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TRANSCENDENCE OVER MEROMORPHIC FUNCTIONS

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Dedicated to Richard P. Brent on his 70th birthday

Abstract

In this short note, considering functions, we show that taking an asymptotic viewpoint allows one to prove strong transcendence statements in many general situations. In particular, as a consequence of a more general result, we show that if $F(z) \in \mathbb{C}[[z]]$ is a power series with coefficients from a finite set, then $F(z)$ is either rational or it is transcendental over the field of meromorphic functions.

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1. Introduction

One of the most important classes of problems and questions in number theory is to *determine if a number ξ—or a function F—is transcendental*. When we ask this question, it is generally meant to be taken over the field of rational numbers $\mathbb{Q}$ or, in the case of functions, over the field of rational functions with coefficients in a given field—usually $\mathbb{Q}$, $\overline{\mathbb{Q}}$ or $\mathbb{C}$. This statement is punctuated by the definitions: a number is algebraic over $\mathbb{Q}$ if it is a zero of a nonzero polynomial with integer coefficients and, if $\mathbb{K}$ is a field, then a function $F(z)$ is algebraic over $\mathbb{K}(z)$ provided there is a nonzero polynomial $P(z, y) \in \mathbb{K}[z, y]$ with $P(z, F(z)) = 0$. A function $F(z)$ is *transcendental* over $\mathbb{K}(z)$ if it is not algebraic over $\mathbb{K}(z)$.

In this short note, considering functions, we show that taking an asymptotic viewpoint allows one to prove strong transcendence statements in many general situations.

To this end, let $\mathcal{M}$ denote the set of complex-valued meromorphic functions in the complex variable z and let $\mathbb{C}(z)(\mathcal{M})$ denote the field of such functions.

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Our main result is the following theorem.

THEOREM 1.1. *Suppose that $F(z)$ is a function of a complex variable with finite radius of convergence $R > 0$. If there are infinitely many complex numbers ξ with $|\xi| = R$ such that $|F(\xi z)| \to \infty$ as $z \to 1^-$, then $F(z)$ is transcendental over $\mathbb{C}(z)(\mathcal{M})$.*

Combining Theorem 1.1 and a classical result of Duffin and Schaeffer [7] gives the following result, which can be applied to many situations.

THEOREM 1.2. *If $F(z) \in \mathbb{C}[[z]]$ is a power series with coefficients from a finite set, then $F(z)$ is either rational or it is transcendental over $\mathbb{C}(z)(\mathcal{M})$.*

The remainder of this note is broken into three further sections. In the first we prove Theorem 1.1 and in the second we prove Theorem 1.2 and list some applications and examples. The final section contains a question for further study.

2. Proof of the main result

PROOF OF THEOREM 1.1. Assume to the contrary that there exists a polynomial

$$P(X) = \sum_{i=0}^{d} p_i(z) X^i$$

such that $P(F(z)) = 0$, where $p_i(z) \in \mathbb{C}(z)(\mathcal{M})$ $(i = 0, 1, \ldots, d)$ and $p_d(z)$ is nonzero. Let M denote the infinite set of complex numbers ξ with $|\xi| = R$ for which $|F(\xi z)| \to \infty$ as $z \to 1^-$. Since the coefficients $p_i(z)$ are meromorphic, there is an infinite subset T of M such that the elements of T are not zeros of the nonzero function $p_d(z)$ and are not poles of the functions $p_i(z)$ for $i = 0, 1, \ldots, d$.

For $\xi \in T$, substituting ξz for z in the equation $P(F(z)) = 0$ and dividing by $F(\xi z)^d$ yields

$$p_d(\xi z) + \sum_{i=0}^{d-1} \frac{p_i(\xi z)}{F(\xi z)^{d-i}} = 0.$$

Taking $z \to 1^-$ in this equation implies that $p_d(\xi) = 0$, which is a contradiction since $p_d(\xi) \neq 0$ for $\xi \in T$. □

3. Applications and examples

In this section, we provide a short proof of Theorem 1.2 and then record some applications and examples.

PROOF OF THEOREM 1.2. This follows as a direct corollary of Theorem 1.1 and an old result of Duffin and Schaeffer [7], who proved that *a power series that is bounded in a sector of the unit disc and has coefficients from a finite set is necessarily a rational function.* □

Theorem 1.2 is a generalisation of a result of Fatou from 1906 [8] that states that *a power series* $F(z) \in \mathbb{C}[[z]]$ *whose coefficients take only finitely many values is either rational or transcendental over* $\mathbb{C}(z)$. See Allouche [1] and Borwein and Coons [4] for two different proofs of Fatou's result. It is worth noting that Theorem 1.2 cannot be extended to general integer power series—there are irrational integer power series that are meromorphic, such as the hypergeometric function

$$\frac{1}{\sqrt{1-4z}} = \sum_{n\geqslant 0} \binom{2n}{n} z^n,$$

and there are transcendental integer power series that are both bounded in a sector of the unit disc and have the unit circle as a natural boundary, such as the function

$$\sum_{n\geqslant 0} (1-z)^n z^{n!}$$

that is bounded in the sector $-\pi/4 \leqslant \arg z \leqslant \pi/4$. (This example was given by Duffin and Schaeffer [7]; see also Borwein *et al.* [5].)

