Canad. Math. Bull. Vol. **54** (3), 2011 pp. 472–486
doi:10.4153/CMB-2011-012-3


# A Semiregularity Map Annihilating Obstructions to Deforming Holomorphic Maps

Donatella Iacono

*Abstract.* We study infinitesimal deformations of holomorphic maps of compact, complex, Kähler manifolds. In particular, we describe a generalization of Bloch's semiregularity map that annihilates obstructions to deform holomorphic maps with fixed codomain.

## 1 Introduction

The investigation of obstruction spaces plays a fundamental role in the study of deformation theory and moduli spaces. For instance, the obstruction theory is used to determine the dimension of moduli spaces or the virtual fundamental class (see, for example, [2–4, 9, 14]). From the local point of view, given an infinitesimal deformation of a geometric object, we would like to know whether it is possible to extend this deformation or not. The idea is to consider the same problem of extension for the associated deformation functor. More precisely, let $F\colon \mathbf{Art} \to \mathbf{Set}$ be a functor of Artin rings, *i.e.,* a covariant functor from the category **Art** of local Artinian $\mathbb{C}$-algebras (with residue field $\mathbb{C}$) to the category **Set** of sets, such that $F(\mathbb{C}) = \{\text{point}\}$.

A (complete) obstruction space for $F$ is a vector space $V$, such that, for each small extension $0 \to J \to B \to A \to 0$ in **Art** and each element $x \in F(A)$, there exists an obstruction element $v_x \in V$, associated with $x$, that is zero if and only if $x$ can be lifted to $F(B)$.

Since this space controls the liftings, we would like to describe it, as well as possible, and know whether the associated obstruction element is zero or not. In general, we just know a vector space that contains the obstructions, but we have no explicit description of which elements of the vector space are actually obstructions. For example, if $W$ is another vector space which contains $V$, then also $W$ is an obstruction space for $F$.

In [8], B. Fantechi and M. Manetti proved the existence of the "smallest" obstruction space for functors associated with deformations of geometric objects. More precisely, they proved the existence of the universal obstruction space for deformation functors, *i.e.,* functors of Artin rings satisfying Schlessinger's conditions $(H_1)$ and a stronger version of $(H_2)$ (see [1, Theorem 2.1] and also [8, Lemma 2.11 and Theorem 6.11]).

Received by the editors June 30, 2008; revised February 9, 2009.
Published electronically February 16, 2011.
AMS subject classification: **13D10**, 14D15, 14B10.
Keywords: semiregularity map, obstruction theory, functors of Artin rings, differential graded Lie algebras.

Since it is quite difficult to determine all the obstruction spaces, the idea is to start by studying some special and easier obstructions. In this setting, a very useful tool is the result known as the Ran–Kawamata $T^1$-lifting theorem: if the functor is pro-representable and if it has no "curvilinear obstructions", then the functor has no obstructions at all. Recall that the curvilinear obstructions are the ones arising from the curvilinear extensions

$$0 \longrightarrow \mathbb{C} \xrightarrow{\cdot x^n} \frac{\mathbb{C}[x]}{(x^{n+1})} \longrightarrow \frac{\mathbb{C}[x]}{(x^n)} \longrightarrow 0.$$

This theorem was generalized by B. Fantechi and M. Manetti: if $F$ is a deformation functor, then $F$ has no obstructions if and only if $F$ has no curvilinear obstructions [8, Corollary 6.4].

Thus, in some cases, this result guarantees that it is enough to study the curvilinear obstructions to determine the obstruction space. More precisely, if the curvilinear obstructions vanish, then all the obstructions are zero.

A fundamental fact to note is that the curvilinear obstructions do not generate the obstruction space. Therefore, if these obstructions do not vanish, we do not have enough information to determine the obstruction space; see [14] and [8, Example 5.7 (1)].

In the case of infinitesimal deformations of complex compact manifolds, an obstruction space is the second cohomology vector space $H^2(X, \Theta_X)$ of the holomorphic tangent bundle $\Theta_X$ of $X$. If $X$ is also Kähler, then A. Beauville and H. Clemens [6] and Z. Ran [19, 20] proved that the obstructions are contained in a subspace of $H^2(X, \Theta_X)$ defined as the kernel of a well defined map. This is the so-called "Kodaira's principle" (see, for example, [6, Theorem 10.1], [16, Corollary 3.4], [10, Corollary 12.6], [19, Theorem 0], or [20, Corollary 3.5]).

In the case of embedded deformations of a submanifold $X$ in a fixed manifold $Y$, the obstructions are naturally contained in the first cohomology vector space $H^1(X, N_{X|Y})$ of the normal bundle $N_{X|Y}$ of $X$ in $Y$. In this case too, if $Y$ is Kähler, then it is possible to define a map on $H^1(X, N_{X|Y})$ called the "semiregularity map", whose kernel contains the curvilinear obstructions; see S. Bloch [3]. Thus, if we can prove that this map is injective, then we can conclude that the deformations are unobstructed.

Recently, M. Manetti studied these deformations using the differential graded Lie algebras (DGLAs) and proved that the semiregularity map annihilates all obstructions [18, Theorem 0.1 and Section 9].

Therefore, even if this map is not injective, we have a control on the obstruction space, *i.e.,* it is contained in the kernel of the map.

Inspired by this work, we follow the approach, via DGLAs, to study the obstructions to infinitesimal deformations of holomorphic maps of complex compact manifolds. In [11], E. Horikawa proved that the obstructions to the deformations of $f \colon X \to Y$, with fixed codomain, are contained in the second cohomology vector space $H^2(C^\cdot_{f_*})$ of the cone $C^\cdot_{f_*}$, associated with the complex morphism

$$f_* \colon A_X^{0,*}(\Theta_X) \to A_X^{0,*}(f^*\Theta_Y).$$

Using the approach via DGLAs, we can give an easy proof of this theorem (Proposition 4.6) and, furthermore, we can improve it in the case of Kähler manifolds. Our main result is the following theorem (Corollary 4.14).

