i, l)}} \backsim \overline{q_{\le k}} \quad \text{and} \quad \overline{\frac{\psi_{\le k}(l, j)}{\mu(l, j)}} \backsim \overline{\overline{p_{\le k}}},$$

we obtain $\overline{r_{0,\le k}(i, l) * \psi_{\le k}(l, j)} \backsim \lambda(i, l)\mu(l, j)\overline{q_{\le k} * p^{(I)}_{\le k}}$. Therefore,

$$\overline{\pi_{\le k}} \backsim x_0 V \bigg(I - \sum_{i=0}^{\infty} \Phi_{-i}\bigg)^{-1} \frac{\mu_p W e \omega(I - R)}{\omega(I - R)(I - \Phi_0) \sum_{j=1}^{\infty} j G_j e} \cdot \overline{q_{\le k} * p^{(I)}_{\le k}}.$$

Note that, since $\{q_k\} \in \mathcal{S}$, $p^{(I)}_k \in \mathcal{S}$ (due to the fact that $\{p_k\} \in \mathcal{S}^*$), and $\lambda q_k + (1 - \lambda) p^{(I)}_k \in \mathcal{S}$ for all $\lambda \in (0, 1)$, it is clear that $\{q_k \circledast p^{(I)}_k\} \in \mathcal{S}$, which is equivalent to saying that the function $q_{\le k} * p^{(I)}_{\le k}$, $k \ge 0$, is in $\mathcal{S}$. Therefore, $\{\pi_k\} \in \mathcal{S}$. This completes the proof.

We now provide a result on regular variation and conclude the paper with some remarks and discussion.

Definition 5.1. (a) A sequence $\{l_n\}$ of nonnegative scalars, with $\sum_{n=0}^{\infty} l_n < \infty$, is called slowly varying if $\overline{l_{\le n}} > 0$ for $n > N$, where N is a sufficiently large positive integer, and $\lim_{n\to\infty} \overline{l_{\le \lfloor \lambda n \rfloor}}/\overline{l_{\le n}} = 1$ for any $\lambda > 0$. Denote by $\mathfrak{R}_0$ the class of slowly varying sequences.

(b) A sequence $\{c_n\}$ of nonnegative scalars with $\sum_{n=0}^{\infty} c_n < \infty$ is called regularly varying, with index $\alpha \in (-\infty, \infty)$, if $\overline{c_{\le n}} = n^{\alpha} \overline{l_{\le n}}$ for all $n \ge N$. Denote by $\mathfrak{R}_\alpha$ the class of regularly varying sequences with index α.

Definition 5.2. A sequence $\{B_n\}$ of nonnegative matrices is called regularly varying, with index $\alpha \in (-\infty, \infty)$, if there exists at least one entry sequence of $\{B_n\}$ that is regularly varying with index α, and all the other entry sequences are either regularly varying with index $\beta \in (-\infty, \alpha]$ or are tail lighter than some entry sequence of $\{B_n\}$ in $\mathfrak{R}_\alpha$. Denote by $\mathfrak{R}_\alpha$ the class of regularly varying matrix sequences, with index α, of all sizes.

Corollary 5.2. *Suppose that the Markov chain of GI/G/1 type is positive recurrent, and that* $\sum_{k=1}^{\infty} k D_k$ *and* $\sum_{k=-\infty}^{\infty} |k| A_k$ *are both finite.*

(i) *If* $\phi_{A+} > 1$ *and* $\{D_k\} \in \mathfrak{R}_{-\beta}$ *for* $\beta \ge 2$, *then* $\{\pi_k\} \in \mathfrak{R}_{-\beta}$.

(ii) *If* $\phi_D > 1$ *and* $\{A_k\} \in \mathfrak{R}_{-\alpha}$ *for* $\alpha \ge 2$, *then* $\{\pi_k\} \in \mathfrak{R}_{-(\alpha-1)}$.

(iii) *If* $\{A_k\} \in \mathfrak{R}_{-\alpha}$ *for* $\alpha \ge 2$ *and* $\{D_k\} \in \mathfrak{R}_{-\beta}$ *for* $\beta \ge 2$, *then* $\{\pi_k\} \in \mathfrak{R}_{-\gamma}$, *where* $\gamma = \min\{\alpha - 1, \beta\}$.

