Can. J. Math., Vol. XXII, No. 5, 1970, pp. 953–966

ON A GENERALIZATION OF ALTERNATIVE RINGS

RAYMOND V. MORGAN, JR.

1. Introduction. Bruck and Kleinfeld [**3**] proved that any alternative ring with characteristic prime to 2 must satisfy the identity

$$(x^2, y, z) = 2x \cdot (x, y, z),$$

where the associator (x, y, z) is defined by $(x, y, z) = (xy)z - x(yz)$ and $x \cdot y = \frac{1}{2}(xy + yx)$. Linearization of the identity $(x^2, y, z) = 2x \cdot (x, y, z)$ yields for characteristic prime to 2 an equivalent identity

$$(x \cdot w, y, z) = x \cdot (w, y, z) + w \cdot (x, y, z). \tag{1}$$

Using the right alternative law $(x, y, z) = -(y, x, z)$ and the flexible law $(x, y, z) = -(z, y, x)$ which is satisfied in any alternative ring we obtain

$$(y, x \cdot w, z) = x \cdot (y, w, z) + w \cdot (y, x, z) \tag{2}$$

and

$$(y, z, x \cdot w) = x \cdot (y, z, w) + w \cdot (y, z, x). \tag{3}$$

In this paper we study the class of rings which satisfy any two of (1), (2), and (3) together with

$$(x, x, x) = 0. \tag{4}$$

These rings are clearly generalizations of alternative rings.

Kosier [**5**] studied rings satisfying (1), (3), and (4). He showed that such rings were power-associative and have the idempotent decomposition $A = A_1 + A_{\frac{1}{2}} + A_0$, where $x \in A_i$ if and only if $ex + xe = 2ix$ and $e^2 = e \neq 0$. He further proved that if A has no nil ideals, then A must have the Peirce decomposition $A = A_{11} + A_{10} + A_{01} + A_{00}$, where $x \in A_{ij}$ if and only if $ex = ix$ and $xe = jx$ and that the subspaces A_{ij} multiply as in the alternative case. His main results were:

(i) if A is simple and e is idempotent with $e \neq 1$, then A is associative or a Cayley-Dickson algebra over its centre and

(ii) if A is a finite-dimensional semisimple algebra, then A has an identity and is the direct sum of simple algebras.

Any ring satisfying (1), (2), and (4) is anti-isomorphic to a ring satisfying (2), (3), and (4); thus it suffices to consider rings satisfying (1), (2), and (4). Throughout this paper, all rings have characteristic prime to 2.

Received September 22, 1969. This paper contains the results of the author's dissertation written at the University of Missouri under the direction of Professor D. J. Rodabaugh.

https://doi.org/10.4153/CJM-1970-109-x Published online by Cambridge University Press

In § 2 we show that any ring satisfying (1), (2), and (4) must be power-associative. In § 3 we prove that if A has no nil ideals, then A has the Peirce decomposition with the same multiplication on the subspaces as an alternative ring. Using this result we show that if A is simple and e is an idempotent in A with $e \neq 1$, then A is either associative or a Cayley-Dickson algebra over its centre. In § 4 we prove that if A is a finite-dimensional semi-simple algebra, then A has an identity and is the direct sum of simple algebras. Finally, in § 5 we show that certain algebras satisfying any two of (1), (2), (3) together with (4) must have a Wedderburn decomposition.

We suppose in the remainder of this paper that the ring A satisfies (1), (2), and (4).

2. Preliminaries.

We begin this section with the following result.

THEOREM 1. *A is power-associative.*

Proof. Identity (4) yields $x^2x = xx^2$ and this along with (1) and (2) yields $x^2x^2 = x^3x = xx^3$. We define x^{n+1} inductively by requiring that $x^1 = x$ and $x^{n+1} = xx^n$ for $n = 1, 2, \ldots$. Then we have $x^ix^j = x^{i+j}$ for $i + j = 3$ and $0 < i, j < 3$ and for $i + j = 4$ and $0 < i, j < 4$. We proceed now by induction on $i + j$. Assume that $x^ix^j = x^{i+j}$ for all i, j with $i + j < n$ and $0 < i, j$ and let $i + j = n$ with $n \geqq 5$. By (1) with $y = x^{n-2-i}$, $z = x^i$, $w = x$, we have $(x^2, x^{n-2-i}, x^i) = 2x \cdot (x, x^{n-2-i}, x^i) = 0$ so that $x^{n-i}x^i = x^2x^{n-2}$ for $0 < i < n - 2$. But now $(x^2, x, x^{n-3}) = 2x \cdot (x, x, x^{n-3}) = 0$ and

$$(x, x^2, x^{n-3}) = 2x \cdot (x, x, x^{n-3}) = 0$$

so that $x^3x^{n-3} = x^2x^{n-2}$ and $x^3x^{n-3} = xx^{n-1} = x^n$. Thus $x^n = x^2x^{n-2}$ and we have $x^{n-i}x^i = x^n$ for $0 < i < n - 2$. If $i = n - 2$, then we have also shown that $x^2x^{n-2} = x^n$ and if $i = n - 1$, then $xx^{n-1} = x^n$ by definition. Thus we have $x^ix^j = x^{i+j}$ for $i + j = n$ and $0 < i, j < n$ and A is power-associative.

Let e be an idempotent in A. It is known [**1**] that if A is commutative and power-associative, then A has the decomposition $A = A_1 + A_{\frac{1}{2}} + A_0$, where $A_i = \{x: xe = ix\}$ for $i = 1, \frac{1}{2}, 0$. Define the ring A^+ to be the same vector space as A but with the multiplication $\cdot$ defined by $a \cdot b = \frac{1}{2}(ab + ba)$. If A is power-associative then powers in A and A^+ coincide so that A^+ is a commutative, power-associative ring. Hence we have the decomposition $A = A_1 + A_{\frac{1}{2}} + A_0$, where $A_i = \{x: ex + xe = 2ix\}$ for $i = 0, \frac{1}{2}, 1$. Albert further showed that the subspaces A_i multiply as follows: A_1, A_0 are orthogonal; $A_i \cdot A_i \subseteq A_i$ for $i = 0, 1$; $A_{\frac{1}{2}} \cdot A_{\frac{1}{2}} \subseteq A_1 + A_0$; $A_i \cdot A_{\frac{1}{2}} \subseteq A_{1-i} + A_{\frac{1}{2}}$ for $i = 0, 1$. It follows from power-associativity that $A_i = \{x: ex = xe = ix\}$, $i = 0, 1$, and $A_{\frac{1}{2}} = \{x: ex + xe = x\}$.

The following is the associator form of the linearization of $(x, x, x) = 0$ that appeared in [**1**]. For all x, y, and z in A we have

(5) $$(x, y, z) + (x, z, y) + (y, z, x) + (y, x, z) + (z, x, y) + (z, y, x) = 0.$$

LEMMA 1. *If $a \in A$, then $(e, a, e) = 0$ and for each $x \in A_{\frac{1}{2}}$, ex and xe are in $A_{\frac{1}{2}}$.*

Proof. If $a \in A_i$ for $i = 0, 1$, clearly $(e, a, e) = 0$. Let $a \in A_{\frac{1}{2}}$; then by (1), $(e, a, e) = (e \cdot e, a, e) = 2e \cdot (e, a, e)$ so that $(e, a, e) \in A_{\frac{1}{2}}$. Let

$$ea = x_1 + x_{\frac{1}{2}} + x_0.$$

Then

$$(e, e, a) = ea - e(ea) = x_1 + x_{\frac{1}{2}} + x_0 - x_1 - ex_{\frac{1}{2}} = x_{\frac{1}{2}}e + x_0.$$

But $ea - e(ea) = e(ae)$ so that $e(ae) = x_{\frac{1}{2}}e + x_0$. Also

$$(a, e, e) = (ae)e - ae = -(ea)e = -x_1 - x_{\frac{1}{2}}e$$

and we have $(ea)e = x_1 + x_{\frac{1}{2}}e$. Thus we have

$$(e, a, e) = (ea)e - e(ae) = x_1 + x_{\frac{1}{2}}e - x_{\frac{1}{2}}e - x_0 = x_1 - x_0.$$

Since $(e, a, e) \in A_{\frac{1}{2}}$, it follows that $(e, a, e) = x_1 = x_0 = 0$.