***Theorem*** *Let $f\colon X \to Y$ be a holomorphic map of compact Kähler manifolds. Let $p = \dim Y - \dim X$. Then the obstruction space to the infinitesimal deformations of $f$ with fixed $Y$ is contained in the kernel of the map*

$$\sigma\colon H^2(C^{\cdot}_{f_*}) \longrightarrow H^{p+1}(Y, \Omega_Y^{p-1}).$$

In the case of an inclusion $X \hookrightarrow Y$, the previous map reduces to Bloch's semiregularity map. We remark that this map annihilates all obstructions.

In [4], R.-O. Buchweitz and H. Flenner studied deformations of coherent modules and, as a particular case, deformations of holomorphic maps. They used very different techniques and they also produced a semiregularity map [4, Theorem 7.23], but they did not explicitly state that their map annihilates all obstructions (and not merely the curvilinear ones).

## 2 Notation

We will work on the field of complex number $\mathbb{C}$, and all vector spaces, linear maps, tensor products etc. are intended over $\mathbb{C}$.

If $A$ is an object in **Art**, then $m_A$ denotes its maximal ideal.

Unless otherwise specified, by a manifold we mean a compact, (complex) connected, and smooth variety.

Given a manifold $X$, we denote by $\Theta_X$ the holomorphic tangent bundle, by $\mathcal{A}_X^{p,q}$ the sheaf of differentiable $(p,q)$-forms on $X$ and by $A_X^{p,q} = \Gamma(X, \mathcal{A}_X^{p,q})$ the vector space of its global sections. More generally, $\mathcal{A}_X^{p,q}(\Theta_X)$ is the sheaf of differentiable $(p,q)$-forms on $X$ with values in $\Theta_X$, and $A_X^{p,q}(\Theta_X) = \Gamma(X, \mathcal{A}_X^{p,q}(\Theta_X))$ is the vector space of its global sections.

Finally, by a map $f\colon X \to Y$ we always mean a holomorphic morphism of (complex compact) manifolds, and we denote by $f^*$ and $f_*$ the induced maps, *i.e.,*

$$f^*\colon A_Y^{p,q}(\Theta_Y) \longrightarrow A_X^{p,q}(f^*\Theta_Y) \quad \text{and} \quad f_*\colon A_X^{p,q}(\Theta_X) \longrightarrow A_X^{p,q}(f^*\Theta_Y).$$

The cone $C^{\cdot}_{f_*}$ is the complex $(C^{\cdot}_{f_*}, D)$ with

$$C^i_{f_*} := A_X^{0,i}(T_X) \oplus A_X^{0,i-1}(f^*T_Y)$$

and

$$D\colon C^i_{f_*} \longrightarrow C^{i+1}_{f_*},$$

$$(l,n) \mapsto (\overline{\partial} l, f_*(l) - \overline{\partial} n) \in A_X^{0,i+1}(T_X) \oplus A_X^{0,i}(f^*T_Y).$$

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## 3 The Semiregularity Map

Let $f\colon X \to Y$ be a map of Kähler manifolds, $n = \dim X$ and $p = \dim Y - \dim X$. Let $\mathcal{H}$ be the space of harmonic forms on $Y$ of type $(n+1, n-1)$. By Dolbeault's theorem and Serre's duality, we have $\mathcal{H}^\nu = (H^{n-1}(Y, \Omega_Y^{n+1}))^\nu = H^{p+1}(Y, \Omega_Y^{p-1})$.

Using the contraction $\lrcorner$ of vector fields with differential forms, for each $\omega \in \mathcal{H}$, we can define the following map

$$A_X^{0,*}(f^*\Theta_Y) \xrightarrow{\lrcorner\,\omega} A_X^{n,*+n-1},$$

$$\lrcorner\,\omega(\phi f^*\chi) = \phi f^*(\chi \lrcorner\,\omega) \in A_X^{n,p+n-1} \qquad \forall\, \phi f^*\chi \in A_X^{0,p}(f^*\Theta_Y).$$

It can be proved (see Lemma 4.7) that if $f^*\omega = 0$, then the following diagram

$$\begin{tikzcd} A_X^{0,*}(f^*\Theta_Y) \arrow[r, "\lrcorner\,\omega"] & A_X^{n,*+n-1} \\ A_X^{0,*}(\Theta_X) \arrow[u, "f_*"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

is commutative. Thus, for each $\omega$, we get a morphism

$$H^2(C^{\cdot}_{f_*}) \longrightarrow H^n(X, \Omega_X^n),$$

which, composed with the integration on $X$, gives the semiregularity map

$$\sigma\colon H^2(C^{\cdot}_{f_*}) \longrightarrow H^{p+1}(Y, \Omega_Y^{p-1}).$$

If $f$ is the inclusion map $X \hookrightarrow Y$, then $H^2(C^{\cdot}_{f_*}) \cong H^1(X, N_{X|Y})$, where $N_{X|Y}$ is the normal bundle of $X$ in $Y$. In this case, the previous map $\sigma$ reduces to Bloch's semiregularity map (see [3] or [18, Section 9]), *i.e.,*

$$\sigma\colon H^1(X, N_{X|Y}) \longrightarrow H^{p+1}(Y, \Omega_Y^{p-1}).$$

***Example 3.1*** Let $S$ be a K3 surface. Then the canonical bundle is trivial, $\Theta_S \cong \Omega_S^1$, $q(s) = \dim H^1(S, \mathcal{O}_S) = 0$, and $p_g(S) = \dim H^2(S, \mathcal{O}_S) = 1$.