We focus our first examples on power series with coefficients from a finite set.

One of the interesting classes of functions from the perspective of Diophantine approximation and transcendence is the set of *automatic functions*. In this setting, one has a positive integer $k \geqslant 2$, a finite set of complex numbers $\mathcal{A}$ and a sequence $f : \mathbb{Z}_{\geqslant 0} \to \mathcal{A}$ such that the value $f(n)$ is generated by a finite-state automaton that reads as input the base-k expansion of n and outputs a number in the range. We call such a sequence $\{f(n)\}_{n\geqslant 0}$ *k-automatic* (or just *automatic*) and we call the function $\sum_{n\geqslant 0} f(n)z^n$ a *k-automatic function* (or just *automatic*). Theorems 1.1 and 1.2 imply the following result, which, while not a direct generalisation, should be compared to results of Bézivin [3] and Randé [10].

Theorem 3.1. *An irrational automatic function is transcendental over* $\mathbb{C}(z)(\mathcal{M})$.

Theorem 3.1 can be considered as an extension of a recent result of Coons [6], who proved that a special class of Mahler functions—that does not include automatic functions—is transcendental over $\mathbb{C}(z)(\mathcal{M})$.

We give a few specific examples as corollaries to Theorem 3.1 that may be of particular interest. The first is inspired by a question posed by K. Nishioka to one of the authors (Tachiya) on the algebraic independence of the number e and the Kempner number $\sum_{n\geqslant 0} 2^{-2^n}$.

Example 3.2. Let $d \geqslant 2$ be an integer and $K_d(z) := \sum_{n\geqslant 0} z^{d^n}$ be Kempner's series. Then $\mathrm{tr.deg.}_{\mathbb{C}(z)} \mathbb{C}(z)(e^z, K_d(z)) = 2$.

Our final automatic example concerns the Thue–Morse sequence $\{t(n)\}_{n\geqslant 0}$ on the alphabet $\{-1, 1\}$, which is given by the 2-automaton recorded in Figure 1. (See Allouche and Shallit's monograph [2] for details on the Thue–Morse sequence as well as general conventions and definitions surrounding automatic sequences.)

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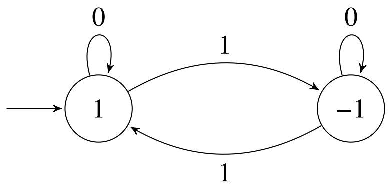

FIGURE 1. The 2-automaton that produces the Thue–Morse sequence $\{t(n)\}_{n\geqslant 0}$.

THEOREM 3.3. *The function* $T(z)=\sum_{n\geqslant 0} t(n)z^n$ *is transcendental over* $\mathbb{C}(z)(\mathcal{M})$.

Our next examples are inspired by the recent talk of Timothy Trudgian at the annual *Number Theory Down Under (NTDU)* meeting[1] of the NTDU Special Interest Group of the Australian Mathematical Society.

Let $\Omega(n)$ denote the number of prime factors (counting multiplicities) of the integer n. *Liouville's function* is given by $\lambda(n)=(-1)^{\Omega(n)}$. As usual, we denote the Riemann zeta function by $\zeta(z)$. It is worth noting that the Riemann zeta function and Liouville's function have some interesting connections. In particular,

$$\sum_{n\geqslant 1}\frac{\lambda(n)}{n^z}=\frac{\zeta(2z)}{\zeta(z)}$$

for $\Re(z)>1$, and the Riemann hypothesis is equivalent to the statement

$$\sum_{n\leqslant x}\lambda(n)=O(x^{1/2+\varepsilon})$$

for any $\varepsilon>0$.

THEOREM 3.4. *Let* $\lambda(n)$ *be Liouville's function and* $\zeta(z)$ *be the Riemann zeta function. Then* $\sum_{n\geqslant 1}\lambda(n)z^n$ *and* $\zeta(z)$ *are algebraically independent over* $\mathbb{C}(z)$.

THEOREM 3.5. *Let* $\lambda(n)$ *be Liouville's function. Then* $\sum_{n\geqslant 1}\lambda(n)z^n$ *and* $\sum_{n\geqslant 1}\lambda(n)/n^z$ *are algebraically independent over* $\mathbb{C}(z)$.

