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Existence of a bounded approximate identity in a tensor product

David A. Robbins

It has been shown that the existence of a (left) approximate identity in the tensor product $A \otimes_\alpha B$ of Banach algebras A and B, where α is an admissible algebra norm on $A \otimes B$, implies the existence of approximate identities in A and B. The question has been raised as to whether the boundedness of the approximate identity in $A \otimes_\alpha B$ implies the boundedness of the approximate identities in A and B. This paper answers the question affirmatively with α being the greatest cross-norm.

Loy has shown [1] that, for arbitrary Banach algebras A and B, if $A \otimes_\alpha B$ has a left (right) approximate identity, then so do A and B, where α is any admissible algebra norm on $A \otimes B$. He raises the question of whether existence of a bounded approximate identity in $A \otimes_\alpha B$ would ensure the boundedness of the approximate identities in A and B.

We are able to answer this question affirmatively when α is the greatest cross-norm (which we denote by γ), as follows:

THEOREM. *Let A and B be Banach algebras, and let $A \otimes_\gamma B$ have a bounded (left) approximate identity. Then both A and B have bounded*

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(left) approximate identities.

Proof. Let $\left\{\sum_{i=1}^{\infty} a_i^{(j)} \otimes b_i^{(j)}\right\}_{j\epsilon J}$ be a bounded left approximate identity in $A \otimes_\gamma B$, with $\left\|\sum_{i=1}^{\infty} a_i^{(j)} \otimes b_i^{(j)}\right\| \le C$ for all $j \in J$.

Fix $b \in B$ such that $\|b\| = 1$. Choose $G \in B^*$ such that $\|G\| = G(b) = 1$. By properties of the norm γ [2, p. 36], we have that $\sum_{i=1}^{\infty} G\left(b_i^{(j)}b\right)a_i^{(j)} \in A$ for all $j \in J$, and, moreover, that the net $\left\{\sum_{i=1}^{\infty} G\left(b_i^{(j)}b\right)a_i^{(j)}\right\}_{j\epsilon J}$ is bounded in A, for

$$\left\|\sum_{i=1}^{\infty} G\left(b_i^{(j)}b\right)a_i^{(j)}\right\| \le \|G\|\left\|\sum_{i=1}^{\infty} a_i^{(j)} \otimes b_i^{(j)}b\right\|$$

$$\le \|b\|\left\|\sum_{i=1} a_i^{(j)} \otimes b_i^{(j)}\right\| \le C .$$

We claim that $\left\{\sum_{i=1}^{\infty} G\left(b_i^{(j)}b\right)a_i^{(j)}\right\}_{j\epsilon J}$ is a left approximate identity in A: let $a \in A$. Then, again depending on the properties of γ, we have

$$\left\|\sum_{i=1}^{\infty} G\left(b_i^{(j)}b\right)a_i^{(j)}a-a\right\| = \left\|\sum_{i=1}^{\infty} G\left(b_i^{(j)}b\right)a_i^{(j)}a-G(b)a\right\|$$

$$\le \left\|\sum_{i=1}^{\infty} a_i^{(j)}a \otimes b_i^{(j)}b-a \otimes b\right\|$$

$$= \left\|\left(\sum_{i=1}^{\infty} a_i^{(j)} \otimes b_i^{(j)}\right)(a \otimes b)-a \otimes b\right\| \xrightarrow{j} 0 ,$$

so that our net is a left approximate identity in A.

The situation is obviously symmetric with respect to B.

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Duke University,
Durham,
North Carolina,
USA.