Let $f\colon C \to S$ be a non-constant holomorphic map from a smooth curve $C$ in $S$ (the differential $f_*\colon \Theta_C \to f^*\Theta_S$ is non zero at the generic point). If we consider the deformations of $f$ with fixed codomain $S$, then the semiregularity map

$$\sigma\colon H^2(C^{\cdot}_{f_*}) \longrightarrow H^2(S, \mathcal{O}_S) \cong \mathbb{C}$$

is surjective. Indeed, let $N_f$ be the cokernel of $f_*$, *i.e.,*

$$\Theta_C \xrightarrow{f_*} f^*\Theta_S \longrightarrow N_f \longrightarrow 0.$$

The hypothesis on $f_*$ implies that the sequence

$$0 \longrightarrow \Theta_C \xrightarrow{f_*} f^*\Theta_S \longrightarrow N_f \longrightarrow 0$$

is also exact. Therefore, $H^i(C, N_f) \cong H^{i+1}(C^{\cdot}_{f_*})$ for each $i \geq 0$, and the induced map

$$H^1(C, f^*\Theta_S) \longrightarrow H^1(C, N_f)$$

is surjective.

Consider the pull-back $f^*\Omega^1_S \to \Omega^1_C$ and denote by $\mathcal{L}$ and $\Delta$ the kernel and the cokernel, respectively, *i.e.*,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L} \arrow[r] & f^*\Omega^1_S \arrow[rr] \arrow[dr] & & \Omega^1_C \arrow[r] & \Delta \arrow[r] & 0 \\
& & & K \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0. & &
\end{tikzcd}$$

By hypothesis on $f$, $\Delta$ is a torsion sheaf, and so $H^1(C, \Delta) = 0$. Therefore, $H^1(C, K) \to H^1(C, \Omega^1_C)$ is surjective.

Moreover, $H^2(C, \mathcal{L}) = 0$ and so $H^1(C, f^*\Omega^1_S) \to H^1(K)$ is surjective. In conclusion, the induced map

$$H^1(C, f^*\Omega^1_S) \longrightarrow H^1(C, \Omega^1_C)$$

is surjective. By the integration on $C$, we get a surjective map

$$H^1(C, f^*\Omega^1_S) \longrightarrow \mathbb{C}.$$

Since the diagram

$$\begin{tikzcd}
& H^1(C, f^*\Omega^1_S) \arrow[dl] \arrow[dr] & \\
H^1(C, N_f) \arrow[rr, "\sigma"] & & \mathbb{C}
\end{tikzcd}$$

is commutative, the semiregularity map is surjective.

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## 4 Proof of the Main Theorem

Nowadays, the approach to deformation theory via DGLAs is quite standard (see for example [5, 15, 17]).

In [18], M. Manetti used the DGLAs to study the obstructions of the inclusion map and Bloch's semiregularity map.

Inspired by his work, we also prove our main theorem using the DGLAs and, in particular, the techniques developed in [12, 13].

For the reader's convenience, we recall the main results of these papers.

To study deformations of holomorphic maps via DGLAs, it is convenient to define a deformation functor associated with a pair of morphisms of DGLAs. More precisely, let $h\colon L \to M$ and $g\colon N \to M$ be morphisms of DGLAs, with $M$ concentrated in non negative degrees, *i.e.*,

$$\begin{array}{ccc} & & L \\ & & \downarrow{\scriptstyle h} \\ N & \xrightarrow{g} & M. \end{array}$$

Then the deformation functor associated with the pair $(h, g)$ is

$$\mathrm{Def}_{(h,g)}\colon \mathbf{Art} \longrightarrow \mathbf{Set}, \quad \mathrm{Def}_{(h,g)}(A) = \frac{\mathrm{MC}_{(h,g)}(A)}{\text{gauge}},$$

where

$$\mathrm{MC}_{(h,g)}(A) = \big\{ (x, y, e^p) \in (L^1 \otimes m_A) \times (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A) \mid \\ dx + \tfrac{1}{2}[x, x] = 0,\ dy + \tfrac{1}{2}[y, y] = 0,\ g(y) = e^p * h(x) \big\},$$

and the gauge equivalence is induced by the gauge action of $\exp(L^0 \otimes m_A) \times \exp(N^0 \otimes m_A)$ on $\mathrm{MC}_{(h,g)}(A)$, given by

$$(e^a, e^b) * (x, y, e^p) = (e^a * x, e^b * y, e^{g(b)} e^p e^{-h(a)}).$$

Let $(\mathrm{C}^{\cdot}_{(h,g)}, D)$ be the differential graded vector space with

$$\mathrm{C}^i_{(h,g)} = L^i \oplus N^i \oplus M^{i-1} \quad \text{and} \quad D(l, n, m) = (dl, dn, -dm - g(n) + h(l)).$$

Then the tangent space of $\mathrm{Def}_{(h,g)}$ is $H^1(\mathrm{C}^{\cdot}_{(h,g)})$, and the obstruction space of $\mathrm{Def}_{(h,g)}$ is naturally contained in $H^2(\mathrm{C}^{\cdot}_{(h,g)})$ ([12, Lemma III.1.19] or [13, Section 4.2]).