Remark 5.4. The model studied in this paper is a generalization of the model of [4]. We allow for a more general boundary sequence, which makes the study much more challenging. The interpretation or determination of results of [4] can be easily obtained from results in this paper, as follows. Consider the Markov chain with $\{A_k\} \in \mathcal{S}^*$ and $p_k = q_k$ for all $k \ge 1$. Since

$\lim_{k\to\infty} \overline{q_{\le k}}/\overline{\overline{p_{\le k}}} = (1/\mu_p)\overline{q_{\le k}}/\overline{p^{(I)}_{\le k}} = 0$ according to Lemma 3.1 of [46], it follows from Theorem 5.2(i) that

$$\lim_{k\to\infty} \frac{\overline{\pi_{\le k}}}{\overline{\overline{p_{\le k}}}} = x_0 V\left(I - \sum_{i=0}^{\infty} \Phi_{-i}\right)^{-1} \frac{(I-R)^{-1} W e \omega}{\omega(I-R)(I-\Phi_0)\sum_{j=1}^{\infty} j G_j e}.$$

Comparing this expression to [4, Equation (1.3)] leads to the determination of c_2 (in [4]):

$$c_2 = x_0 V\left(I - \sum_{i=0}^{\infty} \Phi_{-i}\right)^{-1} \frac{(I-R)^{-1} W e}{\omega(I-R)(I-\Phi_0)\sum_{j=1}^{\infty} j G_j e}.$$

Furthermore, for a Markov chain of M/G/1 type, it is clear that $G_j = 0$ and $\Phi_{-j} = 0$ for $j \ge 2$. In this case, we have

$$c_2 = \frac{x_0 V(I-\Phi_0)(I-G_1)(I-R)^{-1} W e}{\omega(I-R)(I-\Phi_0) G_1 e}.$$

Remark 5.5. Borovkov and Korshunov [9] considered a partially homogeneous Markov chain defined by the following recursion:

$$X(n+1) = \begin{cases} [X(n) + \xi_n]^+ & \text{for } X(n) > 0, \\ \eta_n & \text{for } X(n) = 0. \end{cases}$$

(This recursion is more general than Lindley's; see Subsection 2.2 of [9] for details.) Obviously, the partially homogeneous Markov chain provides the scalar case of the Markov chain of GI/G/1 type, where $\{\xi_n\}$ and $\{\eta_n\}$ correspond to the repeating matrix sequence $\{A_k\}$ and the boundary matrix sequence $\{D_k\}$, respectively. The tail of the stationary probability given in Theorem 2 of [9] plays the same role as that of our Lemma 3.1. We make the following comparisons.

(i) If $\int_x^\infty \mathrm{P}\{\eta > u\}\,\mathrm{d}u$ is a u.p. function (see [9]) and $\int_x^\infty \mathrm{P}\{\eta > u\}\,\mathrm{d}u = o(\int_x^\infty \mathrm{P}\{\xi > u\}\,\mathrm{d}u)$ as $x \to \infty$, then Corollary 1(c) of [9] is the same as our Theorems 5.1(a) and 5.2(i). However, the conditions in Corollary 1 of [9] are stronger than those of our Theorem 5.2(i), because $\int_x^\infty \mathrm{P}\{\eta > u\}\,\mathrm{d}u$ is always tail heavier than $\mathrm{P}\{\eta > x\}$ (or $\overline{\overline{q_{\le k}}}$ is always tail heavier than $\overline{q_{\le k}}$).

(ii) Our Theorems 5.1(b) and 5.2(ii) are better than Corollary 2(c) of [9] in the sense that $\int_x^\infty \mathrm{P}\{\eta > u\}\,\mathrm{d}u$ is part of the conditions and part of the tailed expression in [9], while Theorems 5.1(b) and 5.2(ii) here only require the tail $\overline{q_{\le k}}$.

Remark 5.6. Foss and Zachary [23] studied the subexponential asymptotics of the maximum of a more general random walk, whose increments ζ^{X_n}, $n \ge 0$, are subexponential. Here, ζ^{X_n} is modulated by an independent sequence $\{X_n\}$, which is a χ-valued discrete-time random process with finite states, including Markov chains as a special case. The results of [23] can be used to study the subexponential asymptotics of the R-measure for the repeating matrix sequences (see, for example, Lemma 5.2 above), but cannot be used to deal with the general boundary condition imposed in this paper.

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