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A Riemann–Hurwitz Theorem for the Algebraic Euler Characteristic

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Abstract. We prove an analogue of the Riemann–Hurwitz theorem for computing Euler characteristics of pullbacks of coherent sheaves through finite maps of smooth projective varieties in arbitrary dimensions, subject only to the condition that the irreducible components of the branch and ramification locus have simple normal crossings.

1 Introduction

Consider a finite map $\pi: X \to Y$ of degree μ. Let $B = \bigcup B_i$ be the branch locus and its irreducible decomposition. Let $R = \pi^{-1}(B) = \bigcup R_j$ be the ramification locus and the irreducible decomposition of its reduction. Note that we are taking here the potentially non-standard choice to include in R even those components of $\pi^{-1}(B)$ that are not ramified; this convention will be consistent throughout. The Riemann–Hurwitz formula for the topological Euler characteristic of curves can roughly be interpreted as saying that $\chi(X) - \mu \cdot \chi(Y) = \sum_i r_i \chi(R_i)$ for some integers r_i determined by local data. This formula can be generalized both to higher dimensional manifolds, but also to the algebraic Euler characteristic. However, in the higher dimensional algebraic setting, such a formula typically requires an additional hypothesis on the ramification and/or branch locus, such as one of the following:

- The ramification locus is non-singular.
- The irreducible components of the branch locus do not intersect.
- The irreducible components of the ramification locus are non-singular.
- The irreducible components of the ramification locus have trivial self intersection.

The formula is cleanest if the last condition holds. Note that Izawa [Iza03] handled the case where this last condition is not true, but requires the previous conditions.

We would like to be able to reduce these conditions to the requirement that the branch and ramification locus consist of divisors with simple normal crossings.

The result that we obtain is a formula of the form

$$\chi(X) - \mu \cdot \chi(Y) = \sum_{\alpha} r_\alpha \chi(R^\alpha),$$

where the R^α are irreducible components of the (possibly repeated) intersections, that is the strata, of the ramification locus. The r_α are constants defined in terms of the

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ramification structure along R^{α} by universal equations determined by α. Precise statements are given in Theorem 5.1 and Corollary 5.2; note that terminology introduced elsewhere will be necessary to understand them. Propositions 5.3 and 5.4 give alternative expressions for some of the coefficients r_{α}. Other results that may be of interest are Theorems 2.18 and 3.2 which describe the functoriality of the pullback of logarithmic Chern classes and the logarithmic Euler characteristic through finite maps. It is likely that both of these results admit generalizations outside the context in which the author is able to prove them.

Moreover, our argument works virtually identically in each of the following cases.

- $\chi(X)$ is the topological Euler characteristic, in this case the above results are then classical and follow from excision.
- $\chi(X) = \chi(X, \mathcal{O}_X)$ is the algebraic Euler characteristic.

  This result is well known when the map is étale [Ful98, Example 18.3.9]. Izawa [Iza03] handled the case of no intersection between components of branch/ramification locus.
- $\chi(X) = \chi(X, \mathcal{F})$ is the Euler characteristic of a coherent sheaf $\mathcal{F}$.
- The same argument should apply formally to any characteristic defined by a multiplicative sequence on the Chern classes. The formulas for the coefficients r_{α} naturally depend on this choice of characteristic.

We will only present the argument for the case of $\chi(X, \mathcal{F})$; the main results are Theorem 5.1 and Corollary 5.2. The proof strategy uses primarily formal properties of logarithmic Chern classes and formal properties of multiplicative sequences.

The paper is organized as follows. In Section 2 we introduce our notation and the key results we will make use of. This includes in particular Lemmas 2.15–2.17 and Theorem 2.18. In Section 3 we introduce our definition of the logarithmic Euler *characteristic*. Section 4 contains the key calculations that compare the classical Euler characteristic to the logarithmic Euler characteristic. Section 5 applies the results of Section 4 to the problem of deriving the Riemann–Hurwitz theorem discussed above. In Section 6 we discuss computing the contribution of the self-intersection terms to the logarithmic Euler characteristics.

We should mention that our original motivation for considering the objects being introduced is to compute dimension formulas for spaces of modular forms. For this application, it is actually the results of Section 4 and Section 6 that, by way of the work of Mumford [Mum77], play a significant role. Though actual dimension formulas require additional arithmetic and/or combinatorial input, the results of these two sections can be seen as a generalization of a key ingredient for the approach used in [Tsu80].