***Lemma 4.1*** *Let $h\colon L \to M$ and $g\colon N \to M$ be morphisms of abelian DGLAs. Then the functor $\mathrm{MC}_{(h,g)}$ is smooth, that is, it has no obstructions.*

**Proof** See [12, Lemma II.1.20]. ∎

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***Remark 4.2*** Every commutative diagram of morphisms of DGLAs

$$\begin{tikzcd}
& L \arrow[r, "\alpha'"] \arrow[d, "h"'] & P \arrow[d, "\eta"] \\
& M \arrow[r, "\alpha"] & Q \\
N \arrow[ur, "g"] \arrow[r, "\alpha''"] & R \arrow[ur, "\mu"'] &
\end{tikzcd}$$

induces a *morphism* $\varphi^{\cdot}$ *of complexes*

$$\mathrm{C}^{i}_{(h,g)} \ni (l,n,m) \xmapsto{\varphi^{i}} (\alpha'(l), \alpha''(n), \alpha(m)) \in \mathrm{C}^{i}_{(\eta,\mu)}$$

and a natural *transformation* $F$ of the associated deformation functors, *i.e.,*

$$F\colon \operatorname{Def}_{(h,g)} \longrightarrow \operatorname{Def}_{(\eta,\mu)}\,.$$

***Proposition 4.3*** *If* $\varphi^{\cdot}\colon \mathrm{C}^{\cdot}_{(h,g)} \to \mathrm{C}^{\cdot}_{(\eta,\mu)}$ *is a quasi-isomorphism of complexes, then* $F\colon \operatorname{Def}_{(h,g)} \to \operatorname{Def}_{(\eta,\mu)}$ *is an isomorphism of functors.*

**Proof** See [12, Theorem III.1.23]. ■

***Proposition 4.4*** *Let*

$$\begin{tikzcd}
& L \arrow[r, "\alpha'"] \arrow[d, "h"'] & P \arrow[d, "\eta"] \\
& M \arrow[r, "\alpha"] & Q \\
N \arrow[ur, "g"] \arrow[r, "\alpha''"] & R \arrow[ur, "\mu"'] &
\end{tikzcd}$$

*be a commutative diagram of differential graded Lie algebras. If the functor* $\operatorname{Def}_{(\eta,\mu)}$ *is smooth, then the obstruction space of* $\operatorname{Def}_{(h,g)}$ *is contained in the kernel of the map*

$$H^{2}(\mathrm{C}^{\cdot}_{(h,g)}) \longrightarrow H^{2}(\mathrm{C}^{\cdot}_{(\eta,\mu)}).$$

**Proof** The natural transformation $F\colon \operatorname{Def}_{(h,g)} \to \operatorname{Def}_{(\eta,\mu)}$ induces a linear map between the obstruction spaces. If $\operatorname{Def}_{(\eta,\mu)}$ is smooth, then its obstruction space is zero. ■

By a suitable choice of the morphisms $h$ and $g$, we can study the infinitesimal deformations of holomorphic maps.

Indeed, let $f\colon X \to Y$ be a holomorphic map, $Z = X \times Y$ and $\Gamma \subset Z$ the graph of $f$. Let

$$F\colon X \longrightarrow \Gamma \subseteq Z := X \times Y, \quad x \longmapsto (x, f(x)),$$

and $p\colon Z \to X$ and $q\colon Z \to Y$ be the natural projections.

Then we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "F"] \arrow[dr, "\mathrm{id}"'] \arrow[drrr, "f"] & & Z \arrow[dl, "p"] \arrow[dr, "q"] & \\
& X & & Y.
\end{tikzcd}$$

In particular, $F^* \circ p^* = \mathrm{id}$ and $F^* \circ q^* = f^*$. Since $\Theta_Z = p^*\Theta_X \oplus q^*\Theta_Y$, it follows that $F^*(\Theta_Z) = \Theta_X \oplus f^*\Theta_Y$. Define the morphism $\gamma\colon \Theta_Z \to f^*\Theta_Y$ as the product

$$\gamma\colon \Theta_Z \xrightarrow{F^*} \Theta_X \oplus f^*\Theta_Y \xrightarrow{(f_*, -\,\mathrm{id})} f^*\Theta_Y;$$

moreover, let $\pi$ be the surjective morphism

$$A_Z^{0,*}(\Theta_Z) \xrightarrow{\pi} A_X^{0,*}(f^*\Theta_Y) \longrightarrow 0,$$
$$\pi(\omega\, u) = F^*(\omega)\gamma(u), \qquad \forall\, \omega \in A_Z^{0,*}, \quad u \in \Theta_Z.$$

Since each $u \in \Theta_Z$ can be written as $u = p^*v_1 + q^*v_2$, for some $v_1 \in \Theta_X$ and $v_2 \in \Theta_Y$, we also have

$$\pi(\omega u) = F^*(\omega)(f_*(v_1) - f^*(v_2)).$$

The algebra $A_Z^{0,*}(\Theta_Z)$ is the Kodaira–Spencer (differential graded Lie) algebra of $Z$ and we denote by $A_Z^{0,*}(\Theta_Z(-\log\Gamma))$ its differential graded Lie subalgebra defined by the following exact sequence

$$0 \longrightarrow A_Z^{0,*}(\Theta_Z(-\log\Gamma)) \longrightarrow A_Z^{0,*}(\Theta_Z) \xrightarrow{\pi} A_X^{0,*}(f^*\Theta_Y) \longrightarrow 0. \tag{4.1}$$

The DGLA $A_Z^{0,*}(\Theta_Z)$ controls the infinitesimal deformations of $Z$, and $A_Z^{0,*}(\Theta_Z(-\log\Gamma))$ controls the infinitesimal deformations of the pair $\Gamma \subset Z$, *i.e.*, each solution of the Maurer-Cartan equation in $A_Z^{0,*}(\Theta_Z(-\log\Gamma))$ defines a deformation of both $\Gamma$ and $Z$ [18].