THEOREM 2. *Suppose that $A = A_1 + A_{\frac{1}{2}} + A_0$ with respect to the idempotent e of A. Then A_1 and A_0 are orthogonal subrings and $A_iA_{\frac{1}{2}} + A_{\frac{1}{2}}A_i \subseteq A_{\frac{1}{2}}$ for $i = 0, 1$. Furthermore, for $x, y \in A_{\frac{1}{2}}$ we have $x^2 \in A_1 + A_0$ and $(xy)_{\frac{1}{2}} = -(yx)_{\frac{1}{2}}$.*

Proof. Let $x, y \in A_1$. Then

$$(x, y, e) = (e \cdot x, y, e) = e \cdot (x, y, e) + x \cdot (e, y, e) = e \cdot (x, y, e)$$

by (1) so that $(x, y, e) \in A_1$. Also $(e, x, y) = 2e \cdot (e, x, y)$ by (1) and $(e, x, y) = e \cdot (e, x, y) + x \cdot (e, e, y) = e \cdot (e, x, y)$ by (2); thus $(e, x, y) = 0$. Let $xy = a_1 + a_{\frac{1}{2}} + a_0$. Then $xy - e(xy) = 0$, and so $a_0 = 0$ and $a_{\frac{1}{2}} - ea_{\frac{1}{2}} = 0$. But $(x, y, e) = (xy)e - xy \in A_1$ so that

$$a_1 + a_{\frac{1}{2}}e - a_1 - a_{\frac{1}{2}} = a_{\frac{1}{2}}e - a_{\frac{1}{2}} = -ea_{\frac{1}{2}} = -a_{\frac{1}{2}} \in A_1.$$

Thus $a_{\frac{1}{2}} = 0$ and we have $xy \in A_1$. Similarly, $x, y \in A_0$ implies $xy \in A_0$.

Next, let $x \in A_1$ and $y \in A_{\frac{1}{2}}$, then

$$(x, y, e) = e \cdot (x, y, e) + x \cdot (e, y, e) = e \cdot (x, y, e)$$

by (1) and we have $(x, y, e) \in A_1$. But by (2),

$$(x, y, e) = (x, 2y \cdot e, e) = 2e \cdot (x, y, e) + 2y \cdot (x, e, e) = 2e \cdot (x, y, e)$$

so that $(x, y, e) \in A_{\frac{1}{2}}$. Thus $(x, y, e) = 0$. We also have $(e, x, y) = 2e \cdot (e, x, y)$ by (1) and $(x, e, y) = 2e \cdot (x, e, y)$ by (2). Hence (e, x, y) and (x, e, y) are in $A_{\frac{1}{2}}$. Now $-(x, y, e) + (e, x, y) - (x, e, y) = -(xy)e - e(xy) + xy \in A_{\frac{1}{2}}$. If $xy = a_1 + a_{\frac{1}{2}} + a_0$, we then have $-a_1 + a_0 \in A_{\frac{1}{2}}$. Thus $a_1 = a_0 = 0$ and $xy \in A_{\frac{1}{2}}$. Now $xy + yx \in A_{\frac{1}{2}} + A_0$ so that if $yx = a_1 + a_{\frac{1}{2}} + a_0$, then $a_1 = 0$. But $(y, x, e) \in A_{\frac{1}{2}}$ by (1) so that $(yx)e - yx \in A_{\frac{1}{2}}$. Thus $a_0 \in A_{\frac{1}{2}}$ so that $a_0 = 0$ and we have $yx \in A_{\frac{1}{2}}$. Similarly, if $x \in A_0$ and $y \in A_{\frac{1}{2}}$, then xy and yx are in $A_{\frac{1}{2}}$.

Finally, let $x, y \in A_{\frac{1}{2}}$. Then from the decomposition of commutative

https://doi.org/10.4153/CJM-1970-109-x Published online by Cambridge University Press

power-associative rings, $x^2 = x \cdot x \in A_1 + A_0$ and $x \cdot y \in A_1 + A_0$ so that $(xy)_{\frac{1}{2}} = -(yx)_{\frac{1}{2}}$.

3. Ideals and simple rings.

THEOREM 3. *Let* $\mathscr{L} = \{x \in A_{\frac{1}{2}} | \, xa, ax \in A_{\frac{1}{2}}$ *for all* $a \in A\}$. *Then* $\mathscr{L}$ *is an ideal in* A *and* $x^2 = 0$ *for each* $x \in \mathscr{L}$.

Proof. Clearly, for any $z \in A$ and $x \in \mathscr{L}$, xz and zx are in $A_{\frac{1}{2}}$. Let $x \in \mathscr{L}$. We consider cases when $z \in A_i$ for $i = 0, \frac{1}{2}, 1$.

Case 1. Let $z \in A_{\frac{1}{2}}$. If $y \in A_i$ for $i = 0, 1$, then since $xz, zx \in A_{\frac{1}{2}}$, by Theorem 2 we have $y(xz)$, $y(zx)$, $(xz)y$, and $(zx)y$ in $A_{\frac{1}{2}}$. Let $y \in A_{\frac{1}{2}}$; then $(x, z, y) = 2e \cdot (x, z, y) + 2x \cdot (e, z, y)$ by (1). Set $(x, z, y) = a_1 + a_{\frac{1}{2}} + a_0$. Then $a_1 + a_{\frac{1}{2}} + a_0 = 2a_1 + a_{\frac{1}{2}} + 2x \cdot (e, z, y)$. But $x \in \mathscr{L}$, and so

$$x \cdot (e, z, y) \in A_{\frac{1}{2}}$$

and we have $a_1 = a_0 = 0$. Thus $(x, z, y) = (xz)y - x(zy) \in A_{\frac{1}{2}}$ and since $x(zy) \in A_{\frac{1}{2}}$ we have $(xz)y \in A_{\frac{1}{2}}$. Now $xz + zx \in A_1 + A_0$ so that

$$(xz)y + (zx)y \in A_{\frac{1}{2}},$$

and hence $(zx)y \in A_{\frac{1}{2}}$. Next $(y, x, z) = 2e \cdot (y, x, z) + 2x \cdot (y, e, z)$, and so as before, $(y, x, z) \in A_{\frac{1}{2}}$. But by the first part of the proof with y, z interchanged, $(yx)z \in A_{\frac{1}{2}}$ so that $y(xz) \in A_{\frac{1}{2}}$ also. Thus, since

$$y(xz + zx) \in A_{\frac{1}{2}}(A_1 + A_0) \subseteq A_{\frac{1}{2}},$$

we have $y(zx) \in A_{\frac{1}{2}}$ to complete Case 1.