2 Background and Notation

Notation 2.1 We will make use of the following notation.

(1) X and Y will always be varieties, typically assumed to be smooth and projective.

(2) Given a variety X, we will denote by Ω^1_X the cotangent bundle of X.

(3) $\Delta = \bigcup_i \{D_i\}$ will always be a collection of (reduced irreducible) divisors on a variety. These will typically be assumed to have simple normal crossings.

(4) Given $\Delta = \bigcup_i \{D_i\}$, a collection of divisors on X, we denote by $\Omega^1_X(\log\Delta)$ the logarithmic cotangent bundle of X relative to Δ.

(5) For any $Y \subset X$, we denote by $\Omega^1_Y(\log\Delta')$ the logarithmic cotangent bundle of Y, relative to $\Delta' = \bigcup_i \{D_i \cap Y\}$, where we consider only those i such that $Y \not\subset D_i$. When we write this, we will always assume that Y meets the relevant D_i transversely. Whenever we write Δ', the relevant Y will be understood.

(6) Given any coherent sheaf $\mathcal{F}$ on X we denote by $\mathrm{c}(\mathcal{F}) = \sum_i \mathrm{c}_i(\mathcal{F})$ the total Chern class and the i-th Chern class [Ful98, Chapter 3].

We denote by $\mathrm{ch}(\mathcal{F})$ and $\mathrm{Todd}(\mathcal{F})$ the Chern character and Todd class, respectively. The Todd class $\mathrm{Todd}(\mathcal{F})$ has a universal expression in terms of the $\mathrm{c}_i(\mathcal{F})$, whereas $\mathrm{ch}(\mathcal{F})$ additionally requires the rank, $\mathrm{rk}(\mathcal{F})$, specifically the constant part of the Chern character. These classes can be interpreted as being in the cohomology ring or the Chow ring as appropriate from context.

We can interpret $\mathrm{ch}(\mathcal{F})$ as a vector determining all of $\mathrm{rk}(\mathcal{F}), \mathrm{c}_1(\mathcal{F}), \dots, \mathrm{c}_n(\mathcal{F})$. Conversely, given a vector $\underline{x} = (x_0, \dots, x_n)$, we write $\mathrm{ch}(\underline{x})$ to indicate the formal expression in the x_i, where we replace c_i by x_i and $\mathrm{rk}(\mathcal{F})$ by x_0 in the formal expression for $\mathrm{ch}(\mathcal{F})$. For brevity, and to make clear the connection to the role of the Chern character, we will often write $\mathrm{ch}(\underline{x})$ or $\mathrm{ch}(\mathcal{F})$ when evaluating a function on the vectors $(x_0, \dots, x_n)$ or $(\mathrm{rk}(\mathcal{F}), \mathrm{c}_1(\mathcal{F}), \dots, \mathrm{c}_n(\mathcal{F}))$ when it is defined through $\mathrm{ch}(\mathcal{F})$ (see Theorem 2.2).

(7) Given a collection $\Delta = \bigcup_i \{D_i\}$ of divisors on X, we denote by Δ_k the k-th elementary symmetric polynomial in the D_i, so that $\prod_i (1 - D_i) = \sum_k (-1)^k \Delta_k$. These products take place in either the cohomology ring or the Chow ring, as appropriate from context.

(8) When we say that α is a partition of m, we mean that $m = \sum_i \alpha_i i$. Given a partition α, we denote by $|\alpha|$ the value m it is partitioning, *i.e.*, $|\alpha| = \sum_i \alpha_i i$. Moreover, given such a partition, we will denote by $\mathrm{c}^\alpha(\mathcal{F}) = \prod_i \mathrm{c}_i(\mathcal{F})^{\alpha_i}$ and by $\Delta^\alpha = \prod_i \Delta_i^{\alpha_i}$.

(9) Given a monomial exponent $\underline{b} = (b_1, \dots, b_\ell) \in \mathbb{N}^\ell$ of total degree $|\underline{b}| = \sum b_i$ we will denote by $D^{\underline{b}} = \prod D_i^{b_i}$. The products above take place in either the cohomology ring or the Chow ring, as appropriate from context. Whenever we write this, the choice of base D will make clear the relevant Δ to which D_i belong. Do not confuse D^ℓ with $D^{\underline{b}}$; the former will always be the self intersection of a particular divisor $D \in \Delta$.