Consider the morphism of DGLAs

$$g = (p^*, q^*)\colon A_X^{0,*}(\Theta_X) \times A_Y^{0,*}(\Theta_Y) \to A_Z^{0,*}(\Theta_Z).$$

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The solutions $n = (n_1, n_2)$ of the Maurer–Cartan equation in $N = A_X^{0,*}(\Theta_X) \times A_Y^{0,*}(\Theta_Y)$ correspond to infinitesimal deformations of both $X$ (induced by $n_1$) and $Y$ (induced by $n_2$). Moreover, the image $g(n)$ satisfies the Maurer–Cartan equation in $M = A_Z^{0,*}(\Theta_Z)$, and so it is associated with an infinitesimal deformation of $Z$, that is exactly the one obtained as product of the deformations of $X$ (induced by $n_1$) and of $Y$ (induced by $n_2$).

Next, fix $M = A_Z^{0,*}(\Theta_Z)$, $L = A_Z^{0,*}(\Theta_Z(-\log \Gamma))$, $h$ the inclusion $L \hookrightarrow M$, $N = A_X^{0,*}(\Theta_X) \times A_Y^{0,*}(\Theta_Y)$, and $g = (p^*, q^*) \colon N \to M$, *i.e.*,

$$\begin{tikzcd} & A_Z^{0,*}(\Theta_Z(-\log\Gamma)) \arrow[d, hook, "h"] \\ A_X^{0,*}(\Theta_X) \times A_Y^{0,*}(\Theta_Y) \arrow[r, "g=(p^*,q^*)"] & A_Z^{0,*}(\Theta_Z). \end{tikzcd} \tag{4.2}$$

If $\mathrm{Def}(f)$ is the functor of the infinitesimal deformations of the map $f$, then the following theorem holds.

**Theorem 4.5** *Let $f \colon X \to Y$ be a holomorphic map of compact complex manifold. Then with the notation above, there exists an isomorphism of functors*

$$\mathrm{Def}_{(h,g)} \cong \mathrm{Def}(f).$$

**Proof** See [12, Theorem IV.2.5] or [13, Theorem 5.11]. ■

Furthermore, for each choice of the pair $(h, g)$, there exist a DGLA $H_{(h,g)}$ and an isomorphism $\mathrm{Def}_{\mathrm{H}_{(h,g)}} \cong \mathrm{Def}_{(h,g)}$ [13, Corollary 6.18]. In particular, there exists an explicit description of a DGLA $H_{(h,g)}$ that controls the infinitesimal deformations of $f$, *i.e.*, $\mathrm{Def}(f) \cong \mathrm{Def}_{H(h,g)}$ [13, Theorem 6.19].

In general, it is not easy to handle the DGLA $H_{(h,g)}$, and so it is convenient to use the functor $\mathrm{Def}_{(h,g)}$, associated with the previous diagram (4.2).

Indeed, for example, if we want to study the infinitesimal deformations of $f$ with fixed domain, it suffices to take $N = A_Y^{0,*}(\Theta_Y)$.

Analogously, in the case of deformations of a map $f$ with fixed codomain $Y$, the DGLA $N$ reduces to $A_X^{0,*}(\Theta_X)$, and so diagram (4.2) reduces to

$$\begin{tikzcd} & L \arrow[d, "h"] & \\ A_X^{0,*}(\Theta_X) \arrow[r, "p^*"] \arrow[rrd, bend right, "f_*"'] & A_Z^{0,*}(\Theta_Z) \arrow[rd, "\pi"] & \\ & & A_X^{0,*}(f^*\Theta_Y), \end{tikzcd}$$

where $f_*$ is the product $\pi \circ p^*$.

Using this diagram and Theorem 4.5, we can easily prove the following proposition due to E. Horikawa [11].

***Proposition 4.6*** *The tangent space to the infinitesimal deformations of a holomorphic map $f\colon X \to Y$, with fixed codomain $Y$, is $H^1(C^{\cdot}_{f_*})$, and the obstruction space is naturally contained in $H^2(C^{\cdot}_{f_*})$.*

**Proof** Theorem 4.5 implies that the infinitesimal deformation functor of $f$, with $Y$ fixed, is isomorphic to $\mathrm{Def}_{(h,p^*)}$. Therefore, the tangent space is $H^1(C^{\cdot}_{(h,p^*)})$ and the obstruction space is naturally contained in $H^2(C^{\cdot}_{(h,p^*)})$. Since $h$ is injective, we have isomorphisms $H^i(C^{\cdot}_{(h,p^*)}) \cong H^i(C^{\cdot}_{\pi\circ p^*}) = H^i(C^{\cdot}_{f_*})$ for each $i$. ■

Our main theorem improves this result in the case of Kähler manifolds. To prove it, we need some preliminary lemmas.

***Lemma 4.7*** *Let $f\colon X \to Y$ be a holomorphic map of complex compact manifolds. Let $\chi \in \mathcal{A}^{0,*}_Y(\Theta_Y)$ and $\eta \in \mathcal{A}^{0,*}_X(\Theta_X)$ such that $f^*\chi = f_*\eta \in \mathcal{A}^{0,*}_X(f^*\Theta_Y)$. Then for any $\omega \in \mathcal{A}^{*,*}_Y$*

$$f^*(\chi \lrcorner \omega) = \eta \lrcorner f^*\omega.$$

**Proof** See [12, Lemma II.6.1]. It follows from an easy calculation in local holomorphic coordinates. ■

Let $f\colon X \to Y$ be a holomorphic map, $Z = X \times Y$, and $\Gamma \subset Z$ the graph of $f$.

***Lemma 4.8*** *If $X$ and $Y$ are compact Kähler manifolds, then the sub-complexes $\mathrm{Im}(\partial) = \partial A_Z$, $\partial A_\Gamma$, $\partial A_Z \cap q^*A_Y$ and $\partial A_Z \cap p^*A_X$ are acyclic.*

**Proof** See [12, Lemma II.2.2]. It follows from the $\partial\overline{\partial}$-Lemma. ■

***Remark 4.9*** In the previous lemma, the Kähler hypothesis on $X$ and $Y$ can be substituted by the validity of the $\partial\overline{\partial}$-lemma in $A_X$, $A_Y$, $A_Z = A_{X\times Y}$ and $A_\Gamma$. In particular, it holds for every compact complex manifold bimeromorphic to a Kähler manifold [7, Corollary 5.23].