Case 2. Let $z \in A_1$. If $y \in A_i$ for $i = 0, 1$, then since $x \in \mathscr{L}$, $xz, zx \in A_{\frac{1}{2}}$, and so all products with y are in $A_{\frac{1}{2}}$. Let $y \in A_{\frac{1}{2}}$. By (1),

$$(x, z, y) = 2e \cdot (x, z, y) + 2x \cdot (e, z, y)$$

so that as in Case 1, $(x, z, y) \in A_{\frac{1}{2}}$. Since $x \in \mathscr{L}$, $x(zy) \in A_{\frac{1}{2}}$, so that $(xz)y \in A_{\frac{1}{2}}$. Also, $(z, x, y) = 2e \cdot (z, x, y) + 2x \cdot (z, e, y)$ by (2) so that $(z, x, y) \in A_{\frac{1}{2}}$. But $z \in A_1$ and $xy \in A_{\frac{1}{2}}$; thus by Theorem 2, $z(xy) \in A_{\frac{1}{2}}$ and hence $(zx)y \in A_{\frac{1}{2}}$. Now $(y, x, z) = 2e \cdot (y, x, z) + 2x \cdot (y, e, z)$, and so as before $(y, x, z) \in A_{\frac{1}{2}}$, and since $(yx)z \in A_{\frac{1}{2}}A_1 \subseteq A_{\frac{1}{2}}$, it follows that $y(xz) \in A_{\frac{1}{2}}$. So far we have shown that (x, z, y), (z, x, y), and (y, x, z) are in $A_{\frac{1}{2}}$. Now $(z, y, x) = -z(yx) + (zy)x \in A_{\frac{1}{2}}$ since $x \in \mathscr{L}$, $yx \in A_{\frac{1}{2}}$, and $z \in A_1$. Also, $(x, y, z) = (xy)z - x(yz) \in A_{\frac{1}{2}}$ for the same reason. It then follows by (5) that $(y, z, x) \in A_{\frac{1}{2}}$ also and since $(yz)x \in A_{\frac{1}{2}}$, we have $y(zx) \in A_{\frac{1}{2}}$.

Case 3. Let $z \in A_0$. The proof here follows as in Case 2 with $z \in A_0$ instead of A_1.

Finally, if $x \in \mathscr{L}$, then $x \in A_{\frac{1}{2}}$; thus $x \cdot x = x^2 \in A_1 + A_0$. But $x^2 \in A_{\frac{1}{2}}$ by definition of $\mathscr{L}$ so that $x^2 = 0$.

We are now ready to show that under the additional hypothesis that A contains no ideals $\mathscr{L}$ with $x^2 = 0$ for each $x \in \mathscr{L}$ that A must have a Peirce decomposition.

THEOREM 4. *Suppose that A has no ideals $\mathscr{L} \neq 0$ such that $x^2 = 0$ for each $x \in \mathscr{L}$. Then for e an idempotent of A we have $A = A_{11} + A_{10} + A_{01} + A_{00}$, where $x \in A_{ij}$ if and only if $ex = ix$ and $xe = jx$.*

Proof. It is well known that a necessary and sufficient condition that the decomposition of the theorem holds in A is that

$$(x, e, e) = (e, x, e) = (e, e, x) = 0 \quad \text{for all } x \in A.$$

Since we already have $(e, x, e) = 0$ by Lemma 1 and $(e, e, x) + (x, e, e) = 0$ by (5), it suffices to show that $(x, e, e) = 0$ for $x \in A_{\frac{1}{2}}$. If $x \in A_{\frac{1}{2}}$, we have

$$e(xe) = (e, e, x) = -(x, e, e) = (ex)e.$$

We complete the proof by showing that $e(xe) \in \mathscr{L}$, the ideal of Theorem 3. This will follow from the next lemma.

LEMMA 2. *Let A be a ring with idempotent e, and suppose that $x, y \in A_{\frac{1}{2}}$. Then*

$$(xy)_1 = [(ex)(ye)]_1, \qquad (xy)_0 = [(xe)(ey)]_0;$$
$$(ex)(ey),\ (xe)(ye) \in A_{\frac{1}{2}}.$$

Proof. Identity (1) and Lemma 1 yield

$$(e, x, y) = 2e \cdot (e, x, y),\ (x, y, e) = 2e \cdot (x, y, e).$$

Hence

$$(e, x, y)_1 = (e, x, y)_0 = (x, y, e)_1 = (x, y, e)_0 = 0$$

so that

$$[(ex)y]_1 = (xy)_1, \qquad [(ex)y]_0 = 0,$$
$$[x(ye)]_1 = (xy)_1, \qquad [x(ye)]_0 = 0.$$

The lemma follows when we note that $ex + xe = x$.

We are now ready to determine the multiplication properties for the subsets A_{ij}. Note that $A_{11} = A_1$, $A_{00} = A_0$ so that $A_{10} + A_{01} = A_{\frac{1}{2}}$ and the properties of Theorem 2 hold for A_{10} and A_{01}.

Let $x \in A_{11}$ and $y \in A_{10}$. By (1) and (2), $(x, y, e) = 0$ so that $(xy)e = 0$. But $A_1 A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}$; thus $(xy)_{00} = 0$ and $xy = (xy)_{10}$. Similarly, if $x \in A_{00}$ and $y \in A_{01}$, we have $xy \in A_{01}$. If $x \in A_{11}$ and $y \in A_{01}$, by (1) and (2), $(x, e, y) = 0$ so that $xy = 0$ and $x \in A_{00}$, $y \in A_{10}$ yields $xy = 0$. Thus $A_{ii}A_{km} \subseteq \delta_{ik}A_{im}$.

Now let $x \in A_{01}$ and $y \in A_{11}$. Then by (1) and (2), $(x, y, e) = 0$ and we have $(xy)e = xy$. But $xy \in A_{10} + A_{01}$ by Theorem 2 so that $xy = (xy)_{01}$. Similarly $A_{10}A_{00} \subseteq A_{10}$. Thus $A_{ij}A_{jj} \subseteq A_{ij}$. Next let $x \in A_{10}$ and $y \in A_{11}$. Then $(x, y, e) = 0$ by (1) and (2) so that $(xy)e = xy$. But again $xy \in A_{10} + A_{01}$ so that $xy = (xy)_{01}$. If $x \in A_{01}$ and $y \in A_{00}$, then $xy = (xy)_{10}$ and we have $A_{ij}A_{ii} \subseteq A_{ji}$.

If $x, y \in A_{01}$, then, by (2), $(x, e, y) = 2e \cdot (x, e, y)$, and so

$$(x, e, y) \in A_{10} + A_{01}$$

so that $xy \in A_{10} + A_{01}$. By symmetry, $x, y \in A_{10}$ implies $xy \in A_{10} + A_{01}$. Thus $A_{ij}A_{ij} \subseteq A_{10} + A_{01}$. If $x \in A_{10}$ and $y \in A_{01}$, then by (1) and Lemma 1, $(x, y, e) = 2e \cdot (x, y, e)$, and so $(xy)e \in A_{10} + A_{01}$. Hence $(xy)_{00} = 0$ and we have $A_{10}A_{01} \subseteq A_{11} + A_{10} + A_{01}$. Similarly, $A_{01}A_{10} \subseteq A_{00} + A_{10} + A_{01}$.

Finally we show that $(x_{01}y_{10})_{10}$, $(y_{10}x_{01})_{01}$, $(x_{01}y_{01})_{01}$, $(x_{10}y_{10})_{10}$, $(x_{10}y_{01})_{10}$, and $(y_{01}x_{10})_{01}$ are in $\mathscr{L}$, the ideal of Theorem 3, and hence must be zero. Let $x_{ij} \in A_{ij}$ and $y_{km} \in A_{km}$ for the remainder of this proof.