Theorem 2.2 (Riemann–Roch Theorem) *For each $n \in \mathbb{N}$ there is a universal polynomial $Q_n(x_0, \dots, x_n\,;\, y_1, \dots, y_n) = Q_n(\mathrm{ch}(\underline{x})\,;\, y_1, \dots, y_n)$ such that for every smooth projective variety X of dimension n and coherent sheaf $\mathcal{F}$ on X, the Euler characteristic of $\mathcal{F}$ is*

$$\begin{aligned}\chi(X, \mathcal{F}) &= Q_n\big(\mathrm{rk}(\mathcal{F}), \mathrm{c}_1(\mathcal{F}), \dots, \mathrm{c}_n(\mathcal{F})\,;\, \mathrm{c}_1(\Omega^1_X), \dots, \mathrm{c}_n(\Omega^1_X)\big)\\ &= Q_n\big(\mathrm{ch}(\mathcal{F})\,;\, \mathrm{c}_1(\Omega^1_X), \dots, \mathrm{c}_n(\Omega^1_X)\big).\end{aligned}$$

The polynomial is given explicitly by

$$Q_n\big(\mathrm{rk}(\mathcal{F}), \mathrm{c}_1(\mathcal{F}), \dots, \mathrm{c}_n(\mathcal{F})\,;\, \mathrm{c}_1(\Omega^1_X), \dots, \mathrm{c}_n(\Omega^1_X)\big) = \deg_n(\mathrm{ch}(\mathcal{F})\mathrm{Todd}(\Omega^1_X)).$$

Recall that we interpret $\mathrm{ch}(\mathcal{F})$ as a vector determining all of $\mathrm{rk}(\mathcal{F}), \mathrm{c}_1(\mathcal{F}), \dots, \mathrm{c}_n(\mathcal{F})$, and $\mathrm{ch}(\underline{x})$ as the corresponding vector, where the x_i are substituted in the universal expression for $\mathrm{ch}(\mathcal{F})$.

We have the following explicit formulas for Q_n for small n.

$$
\begin{aligned}
Q_0(x_0; y_0) &= x_0,\\
Q_1(x_0, x_1; y_1) &= \frac{1}{2}x_0 y_1 + x_1,\\
Q_2(x_0, x_1, x_2; y_1, y_2) &= \frac{1}{12}x_0(y_1^2 + y_2) + \frac{1}{2}x_1 y_1 + \frac{1}{2}(x_1^2 - 2x_2),\\
Q_3(x_0, x_1, x_2, x_3; y_1, y_2, y_3) &= \frac{1}{24}x_0 y_1 y_2 + \frac{1}{12}x_1(y_1^2 + y_2) + \frac{1}{4}(x_1^2 - 2x_2)y_1\\
&\quad + \frac{1}{6}(x_1^3 - 3x_1x_2 + 3x_3),\\
Q_4(x_0, \dots, x_4; y_1, \dots, y_4) &= \frac{1}{720}x_0(-y_1^4 + 4y_1^2 y_2 + y_1 y_3 + 3y_2^2 - y_4)\\
&\quad + \frac{1}{24}x_1 y_1 y_2 + \cdots .
\end{aligned}
$$

Remark 2.3 The most important feature of the explicit description we will use is that $\mathrm{Todd}(\mathcal{E}_1 \oplus \mathcal{E}_2) = \mathrm{Todd}(\mathcal{E}_1)\mathrm{Todd}(\mathcal{E}_2)$, so that Q_n is effectively multiplicative in the $\mathrm{c}_i(\Omega_X^1)$ set of parameters.

The following proposition makes precise what we mean by multiplicative.

Proposition 2.4 *For notational convenience in the following we use the constants $u_0 = v_0 = 1$ and $u_i = v_i = 0$ for $i < 0$. Consider formal variables $u_1, \dots, u_n$ and $v_1, \dots, v_n$ and set $y_i = \sum_{j+k=i} u_j v_k$. Then*

$$
Q_n(\mathrm{ch}(\underline{x})\,;y_1, \dots, y_n) = \sum_{\ell+m=n} Q_\ell(\mathrm{ch}(\underline{x})\,;u_1, \dots, u_\ell)Q_m(1\,;v_1, \dots, v_m).
$$