PROOF OF THEOREMS 3.4 AND 3.5. Borwein and Coons [4] proved that $\sum_{n\geqslant 1}\lambda(n)z^n$ is transcendental over $\mathbb{C}(z)$. Combining this with the fact that $\zeta(z)$ is meromorphic and $\sum_{n\geqslant 1}\lambda(n)/n^z$ is a zeta quotient establishes the theorems. □

Our final example concerns the generating function of the standard divisor function $d(n)$, which records the number of positive divisors of the number n. It is well known that the generating function satisfies

$$D(z):=\sum_{n\geqslant 1}d(n)z^n=\sum_{n\geqslant 1}\frac{z^n}{1-z^n}$$

[1]Trudgian's talk was entitled 'A Tale of Two Omegas' and was given on 23 September 2016 in Newcastle, Australia. He highlighted some of his recent work with Michael Mossinghoff.

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and that the series converges in the unit disc. Series of this form (the right-hand side of the equalities) are called *Lambert series*; see Knopp [9] for details on Lambert series. Knopp [9] showed that $D(z)$ is unbounded as z radially approaches the unit circle along any radius $z = re^{2\pi i p/q}$ with p and q positive integers. In fact, one can show that for any positive integers p and q there is a positive constant $c_{p/q}$ such that

$$|D(re^{2\pi i p/q})| > c_{p/q}\frac{1}{1-r}\log\Big(\frac{1}{1-r}\Big);$$

see Titchmarsh [12, page 160] and Stein and Shakarchi [11, page 68]. Since there are infinitely many positive rational numbers p/q, Theorem 1.1 applies to give the following result.

Theorem 3.6. *The series $\sum_{n\geqslant 0} d(n)z^n$ is transcendental over $\mathbb{C}(z)(\mathcal{M})$.*

4. A question

In this note, we have presented a function-level transcendence result (over meromorphic functions) based on a criterion concerning radial asymptotics.

The question we close with is inspired by the example of Duffin and Schaeffer, that the series $\sum_{n\geqslant 0}(1-z)^n z^{n!}$ is bounded in the sector $-\pi/4 \leqslant \arg z \leqslant \pi/4$ and has the unit circle as a natural boundary.

Question 4.1. Let $F(z)$ be a power series with integer coefficients, converging in the unit disc, that does not represent a rational function. Is it true that there are infinitely many complex numbers ξ with $|\xi| = 1$ such that $|F(\xi z)| \to \infty$ as $z \to 1^-$?

Slightly rephrased, this question asks if there exist integer power series that are both bounded on the unit circle, except at a finite number of exceptional points, and have the unit circle as a natural boundary. A positive answer to Question 4.1 would of course imply that an integer power series that converges in the unit disc is either rational or it is transcendental over the field of meromorphic functions. Duffin's and Schaeffer's series may itself be exceptional here as well.

As a final remark, we note that for a general answer to Question 4.1, it will not be enough to consider only the radial asymptotics as z approaches roots of unity. For example, the function

$$G(z) := \sum_{n\geqslant 0}(1-z^{n!})^3 z^{(n!)^3}$$

can be written as the integer power series

$$A(z) = \sum_{n\geqslant 0} a(n)z^n, \quad |z| < 1,$$

where

$$a(n) = \begin{cases} 1 & \text{if } n = (k!)^3, \\ -3 & \text{if } n = k! + (k!)^3, \\ 3 & \text{if } n = 2(k!) + (k!)^3, \\ -1 & \text{if } n = 3(k!) + (k!)^3. \end{cases}$$

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Using the Fabry gap theorem and a classical result of Fatou [8], the series $A(z)$ has the unit circle as a natural boundary. Additionally, the result of Duffin and Schaeffer shows that the series $A(z)$ is unbounded at a dense set of points on the unit circle. Interestingly, though, the series $A(z)$ is bounded as z radially approaches any root of unity. To see this, note that as $r \to 1^-$,

$$\left|\sum_{n\geqslant q}(1-r^{n!})^3 r^{(n!)^3}\right| \leqslant (1-r)^3 \sum_{n\geqslant 0}(n!)^3 r^{(n!)^3} \leqslant (1-r)^3 \frac{r}{(1-r)^2} = o(1),$$

so that if we let p/q be a reduced rational and set $z = re^{2\pi ip/q}$,

$$\lim_{r\to 1^-} A(re^{2\pi ip/q}) = \lim_{r\to 1^-} G(re^{2\pi ip/q}) = \sum_{n=0}^{q-1}(1-(e^{2\pi ip/q})^{n!})^3(e^{2\pi ip/q})^{(n!)^3}.$$

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