Now from above, $x_{01}y_{10} \in A_{10} + A_{01} + A_{00}$. By the first parts of this proof it suffices to show that $(x_{01}y_{10})_{10}z_{01}$ and $z_{01}(x_{01}y_{10})_{10}$ are in $A_{\frac{1}{2}}$ for $z_{01} \in A_{01}$. Now by (1),

$$(z_{01}, x_{01}, y_{10}) = 2e \cdot (z_{01}, x_{01}, y_{10}) + 2z_{01} \cdot (e, x_{01}, y_{10})$$

and by (2),

$$(z_{01}, x_{01}, y_{10}) = 2e \cdot (z_{01}, x_{01}, y_{10}) + 2x_{01} \cdot (z_{01}, e, y_{10}).$$

Thus $z_{01} \cdot (e, x_{01}, y_{10}) = x_{01} \cdot (z_{01}, e, y_{10}) = 0$. Thus

$$z_{01} \cdot (e(x_{01}y_{10})) = z_{01} \cdot (x_{01}y_{10})_{10} = 0.$$

But $z_{01}(x_{01}y_{10})_{10} \in A_{10} + A_{01} + A_{00}$ and $(x_{01}y_{10})_{10}z_{01} \in A_{11} + A_{01} + A_{10}$. Thus $[z_{01}(x_{01}y_{10})_{10}]_{00} = [(x_{01}y_{10})_{10}z_{01}]_{11} = 0$. Thus $(x_{01}y_{10})_{10} \in \mathscr{L}$. The same procedure with $z_{10} \in A_{10}$ yields $z_{10} \cdot (y_{10}x_{01})_{01} \in A_{\frac{1}{2}}$ so that $(y_{10}x_{01})_{01} \in \mathscr{L}$.

Consider now $(x_{01}y_{01})_{01}$ and let $z_{10} \in A_{10}$. Then

$$(x_{01}, z_{10}, y_{01}) = 2e \cdot (x_{01}, z_{10}, y_{01}) + 2x_{01} \cdot (e, z_{10}, y_{01})$$

by (1) and by (2),

$$(x_{01}, z_{10}, y_{01}) = 2e \cdot (x_{01}, z_{10}, y_{01}) + 2z_{10} \cdot (x_{01}, e, y_{01}).$$

Thus $z_{10} \cdot (x_{01}, e, y_{01}) = x_{01} \cdot (e, z_{10}, y_{01}) = x_{01} \cdot [z_{10}y_{01} - e(z_{10}y_{01})]$. But

$$z_{10}y_{01} - e(z_{10}y_{01}) = (z_{10}y_{01})_{01} \in \mathscr{L} \subseteq A_{\frac{1}{2}}$$

so that $x_{01} \cdot (e, z_{10}, y_{01}) \in A_{\frac{1}{2}}$, and hence $z_{10} \cdot (x_{01}, e, y_{01}) \in A_{\frac{1}{2}}$. Now

$$z_{10} \cdot (x_{01}, e, y_{01}) = z_{10} \cdot (x_{01}y_{01})$$

and since $z_{10} \cdot (x_{01}y_{01})_{10} \in A_{\frac{1}{2}}$ we must have $z_{10} \cdot (x_{01}y_{01})_{01} \in A_{\frac{1}{2}}$. Hence $(x_{01}y_{01})_{01} \in \mathscr{L}$. The same proof with $z_{01} \in A_{01}$ yields $(x_{10}y_{10})_{10} \in \mathscr{L}$.

Finally, $x_{10}y_{01} \in A_{11} + A_{10} + A_{01}$ and $y_{01}x_{10} \in A_{10} + A_{01} + A_{00}$ with $(x_{10}y_{01})_{01}$ and $(y_{01}x_{10})_{10}$ in $\mathscr{L}$. But by Theorem 2, $(x_{10}y_{01})_{\frac{1}{2}} = -(y_{01}x_{10})_{\frac{1}{2}}$ so that $(x_{10}y_{01})_{01} = -(y_{01}x_{10})_{01}$ and $(x_{10}y_{01})_{10} = -(y_{01}x_{10})_{10}$. Hence we have $(y_{01}x_{10})_{01}$ and $(x_{10}y_{01})_{10}$ in $\mathscr{L}$. Combining these remarks we have the following result.

THEOREM 5. *Suppose that A satisfies the hypothesis of Theorem 4. Then for any idempotent e of A, $A = A_{11} + A_{10} + A_{01} + A_{00}$, where $A_{ij}A_{km} \subseteq \delta_{jk}A_{im}$ except when $i \neq j$, $i = k$, $j = m$ or $i \neq j$, $k = m = i$. Then $A_{ij}A_{ij} \subseteq A_{ji}$ and $A_{ij}A_{ii} \subseteq A_{ji}$.*

https://doi.org/10.4153/CJM-1970-109-x Published online by Cambridge University Press

These properties of the subspaces A_{ij} are almost those of an alternative algebra, the only exception being $A_{ij}A_{ii} \subseteq A_{ji}$. We proceed now to show that if A has no ideals J with $x^4 = 0$ for each $x \in J$, then $A_{ij}A_{ii} = 0$.

LEMMA 3. *If A satisfies the hypothesis of Theorem 4, then the following are true for $i \neq j$, $i, j = 0, 1$:*

(i) $(A_{ij}A_{ii})(A_{jj} + A_{ij} + A_{ji}) = 0$,
(ii) $(A_{ij}A_{ii})A_{ii} \subseteq A_{ij}A_{ii}$,
(iii) $(A_{ji} + A_{ii})(A_{ij}A_{ii}) = 0$,
(iv) $A_{jj}(A_{ij}A_{ii}) \subseteq A_{ij}A_{ii}$.

Proof. Throughout this proof, $a_{ij} \in A_{ij}$ and $b_{km} \in A_{km}$. By (2) we have $0 = a_{00} \cdot (a_{01}, e, a_{11} + a_{10}) + e \cdot (a_{01}, a_{00}, a_{11} + a_{10})$ and since

$$(a_{01}, e, a_{11} + a_{10}) = 0,$$

$0 = e \cdot (a_{01}, a_{00}, a_{11} + a_{10}) = e \cdot ((a_{01}a_{00})(a_{11} + a_{10}))$. But

$$(a_{01}a_{00})(a_{11} + a_{10}) \in A_{01}$$

by Theorem 5, and so $(a_{01}a_{00})(a_{11} + a_{10}) = 0$. Now $(a_{01}, a_{00}, b_{01}) \in A_{\frac{1}{2}}$ by (1) and $(a_{01}, a_{00}, b_{01}) = (a_{01}a_{00})b_{01} - a_{01}(a_{00}b_{01}) \in A_{11} + A_{10}$ by Theorem 5. Thus $(a_{01}a_{00})b_{01} \in A_{11} \cap A_{\frac{1}{2}} = 0$. Interchanging 0, 1 in the above proof completes (i).

By (2), $(a_{00}, a_{01}, b_{00}) \in A_{\frac{1}{2}}$ and by (1),

$$0 = 2e \cdot (a_{00}, a_{01}, b_{00}) + 2a_{00} \cdot (e, a_{01}, b_{00}) = (a_{00}, a_{01}, b_{00}) + 2a_{00} \cdot (e, a_{01}, b_{00}).$$

Thus $(a_{00}a_{01})b_{00} - a_{00}(a_{01}b_{00}) - a_{00}(a_{01}b_{00}) - (a_{01}b_{00})a_{00} = 0$ and since $a_{00}(a_{01}b_{00}) = 0$ we have $(a_{01}b_{00})a_{00} = (a_{00}a_{01})b_{00} \in A_{01}A_{00}$ by Theorem 5. By symmetry we have $(a_{10}a_{11})b_{11} \in A_{10}A_{11}$ to complete (ii).