Let $W$ be a manifold and $A^{0,*}_W(\Theta_W)$ its Kodaira–Spencer algebra. Then we define the contraction map $i$ as follows:

$$i\colon A^{0,*}_W(\Theta_W) \longrightarrow \mathrm{Hom}^*(A_W, A_W),$$

$$i_a(\omega) = a \lrcorner \omega, \qquad \forall\, a \in A^{0,*}_W(\Theta_W) \text{ and } \omega \in A^{*,*}_W.$$

Therefore,

$$i(A^{0,j}_W(\Theta_W)) \subset \oplus_{h,l} \mathrm{Hom}^0(A^{h,l}_W, A^{h-1,l+j}_W) \subset \mathrm{Hom}^{j-1}(A_W, A_W).$$

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In order to interpret $i$ as a morphism of DGLAs, the key idea, due to M. Manetti [18, Section 8], is to substitute $\mathrm{Hom}^*(A_W, A_W)$ with the differential graded vector space $\mathrm{Htp}(\ker(\partial), A_W/\partial A_W) = \bigoplus_i \mathrm{Hom}^{i-1}(\ker(\partial), A_W/\partial A_W)$. Consider on $\mathrm{Htp}(\ker(\partial), A_W/\partial A_W)$ the following differential $\delta$ and bracket $\{\cdot, \cdot\}$:

$$\delta(f) = -\overline{\partial} f - (-1)^{\deg(f)} f\overline{\partial},$$

$$\{f, g\} = f\partial g - (-1)^{\deg(f)\deg(g)} g\partial f.$$

***Lemma 4.10*** *$\mathrm{Htp}(\ker(\partial), A_W/\partial A_W)$ is a DGLA and the linear map*

$$i\colon A_W^{0,*}(T_W) \longrightarrow \mathrm{Htp}\Big(\ker(\partial), \frac{A_W}{\partial A_W}\Big)$$

*is a morphism of DGLAs.*

**Proof** See [18, Proposition 8.1]. ∎

***Remark 4.11*** For any pair of graded vector spaces $V$ and $W$, there exists an isomorphism $H^i(\mathrm{Htp}(V, W)) \cong \mathrm{Htp}^i(H^*(V), H^*(W))$ for each $i$.

Next, we apply this construction to $Z = X \times Y$. Let $\Gamma$ be the graph of $f$ in $Z$ and $I_\Gamma \subset A_Z$ the space of the differential forms vanishing on $\Gamma$. The DGLA $L = A_Z^{0,*}(\Theta_Z(-\log\Gamma))$ defined in (4.1) satisfies the property

$$L \subset \{a \in A_Z^{0,*}(\Theta_Z) \mid i_a(I_\Gamma) \subset I_\Gamma\}.$$

Furthermore,

$$p^* A_X^{0,*}(\Theta_X) \subset \{a \in A_Z^{0,*}(\Theta_Z) \mid i_a(q^* A_Y) = 0\},$$

where $p$ and $q$ are the projections of $Z$ onto $X$ and $Y$, respectively.

In conclusion, we can define the following commutative diagram of morphisms of DGLAs

(4.3)

$$\begin{tikzcd}
L \arrow[r] \arrow[d, hook, "h"] & K = \Big\{ f \in \mathrm{Htp}\big(\ker(\partial), \frac{A_Z}{\partial A_Z}\big) \;\Big|\; f(I_\Gamma \cap \ker(\partial)) \subset \frac{I_\Gamma}{I_\Gamma \cap \partial A_Z} \Big\} \arrow[d, hook, "\eta"] \\
A_Z^{0,*}(\Theta_Z) \arrow[r] & \mathrm{Htp}\big(\ker(\partial), \frac{A_Z}{\partial A_Z}\big) \\
A_X^{0,*}(\Theta_X) \arrow[r] \arrow[u, "p^*"] & J = \Big\{ f \in \mathrm{Htp}\big(\ker(\partial), \frac{A_Z}{\partial A_Z}\big) \;\Big|\; f(\ker(\partial) \cap q^* A_Y) = 0 \Big\} \arrow[u, hook, "\mu"]
\end{tikzcd},$$

where the horizontal maps are all given by $i$.

We note that diagram (4.3) induces a natural transformation of deformation functors

$$\mathcal{I}\colon \mathrm{Def}_{(h, p^*)} \longrightarrow \mathrm{Def}_{(\eta, \mu)}.$$

https://doi.org/10.4153/CMB-2011-012-3 Published online by Cambridge University Press

***Lemma 4.12*** *If the differential graded vector spaces $(\partial A_Z, \overline{\partial})$, $(\partial A_\Gamma, \overline{\partial})$ and $(\partial A_Z \cap q^* A_Y, \overline{\partial})$ are acyclic, then the functor* $\mathrm{Def}_{(\eta,\mu)}$ *has no obstructions. In particular, the obstruction space of* $\mathrm{Def}_{(h,p^*)}$ *is naturally contained in the kernel of the map*

$$H^2(C^{\cdot}_{(h,p^*)}) \xrightarrow{\mathfrak{I}} H^2(C^{\cdot}_{(\eta,\mu)}).$$