It follows from Theorem 5 that $A_{11}(A_{10}A_{11}) = 0$ and $A_{00}(A_{01}A_{00}) = 0$. Now by (5),

$$(a_{10}, a_{01}, a_{00}) + (a_{10}, a_{00}, a_{01}) + (a_{00}, a_{10}, a_{01}) + (a_{00}, a_{01}, a_{10}) + (a_{01}, a_{00}, a_{10}) + (a_{01}, a_{10}, a_{00}) = 0.$$

But $(a_{10}, a_{00}, a_{01}) \in A_{11}$, $(a_{00}, a_{10}, a_{01}) = 0$, $(a_{00}, a_{01}, a_{10}) \in A_{00}$,

$$(a_{01}, a_{00}, a_{10}) \in (A_{01}A_{00})A_{10},$$

and $(a_{01}, a_{10}, a_{00}) \in A_{00}$ by Theorem 5, and from (i) above, $(A_{01}A_{00})A_{10} = 0$. Thus we have $(a_{10}, a_{01}, a_{00}) \in A_{11} + A_{00}$. On the other hand,

$$(a_{10}, a_{01}, a_{00}) = (a_{10}, a_{01})a_{00} - a_{10}(a_{01}a_{00}) = -a_{10}(a_{01}a_{00}) \in A_{01}$$

by Theorem 5. Hence $a_{10}(a_{01}a_{00}) \in (A_{11} + A_{00}) \cap A_{01} = 0$ and so $A_{10}(A_{01}A_{00}) = 0$. By symmetry, $A_{01}(A_{10}A_{11}) = 0$, which completes (iii).

Finally, by (5),

$$(a_{11}, a_{01}, a_{00}) + (a_{11}, a_{00}, a_{01}) + (a_{00}, a_{11}, a_{01}) + (a_{00}, a_{01}, a_{11}) + (a_{01}, a_{00}, a_{11}) + (a_{01}, a_{11}, a_{00}) = 0.$$

https://doi.org/10.4153/CJM-1970-109-x Published online by Cambridge University Press

But $(a_{11}, a_{00}, a_{01}) = (a_{00}, a_{11}, a_{01}) = 0$ by Theorem 5 and by (2),

$$(a_{00}, a_{01}, a_{11}) = 2e \cdot (a_{00}, a_{01}, a_{11})$$

while by (1), $0 = 2e \cdot (a_{00}, a_{01}, a_{11}) + 2a_{00} \cdot (e, a_{01}, a_{11}) = 2e \cdot (a_{00}, a_{01}, a_{11})$. Thus $(a_{00}, a_{01}, a_{11}) = 0$. We also have $(a_{01}, a_{00}, a_{11}) = 0$ by (i) above and $(a_{01}, a_{11}, a_{00}) = (a_{01}a_{11})a_{00} \in A_{01}A_{00}$ by Theorem 5. Thus we have

$$(a_{11}, a_{01}, a_{00}) \in A_{01}A_{00}$$

and since $a_{11}a_{01} = 0$, we have $a_{11}(a_{01}a_{00}) \in A_{01}A_{00}$. Interchanging 0, 1 yields $A_{00}(A_{10}A_{11}) \subseteq A_{10}A_{11}$ to complete the proof of the lemma.

Denote $A_{10}A_{11} + A_{01}A_{00}$ by I. By Lemma 3 (i) and (ii) we have $IA \subseteq I$. Also by (iii) and (iv) we have $(A_{11} + A_{00})I \subseteq I$ and

$$A_{10}(A_{01}A_{00}) = A_{01}(A_{10}A_{11}) = 0.$$

Now

$$J = I + AI = I + (A_{11} + A_{10} + A_{01} + A_{00})I$$
$$= I + A_{10}(A_{10}A_{11}) + A_{01}(A_{01}A_{00}).$$

We claim that J is an ideal of A. Since $IA \subseteq I$, it suffices to show that for $i \neq j$, $A(A_{ij}(A_{ij}A_{ii})) \subseteq I + AI$ and $(A_{ij}(A_{ij}A_{ii}))A \subseteq I + AI$.

LEMMA 4. *In A we have for $i \neq j$, $i, j = 0, 1$:*
(i) $A_{ii}(A_{ji}(A_{ji}A_{jj})) = (A_{ji}(A_{ji}A_{jj}))A_{ii} = 0$,
(ii) $A_{ij}(A_{ij}A_{ii}) \subseteq A_{ij}A_{ji}$,
(iii) $(A_{ij}(A_{ij}A_{ii}))A_{ji} = 0$,
(iv) $A_{ii}(A_{ij}(A_{ij}A_{ii})) \subseteq A_{ij}(A_{ij}A_{ii})$,
(v) $(A_{ij}(A_{ij}A_{ii}))A_{ii} \subseteq A_{ij}(A_{ij}A_{ii})$,
(vi) $(A_{ij}(A_{ij}A_{ii}))A_{ij} = 0$,
(vii) $A_{ij}(A_{ji}(A_{ji}A_{jj})) \subseteq A_{ji}A_{jj}$.

Proof. The first three properties follow from Theorem 5.

We have $(a_{00}, a_{01}, b_{01}b_{00}) = 2e \cdot (a_{00}, a_{01}, b_{01}b_{00}) + 2a_{01} \cdot (a_{00}, e, b_{01}b_{00})$ by (2) so that $(a_{00}, a_{01}, b_{01}b_{00}) \in A_{\frac{1}{2}}$ since $a_{00}(b_{01}b_{00}) = 0$ by Lemma 3 (iii). But $(a_{00}, a_{01}, b_{01}b_{00}) = (a_{00}a_{01})(b_{01}b_{00}) - a_{00}(a_{01}(b_{01}b_{00})) \in A_{00}$ by Theorem 3 so that $(a_{00}, a_{01}, b_{01}b_{00}) = 0$ and we have

$$a_{00}(a_{01}(b_{01}b_{00})) = (a_{00}a_{01})(b_{01}b_{00}) \in A_{01}(A_{01}A_{00}).$$

By symmetry, $A_{11}(A_{10}(A_{10}A_{11})) \subseteq A_{10}(A_{10}A_{11})$ and we have (iv).

To prove (v) we note that

$$(a_{01}, b_{01}b_{00}, a_{00}) = 2e \cdot (a_{01}, b_{01}b_{00}, a_{00}) + 2a_{01} \cdot (e, b_{01}b_{00}, a_{00})$$
$$= 2e \cdot (a_{01}, b_{01}b_{00}, a_{00})$$

by (1) and Theorem 5. Thus $(a_{01}, b_{01}b_{00}, a_{00}) \in A_{\frac{1}{2}}$. But $(a_{01}, b_{01}b_{00}, a_{00}) \in A_{00}$ by Theorem 5 so that $(a_{01}, b_{01}b_{00}, a_{00}) = 0$, and hence

$$(a_{01}(b_{01}b_{00}))a_{00} = a_{01}((b_{01}b_{00})a_{00}) \in A_{01}(A_{01}A_{00})$$

by Lemma 3 (ii). By symmetry, $(A_{10}(A_{10}A_{11}))A_{11} \subseteq A_{10}(A_{10}A_{11})$, completing (v).

Next, by (2), $(b_{01}, c_{01} \cdot c_{00}, a_{01}) = c_{01} \cdot (b_{01}, c_{00}, a_{01}) + c_{00} \cdot (b_{01}, c_{01}, a_{01})$. But $(b_{01}, c_{00}, a_{01}) \in A_{10}$ by Lemma 3 (i) and Theorem 5 so that

$$c_{01} \cdot (b_{01}, c_{00}, a_{01}) \in A_{11} + A_{00}.$$

Also, $(b_{01}, c_{01}, a_{01}) \in A_{11} + A_{00}$ by Theorem 5, and so by the same theorem, $c_{00} \cdot (b_{01}, c_{01}, a_{01}) \in A_{00}$. Finally, $(b_{01}, c_{00}c_{01}, a_{01}) \in A_{11} + A_{00}$ by Theorem 5. Thus we have $(b_{01}, c_{01}c_{00}, a_{01}) \in A_{11} + A_{00}$. But $(b_{01}(c_{01}c_{00}))a_{01} \in A_{01}$ by Theorem 5 and $(c_{01}c_{00})a_{01} = 0$ by Lemma 3 (i) so that

$$(b_{01}, c_{01}c_{00}, a_{01}) = (b_{01}(c_{01}c_{00}))a_{01} \in A_{01} \cap (A_{11} + A_{00}) = 0.$$

By symmetry, $(A_{10}(A_{10}A_{11}))A_{10} = 0$, which completes (vi).

Finally, by (5),

$$(a_{10}, a_{01}, b_{01}b_{00}) + (a_{10}, b_{01}b_{00}, a_{01}) + (b_{01}b_{00}, a_{10}, a_{01}) + (b_{01}b_{00}, a_{01}, a_{10}) + (a_{01}, b_{01}b_{00}, a_{10}) + (a_{01}, a_{10}, b_{01}b_{00}) = 0.$$

Now $(a_{10}, b_{01}b_{00}, a_{01}) = 0$ by Lemma 3 (iii) and (i) and $(b_{01}b_{00}, a_{10}, a_{01}) = 0$ by Lemma 3 (i). Also, $(a_{01}, b_{01}b_{00}, a_{10}) = 0$ by (iii) of this lemma and Lemma 3 (i) and $(a_{01}, a_{10}, b_{01}b_{00}) = 0$ by Lemma 3 (iii). But

$$(b_{01}b_{00}, a_{01}, a_{10}) \in A_{01}A_{00}$$

by Lemma 3 (i) and (ii). Therefore $(a_{10}, a_{01}, b_{01}b_{00}) \in A_{01}A_{00}$ and since $(a_{10}a_{01})(b_{01}b_{00}) \in A_{01}A_{00}$ by Lemma 3 (iv) we have $a_{10}(a_{01}(b_{01}b_{00})) \in A_{01}A_{00}$. By symmetry, $A_{01}(A_{10}(A_{10}A_{11})) \subseteq A_{10}A_{11}$ completing the proof of the lemma.

THEOREM 6. *Suppose that A satisfies the hypothesis of Theorem 4. Then for any idempotent e of A the set*

$$J = A_{10}(A_{10}A_{11}) + A_{10}A_{11} + A_{01}A_{00} + A_{01}(A_{01}A_{00})$$

is an ideal of A such that $x^4 = 0$ for each $x \in J$.

Proof. J is an ideal by Lemmas 3 and 4. Recall that $I = A_{10}A_{11} + A_{01}A_{00}$. We claim that $J^2 \subseteq I$. Since $J = A_{10}(A_{10}A_{11}) + I + A_{01}(A_{01}A_{00})$ and $A_{10}(A_{10}A_{11}) \subseteq A_{11}$ and $A_{01}(A_{01}A_{00}) \subseteq A_{00}$ we have from Lemma 3 that $J^2 \subseteq (A_{10}(A_{10}A_{11}))^2 + I + (A_{01}(A_{01}A_{00}))^2$. But now

$$(a_{10}, b_{10}a_{00}, x) = 2e \cdot (a_{10}, b_{10}a_{00}, x) + 2a_{10} \cdot (e, b_{10}a_{00}, x)$$

by (2) for $x \in A_{10}(A_{10}A_{11})$ and since $(e, b_{10}a_{00}, x) = 0$ by Theorem 5 we have $(a_{10}, b_{10}a_{00}, x) \in A_{\frac{1}{2}}$. On the other hand, $(a_{10}(b_{10}a_{00}))x - a_{10}((b_{10}a_{00})x) \in A_{11}$ by Theorem 5. Thus $(a_{10}(b_{10}a_{00}))x = a_{10}((b_{10}a_{00})x) = 0$ by Lemma 3 (i) since $x \in A_{11}$. Since $[A_{10}(A_{10}A_{11})]^2$ consists of finite sums of products of elements of the form $[a_{10}(b_{10}a_{00})]x$, it follows that

$$[A_{10}(A_{10}A_{11})]^2 = 0$$

and by symmetry $[A_{01}(A_{01}A_{00})]^2 = 0$ so that $J^2 \subseteq I$.

Now let $x \in I$. It suffices to consider $x = a_{10}a_{11} + b_{01}b_{00}$. Then

$$x^2 = (a_{10}a_{11})^2 + (a_{10}a_{11})(b_{01}b_{00}) + (b_{01}b_{00})(a_{10}a_{11}) + (b_{01}b_{00})^2 \subseteq (A_{10}A_{11})A_{01} + (A_{10}A_{11})A_{10} + (A_{01}A_{00})A_{01} + (A_{01}A_{00})A_{10} = 0$$

by Lemma 3 (i). Thus if $x \in J$, then $x^2 \in I$ and so $(x^2)^2 = x^4 = 0$, as was desired.

THEOREM 7. *Let A be a ring with characteristic A prime to 2 satisfying* (1), (2), *and* (4) *and let e be an idempotent in A. If A contains no ideals J with $x^4 = 0$ for all $x \in J$, then $A = A_{11} + A_{10} + A_{01} + A_{00}$, where $x \in A_{ij}$ if and only if $ex = ix$ and $xe = jx$ and $A_{ij}A_{km} \subseteq \delta_{jk}A_{im}$ unless $i \neq j$, $i = k$, $j = m$, and then $(A_{ij})^2 \subseteq A_{ji}$.*

Proof. This theorem follows immediately from Theorems 5 and 6.

THEOREM 8. *Suppose that A satisfies the hypothesis of Theorem 4. Then $A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}$ is an ideal of A.*

Proof. By Theorem 5 it suffices to show that $A_{10}A_{01}$ and $A_{01}A_{10}$ are ideals of A_{11} and A_{00}, respectively. Let $a_{11} \in A_{11}$, $a_{10} \in A_{10}$, and $a_{01} \in A_{01}$. Then by (1), $(a_{11}, a_{10}, a_{01}) = e \cdot (a_{11}, a_{10}, a_{01})$ and by (2),

$$(a_{11}, a_{10}, a_{01}) = 2e \cdot (a_{11}, a_{10}, a_{01})$$

so that $(a_{11}, a_{10}, a_{01}) = 0$. Thus $a_{11}(a_{10}a_{01}) = (a_{11}a_{10})a_{01} \in A_{10}A_{01}$. Also by (1), $(a_{10}, a_{01}, a_{11}) \in A_{\frac{1}{2}}$ and by Theorem 5, $(a_{10}, a_{01}, a_{11}) \in A_{11}$ so that $(a_{10}, a_{01}, a_{11}) = 0$ and we have $(a_{10}a_{01})a_{11} = a_{10}(a_{01}a_{11}) \in A_{10}A_{01}$. Interchanging 0s and 1s yields the corresponding results for $A_{01}A_{10}$.