**Proof** This lemma is an extension of [18, Lemma 8.2].

The projection $\ker(\partial) \to \ker(\partial)/\partial A_Z$ induces a commutative diagram

$$\begin{tikzcd}
K \arrow[d, "\eta"] & \{f \in K \mid f(\partial A_Z) = 0\} \arrow[l, "\alpha"'] \arrow[d, "\eta'"'] \\
\mathrm{Htp}\big(\ker(\partial), \frac{A_Z}{\partial A_Z}\big) & \mathrm{Htp}\big(\frac{\ker(\partial)}{\partial A_Z}, \frac{A_Z}{\partial A_Z}\big) \arrow[l, "\beta"'] \\
J \arrow[u, "\mu"] & \{f \in J \mid f(\partial A_Z) = 0\}. \arrow[l, "\gamma"'] \arrow[u, "\mu'"']
\end{tikzcd} \tag{4.4}$$

Since $\partial A_Z$ is acyclic, $\beta$ is a quasi-isomorphism of DGLAs. Since

$$\mathrm{coker}(\alpha) = \Big\{ f \in \mathrm{Htp}\Big(\partial A_Z, \frac{A_Z}{\partial A_Z}\Big) \mid f(I_\Gamma \cap \partial A_Z) \subset \frac{I_\Gamma}{I_\Gamma \cap \partial A_Z} \Big\},$$

there exists an exact sequence

$$0 \to \mathrm{Htp}\Big(\frac{\partial A_Z}{I_\Gamma \cap \partial A_Z}, \frac{A_Z}{\partial A_Z}\Big) \to \mathrm{coker}(\alpha) \to \mathrm{Htp}\Big(I_\Gamma \cap \partial A_Z, \frac{I_\Gamma}{I_\Gamma \cap \partial A_Z}\Big) \to 0.$$

Moreover, the exact sequence

$$0 \longrightarrow I_\Gamma \cap A_Z \longrightarrow \partial A_Z \longrightarrow \partial A_\Gamma \longrightarrow 0$$

implies that $I_\Gamma \cap A_Z$ and $\partial A_Z/(I_\Gamma \cap \partial A_Z) = \partial A_\Gamma$ are acyclic. Thus, the complexes

$$\mathrm{Htp}\Big(\frac{\partial A_Z}{I_\Gamma \cap \partial A_Z}, A_Z/\partial A_Z\Big) \quad \text{and} \quad \mathrm{Htp}\Big(I_\Gamma \cap \partial A_Z, \frac{I_\Gamma}{I_\Gamma \cap \partial A_Z}\Big)$$

are acyclic, and the same holds for $\mathrm{coker}(\alpha)$, *i.e.*, $\alpha$ is a quasi-isomorphism.

As to $\gamma$, we have

$$\begin{aligned}
\mathrm{coker}(\gamma) &= \{f \in \mathrm{Htp}\Big(\partial A_Z, \frac{A_Z}{\partial A_Z}\Big) \mid f(\partial A_Z \cap q^* A_Y) = 0\} \\
&= \mathrm{Htp}\Big(\frac{\partial A_Z}{\partial A_Z \cap q^* A_Y}, \frac{A_Z}{\partial A_Z}\Big).
\end{aligned}$$

By hypothesis, $\partial A_Z \cap q^* A_Y$ and $\partial A_Z$ are acyclic, and so the same holds for $\partial A_Z/(\partial A_Z \cap q^* A_Y)$. Then $\operatorname{coker}(\gamma)$ is acyclic, *i.e.*, $\gamma$ is also a quasi-isomorphism.

Therefore, by Lemma 4.3, there exists an isomorphism of deformation functors $\mathrm{Def}_{(\eta,\mu)} \cong \mathrm{Def}_{(\eta',\mu')}$. We note that the elements of the three algebras of the right column of (4.4) vanish on $\partial A_Z$. Then by the definition of the bracket $\{\cdot,\cdot\}$, these algebras are abelian and so, by Lemma 4.1, the functor $\mathrm{Def}_{(\eta,\mu)} \cong \mathrm{Def}_{(\eta',\mu')}$ has no obstructions.

Therefore, by Proposition 4.4 the obstruction space of $\mathrm{Def}_{(h,p^*)}$ lies in the kernel of $H^2(C^{\cdot}_{(h,p^*)}) \xrightarrow{\mathcal{I}} H^2(C^{\cdot}_{(\eta,\mu)})$. ■

***Theorem 4.13*** *Let $f\colon X \to Y$ be a holomorphic map of compact Kähler manifolds. Then the obstruction space to the infinitesimal deformations of $f$ with fixed codomain is contained in the kernel of the map*

$$H^2(C^{\cdot}_{f_*}) \xrightarrow{\mathcal{I}} H^1\Big(\mathrm{Htp}(I_\Gamma \cap \ker(\partial) \cap q^* A_Y, A_\Gamma)\Big).$$

**Proof** By Lemma 4.8, the complexes $(\partial A_Z, \overline{\partial})$, $(\partial A_\Gamma, \overline{\partial})$, and $(\partial A_Z \cap q^* A_Y, \overline{\partial})$ are acyclic. Then Lemma 4.12 implies that the obstruction space lies in the kernel of the following map

$$H^2(C^{\cdot}_{(h,p^*)}) \xrightarrow{\mathcal{I}} H^2(C^{\cdot}_{(\eta,\mu)}).$$

Since $h$ is injective, as in Proposition 4.6, we have $H^2(C^{\cdot}_{(h,p^*)}) \cong H^2(C^{\cdot}_{f_*})$. Thus, the obstructions lie in the kernel of $\mathcal{I}\colon H^2(C^{\cdot}_{f_*}) \to H^2(C^{\cdot}_{(\eta,\mu)})$.