COROLLARY 1. *$A_{10}A_{01}$ and $A_{01}A_{10}$ are associative subrings of A.*

Proof. Clearly $A_{10}A_{01}$ and $A_{01}A_{10}$ are subrings. Now

$$(a_{11}, a_{10} \cdot a_{01}, b_{11}) = \tfrac{1}{2}(a_{11}, a_{10}a_{01}, b_{11})$$

since A_{11} and A_{00} are orthogonal. But by (2),

$$(a_{11}, a_{10} \cdot a_{01}, b_{11}) = a_{10} \cdot (a_{11}, a_{01}, b_{11}) + a_{01} \cdot (a_{11}, a_{10}, b_{11}) = a_{01} \cdot (a_{11}, a_{10}, b_{11}).$$

But $(a_{11}, a_{10}, b_{11}) = (a_{11}a_{10})b_{11} - a_{11}(a_{10}b_{11}) = (a_{11}a_{10})b_{11}$ by Lemma 3 (iii) and $(a_{11}a_{10})b_{11} \in A_{01}$. Thus $a_{01} \cdot (a_{11}, a_{10}, b_{11}) \in (A_{11} + A_{00}) \cap A_{10} = 0$ and we have $(a_{11}, a_{10}a_{01}, b_{11}) = 0$. Since $A_{10}A_{01}$ consists of sums of elements of the form $a_{10}a_{01}$, we have the desired associativity of $A_{10}A_{01}$. By symmetry, $A_{01}A_{10}$ is an associative subring also.

COROLLARY 2. *If A is simple, then either $e = 1$ or $A_{11} = A_{10}A_{01}$ and $A_{00} = A_{01}A_{10}$.*

We are now in a position to state our main result.

THEOREM 9. *Let A be a simple ring satisfying* (1), (2), *and* (4). *Suppose that A has an idempotent $e \neq 1$. Then A is either an associative ring or a Cayley-Dickson algebra over its centre.*

Proof. A ring is alternative if and only if

$$(x, y, z) = \epsilon(\sigma)(\sigma(x), \sigma(y), \sigma(z)) \tag{6}$$

for all permutations σ, where $\epsilon(\sigma) = 1$ or -1 as σ is even or odd. We prove the theorem by showing that (6) holds for all possible choices of x, y, z belonging to the A_{ij} since then Albert's result is applicable [**2**].

Combining Corollaries 1 and 2 of Theorem 8 we have $(x_{ii}, y_{ii}, z_{ii}) = 0$, $i = 0, 1$. Suppose that $x_{11}, y_{11} \in A_{11}$, $z_{10} \in A_{10}$. Then

$$(z_{10}, x_{11}, y_{11}) = (x_{11}, z_{10}, y_{11}) = (z_{10}, y_{11}, x_{11}) = (y_{11}, z_{10}, x_{11}) = 0.$$

Next by (1),

$$(x_{11}, y_{11}, z_{10}) = e \cdot (x_{11}, y_{11}, z_{10}) + x_{11} \cdot (e, y_{11}, z_{10}) = e \cdot (x_{11}, y_{11}, z_{10})$$

so that $(x_{11}, y_{11}, z_{10}) \in A_{11}$. But $(x_{11}, y_{11}, z_{10}) \in A_{10}$. Thus

$$(x_{11}, y_{11}, z_{10}) = (y_{11}, x_{11}, z_{10}) = 0.$$

Replacing z_{10} by $z_{01} \in A_{01}$ we find the corresponding result, and by symmetry, the corresponding result holds if x_{11}, y_{11} are replaced by $x_{00}, y_{00} \in A_{00}$. Clearly $(x_{11}, y_{11}, z_{00}) = (x_{11}, z_{00}, y_{11}) = (z_{00}, x_{11}, y_{11}) = 0$ for $z_{00} \in A_{00}$ and

$$(x_{00}, z_{11}, y_{00}) = (x_{00}, y_{00}, z_{11}) = (z_{11}, x_{00}, y_{00}) = 0$$

for $x_{00}, y_{00} \in A_{00}$ and $z_{11} \in A_{11}$. Now we examine products involving $x_{11} \in A_{11}$, $y_{10}, z_{10} \in A_{10}$. By (1), $(x_{11}, y_{10}, z_{10}) = e \cdot (x_{11}, y_{10}, z_{10}) + x_{11} \cdot (e, y_{10}, z_{10})$ so that $(x_{11}y_{10})z_{10} = (y_{10}z_{10})x_{11}$. Since $a_{10}b_{10} = -b_{10}a_{10}$ we have

$$(x_{11}y_{10})z_{10} = -z_{10}(x_{11}y_{10}) = (y_{10}z_{10})x_{11} = -(z_{10}y_{10})x_{11} = -(x_{11}z_{10})y_{10}$$
$$= y_{10}(x_{11}z_{10}).$$

Combining these we have $(x_{11}, y_{10}, z_{10}) = \epsilon(\sigma)(\sigma(x_{11}), \sigma(y_{10}), \sigma(z_{10}))$ for all σ. Again replacing y_{10}, z_{10} by y_{01}, x_{01} we have the corresponding results. By symmetry we have the desired result if x_{11} is replaced by x_{00}. Let $x_{11} \in A_{11}$, $y_{10} \in A_{10}$, and $z_{01} \in A_{01}$. Then $(y_{10}, x_{11}, z_{01}) = (x_{11}, z_{01}, y_{10}) = (z_{01}, y_{10}, x_{11}) = 0$ by Theorem 7 while $(z_{01}, x_{11}, y_{10}) = (x_{11}, y_{10}, z_{01}) = 0$ by (1) and (2). Now $(y_{10}, z_{01}, x_{11}) \in A_{\frac{1}{2}}$ by (1) and $(y_{10}, z_{01}. x_{11}) \in A_{11}$ by Theorem 7 so that $(y_{10}, z_{01}, x_{11}) = 0$. Interchanging 0s and 1s yields the corresponding result for $x_{00} \in A_{00}$.

We have reduced the proof to considering $x, y, z \in A_{10} + A_{01}$. First suppose that $x_{10}, y_{10}, z_{10} \in A_{10}$. Then by (1),

$$(x_{10}, y_{10}, z_{10}) = 2e \cdot (x_{10}, y_{10}, z_{10}) + 2x_{10} \cdot (e, y_{10}, z_{10}).$$

Thus equating the A_{00}-components we obtain

$$(x_{10}y_{10})z_{10} = (y_{10}z_{10})x_{10}.$$

Similarly, $(x_{10}z_{10})y_{10} = (z_{10}y_{10})x_{10}$ and since $a_{10}b_{10} = -b_{10}a_{10}$ we have

$$-(y_{10}x_{10})z_{10} = (x_{10}y_{10})z_{10} = (y_{10}z_{10})x_{10} = -(z_{10}y_{10})x_{10} = -(x_{10}z_{10})y_{10} = (z_{10}x_{10})y_{10}.$$

Now by (2), $(x_{10}, y_{10}, z_{10}) = 2e \cdot (x_{10}, y_{10}, z_{10}) + 2y_{10} \cdot (x_{10}, e, z_{10})$ so that equating A_{11}-components we have $x_{10}(y_{10}z_{10}) = -y_{10}(x_{10}z_{10})$. Similarly, $x_{10}(z_{10}y_{10}) = z_{10}(x_{10}y_{10})$ so we have

$$-y_{10}(z_{10}x_{10}) = y_{10}(x_{10}z_{10}) = -x_{10}(y_{10}z_{10}) = x_{10}(z_{10}y_{10}) = z_{10}(x_{10}y_{10}) = -z_{10}(y_{10}x_{10}).$$