As to $H^2(C^{\cdot}_{(\eta,\mu)})$, consider $K$ as in equation (4.3), *i.e.*,

$$K = \Big\{ f \in \mathrm{Htp}\Big(\ker(\partial), \frac{A_Z}{\partial A_Z}) \mid f(I_\Gamma \cap \ker(\partial)\Big) \subset \frac{I_\Gamma}{I_\Gamma \cap \partial A_Z} \Big\}$$

and the exact sequence

$$0 \longrightarrow K \xrightarrow{\eta} \mathrm{Htp}\Big(\ker(\partial), \frac{A_Z}{\partial A_Z}\Big) \xrightarrow{\pi'} \operatorname{coker}(\eta) \longrightarrow 0,$$

with $\operatorname{coker}(\eta) = \mathrm{Htp}(I_\Gamma \cap \ker(\partial), \frac{A_\Gamma}{\partial A_\Gamma})$. Then $H^2(C^{\cdot}_{(\eta,\mu)}) \cong H^2(C^{\cdot}_{\pi'\circ\mu})$. Let $J$ be as in (4.3), *i.e.*,

$$J = \Big\{ f \in \mathrm{Htp}(\ker(\partial), \frac{A_Z}{\partial A_Z}) \mid f(\ker(\partial) \cap q^* A_Y) = 0 \Big\};$$

thus,

$$\pi' \circ \mu\colon J \longrightarrow \mathrm{Htp}(I_\Gamma \cap \ker(\partial), \frac{A_\Gamma}{\partial A_\Gamma}),$$

with

$$\operatorname{coker}(\pi' \circ \mu) = \mathrm{Htp}(I_\Gamma \cap \ker(\partial) \cap q^* A_Y, \frac{A_\Gamma}{\partial A_\Gamma}).$$

Consider the map $\mathcal{I}'\colon H^2(C^{\cdot}_{\pi'\circ\mu}) \to H^1(\operatorname{coker}(\pi'\circ\mu)) = H^1(\operatorname{Htp}(I_\Gamma \cap \ker(\partial) \cap q^*A_Y, \frac{A_\Gamma}{\partial A_\Gamma}))$. Since the complex $\partial A_\Gamma$ is acyclic, the projection

$$\operatorname{Htp}(I_\Gamma \cap \ker(\partial) \cap q^*A_Y, \frac{A_\Gamma}{\partial A_\Gamma}) \longrightarrow \operatorname{Htp}(I_\Gamma \cap \ker(\partial) \cap q^*A_Y, A_\Gamma)$$

is a quasi-isomorphism.

Therefore, the obstruction space is contained in the kernel of $\mathcal{J}\colon H^2(C^{\cdot}_{f_*}) \to H^1(\operatorname{Htp}(I_\Gamma \cap \ker(\partial) \cap q^*A_Y, A_\Gamma))$, *i.e.*,

$$\begin{tikzcd}
H^2(C^{\cdot}_{f_*}) \arrow[r, "\mathcal{I}"] \arrow[drr, "\mathcal{J}"'] & H^2(C^{\cdot}_{\pi'\circ\mu}) \arrow[r, "\mathcal{I}'"] & H^1(\operatorname{Htp}(I_\Gamma \cap \ker(\partial) \cap q^*A_Y, \frac{A_\Gamma}{\partial A_\Gamma})) \arrow[d, "\cong"] \\
& & \oplus_i \operatorname{Hom}(H^i(I_\Gamma \cap \ker(\partial) \cap q^*A_Y), H^i(A_\Gamma)).
\end{tikzcd}$$

■

***Corollary 4.14*** *Let $f\colon X \to Y$ be a holomorphic map of compact Kähler manifolds. Let $p = \dim Y - \dim X$. Then the obstruction space to the infinitesimal deformations of $f$ with fixed $Y$ is contained in the kernel of the map*

$$\sigma\colon H^2(C^{\cdot}_{f_*}) \longrightarrow H^{p+1}(Y, \Omega_Y^{p-1}).$$

**Proof** Let $n = \dim X$, $p = \dim Y - \dim X$ and $\mathcal{H}$ be the space of harmonic forms on $Y$ of type $(n+1, n-1)$. Using the contraction with the forms $\omega \in \mathcal{H}$, we define the semiregularity map $\sigma$ as in Section 3. Since $f^*\omega = 0$, Lemma 4.7 implies that the diagram

$$\begin{tikzcd}
A_X^{0,*}(f^*\Theta_Y) \arrow[r, "\lrcorner\,\omega"] & A_X^{n,*+n-1} \\
A_X^{0,*}(\Theta_X) \arrow[u, "f_*"] \arrow[r] & 0 \arrow[u, "\alpha"']
\end{tikzcd}$$

is commutative, and we get the semiregularity map

$$\sigma\colon H^2(C^{\cdot}_{f_*}) \longrightarrow H^{p+1}(Y, \Omega_Y^{p-1}).$$

Since $q^*\mathcal{H}$ is contained in $I_\Gamma \cap \ker\overline{\partial} \cap \ker\partial \cap q^*A_Y$, we conclude the proof applying Theorem 4.13. ■

***Remark 4.15*** As we already noticed in Remark 4.9, the previous corollary holds if the compact complex manifolds are bimeromorphic to Kähler manifolds.

**Acknowledgments** It is a pleasure for me to show my deep gratitude to the advisor of my Ph.D. thesis, Prof. Marco Manetti, an excellent helpful professor who supports and encourages me every time. I am indebted to him for many useful discussions, advice, and suggestions. Several ideas for this work have grown under his influence. The first draft of this paper was written at the Mittag–Leffler Institute in Stockholm, during the special year on Moduli Spaces. I am very grateful for the support received. I am also grateful to the referee for improvements in the exposition of the paper.

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*Dipartimento di Matematica "G. Castelnuovo", Sapienza Università di Roma, Italy*

Current: *Dipartimento di Matematica, Università degli Studi di Bari "A. Moro", Italy*

*e-mail*: iacono@dm.uniba.it