Combining these results yields $(x_{10}, y_{10}, z_{10}) = \epsilon(\sigma)(\sigma(x_{10}), \sigma(y_{10}), \sigma(z_{10}))$ for all σ. The case $x_{01}, y_{01}, z_{01} \in A_{01}$ is proved in the same way. Finally, we consider $x_{10}, y_{10} \in A_{10}$ and $z_{01} \in A_{01}$. Then

$$(z_{01}, x_{10}, y_{10}) = -(z_{01}, y_{10}, x_{10}) = (y_{10}, x_{10}, z_{01}) = -(x_{10}, y_{10}, z_{01})$$

since $z_{01}(x_{10}y_{10}) = -z_{01}(y_{10}x_{10}) = (y_{10}x_{10})z_{01} = -(x_{10}y_{10})z_{01}$. Consider

$$(x_{10}, z_{01}, y_{10}) + (z_{01}, x_{10}, y_{10}) = w_{10} \in A_{10}.$$

We show that $x_{01}w_{10} = w_{10}x_{01} = 0$ for all $x_{01} \in A_{01}$. Then

$$Aw_{10} + w_{10}A \subseteq A_{10} + A_{01}$$

so that w_{10} belongs to the ideal $\mathscr{L}$ of Theorem 3 and hence must be zero.

$$\begin{aligned} x_{01}w_{10} &= x_{01}(x_{10}, z_{01}, y_{10}) + x_{01}(z_{01}, x_{10}, y_{10}) \\ &= x_{01}(x_{10}, z_{01}, y_{10}) - x_{01}(z_{01}(x_{10}y_{10})) \\ &= x_{01}(x_{10}, z_{01}, y_{10}) - (x_{10}y_{10})(x_{01}z_{01}), \end{aligned}$$

since $x_{10}y_{10} \in A_{01}$ and $a_{01}(b_{01}c_{01}) = c_{01}(a_{01}b_{01})$ from the preceding case. But now by (1) we have

$$(x_{01} \cdot x_{10}, z_{01}, y_{10}) = x_{01} \cdot (x_{10}, z_{01}, y_{10}) + x_{10} \cdot (x_{01}, z_{01}, y_{10})$$

and since $x_{01} \cdot x_{10} \in A_{11} + A_{00}$, we must have

$$x_{01} \cdot (x_{10}, z_{01}, y_{10}) + x_{10} \cdot (x_{01}, z_{01}, y_{10}) = 0.$$

Thus the A_{00}-component of this sum must be zero and we have

$$\begin{aligned} 0 &= x_{01}(x_{10}, z_{01}, y_{10}) + (x_{01}, z_{01}, y_{10})x_{10} \\ &= x_{01}(x_{10}, z_{01}, y_{10}) + [(x_{01}z_{01})y_{10}]x_{10} \\ &= x_{01}(x_{10}, z_{01}, y_{10}) + (y_{10}x_{10})(x_{01}z_{01}) \\ &= x_{01}(x_{10}, z_{01}, y_{10}) - (x_{10}y_{10})(x_{01}z_{01}) \\ &= x_{01}w_{10}. \end{aligned}$$

In a similar fashion we have $w_{10}x_{01} = 0$. Hence from our preceding remarks, $w_{10} = 0$ and $(x_{10}, z_{01}, y_{10}) = -(z_{01}, x_{10}, y_{10})$. Interchanging x_{10} and y_{10} we obtain $(y_{10}, z_{01}, x_{10}) = -(z_{01}, y_{10}, x_{10})$. Combining these results we have $(x_{10}, y_{10}, z_{01}) = \epsilon(\sigma)(\sigma(x_{10}), \sigma(y_{10}), \sigma(z_{01}))$ for all σ. By symmetry, the results are true if $x_{01}, y_{01} \in A_{01}$ and $z_{10} \in A_{10}$ and the theorem is true.

4. Semisimple algebras. Let A be a finite-dimensional algebra over a field F of characteristic not equal to 2 satisfying (1), (2), and (4). We define the radical N of A to be the maximal nil ideal of A. This makes sense since A is power-associative by Theorem 1. A is said to be semisimple if $N = 0 \neq A$.

THEOREM 10. *If A is simple, A has an identity element.*

Proof. Since A is non-nil and power-associative, it follows that A has an idempotent e [**7**]. If e is not the identity element, then by Theorem 9, A is alternative and hence has an identity element [**2**].

THEOREM 11. *For any principal idempotent e of A, $A_{10} + A_{01} + A_{00} \subseteq N$. If A is also semisimple, A has an identity element and is the direct sum of simple algebras.*

Proof. The proof of this theorem is essentially the same as that of the corresponding results given in [**4**] and we do not repeat it here.

5. Wedderburn decomposition. Rodabaugh [**6**] has shown that an algebra A satisfying (1), (3), and (4) over a splitting field F of characteristic not equal to 2 or 3 and having neither nodal subalgebras nor ideals $\mathscr{L}$ with $x \in \mathscr{L}$ implying $x^2 = 0$ has a Wedderburn decomposition. He further showed that the condition of no ideals $\mathscr{L}$ with $x^2 = 0$ for each $x \in \mathscr{L}$ cannot be removed.

THEOREM 12. *Let A be an algebra satisfying* (1), (2), *and* (4) *over a splitting field F of characteristic not equal to* 2 *or* 3. *If A contains neither nodal subalgebras nor ideals J with x in J implying $x^4 = 0$, then A has a Wedderburn decomposition.*

Proof. Rodabaugh [**6**, Theorem 4.1] has shown that under these hypotheses it suffices to show that A has the decomposition of Theorem 7 and that the set $A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}$ be an alternative ideal. The latter condition follows from the proof of Theorem 9. Hence A has a Wedderburn decomposition.

THEOREM 13. *Let A be an algebra satisfying* (1), (2), *and* (4) *over a splitting field F of characteristic not equal to* 2 *or* 3. *If A contains no nodal subalgebras and if $A - N$ contains no simple ideals of degree* 2, *then A has a Wedderburn decomposition.*

Proof. The proof is essentially the same as that given in [**6**, Theorem 5.2] for algebras satisfying (1), (3), and (4) and we do not repeat it here.

https://doi.org/10.4153/CJM-1970-109-x Published online by Cambridge University Press

Acknowledgement. I wish to express my gratitude to my adviser Professor D. J. Rodabaugh.

References

1. A. A. Albert, *Power-associative rings*, Trans. Amer. Math. Soc. *64* (1948), 552–593.
2. ——— *On alternative rings*, Can. J. Math. *4* (1954), 129–135.
3. R. H. Bruck and E. Kleinfeld, *The structure of alternative division rings*, Proc. Amer. Math. Soc. *2* (1951), 878–890.
4. E. Kleinfeld, F. Kosier, J. M. Osborn, and D. J. Rodabaugh, *The structure of associator dependent rings*, Arch. Math. *13* (1962), 203–212.
5. F. Kosier, *A generalization of alternative rings*, Trans. Amer. Math. Soc. *112* (1964), 32–42.
6. D. J. Rodabaugh, *On the Wedderburn principal theorem*, Trans. Amer. Math. Soc. *138* (1969), 343–362.
7. R. D. Schafer, *An introduction to nonassociative algebras* (Academic Press, New York, 1966).

Southern Methodist University,
Dallas, Texas