Proof Denote by $\mathrm{Todd}(\underline{y}), \mathrm{Todd}(\underline{u}), \mathrm{Todd}(\underline{v})$ the universal expressions for the Chern characters or Todd classes where we substitute the appropriate set of variables for the Chern classes. We then have

$$
\begin{aligned}
Q_n(\mathrm{ch}(\underline{x})\,;y_1, \dots, y_n) &= \deg_n(\mathrm{ch}(\underline{x})\mathrm{Todd}(\underline{y}))\\
&= \deg_n(\mathrm{ch}(\underline{x})\mathrm{Todd}(\underline{u})\mathrm{Todd}(\underline{v}))\\
&= \sum_{\ell+m=n} \deg_\ell(\mathrm{ch}(\underline{x})\mathrm{Todd}(\underline{u}))\deg_m(\mathrm{Todd}(\underline{v}))\\
&= \sum_{\ell+m=n} Q_\ell(\mathrm{ch}(\underline{x})\,;u_1, \dots, u_\ell)Q_m(1\,;v_1, \dots, v_m). \qquad \blacksquare
\end{aligned}
$$

Remark 2.5 The same formula holds if we use instead the system of polynomials

$$
Q_n(x_0, x_1, \dots, x_n\,;y_1, \dots, y_n) = y_n
$$

that gives the topological Euler characteristic. The algebraic Euler characteristic of X is just the special case of $\mathcal{F} = \mathcal{O}_X$.

Notation 2.6 We will also need the following terminology and combinatorial quantities. Note that these are all universal and depend only on the choice of multiplicative sequence Q. These constants can all be effectively computed.

(1) Given any monomial exponent $\underline{b}$ we denote by $\delta_{\underline{b}}$ the coefficient of $D^{\underline{b}}$ in $Q_{|b|}(1\,;\Delta_1,\dots,\Delta_{|\underline{b}|})$. Note that these coefficients depend only on the monomial type of $\underline{b}$, *i.e.*, the multi-set $\{b_i \neq 0\}$. In particular, $\delta_{(2,0,1)} = \delta_{(1,2,0)} = \delta_{(2,1)}$.

In the context where Q describes the algebraic Euler characteristic, this is also precisely the coefficient of $D^{\underline{b}}$ in $\prod_{D\in\Delta} \frac{D}{1-e^{-D}}$. For example, given that

$$Q_2(1,\Delta_1,\Delta_2) = \frac{1}{12}(\Delta_1^2+\Delta_2) = \frac{1}{12}\sum_i D_i^2 + \frac{1}{4}\sum_{i\neq j} D_i D_j,$$

we have that $\delta_{(2)} = \frac{1}{12}$ and $\delta_{(1,1)} = \frac{1}{4}$. Likewise, given that

$$Q_3(1,\Delta_1,\Delta_2,\Delta_3) = \frac{1}{24}\Delta_1\Delta_2 = 0\sum_i D_i^3 + \frac{1}{24}\sum_{i\neq j} D_i^2 D_j + \frac{1}{8}\sum_{i\neq j\neq k} D_i D_j D_k,$$

we have that $\delta_{(3)} = 0$, $\delta_{(2,1)} = \frac{1}{24}$, $\delta_{(1,1,1)} = \frac{1}{8}$. We can likewise compute that $\delta_{(0)} = 1$ and $\delta_{(1)} = \frac{1}{2}$.

(2) We may think of the monomial exponents $\underline{b}$ as vectors indexed by the elements D of Δ. As such, given two monomial exponents $\underline{b}$ and $\underline{b}'$ we will write $\underline{b} \le \underline{b}'$ if the inequality holds component-wise, so that we can write $D^{\underline{b}}D^{\underline{b}''} = D^{\underline{b}'}$ for some $\underline{b}''$ with all components $b_i'' \ge 0$. By the *support* of a monomial exponent $\underline{b}$ we mean the collection of D_i for which $b_i \neq 0$. We say $\underline{a}$ and $\underline{b}$ have *disjoint support* if the corresponding collections have no common elements. Given a monomial exponent $\underline{b}$, we say it is *multiplicity free* (MF) if $b_i \le 1$ for all i; otherwise, we say it is *not multiplicity free* (NMF). Note that a monomial exponent is MF precisely when computing $D^{\underline{b}}$ involves no self-intersections. Finally, given a collection of monomial exponents $\underline{b}_j$, we write $\sum_j \underline{b}_j = \underline{b}$ if this is true as a vector sum.

Proposition 2.7 *If $\underline{b}_j$ have disjoint support, then $\delta_{\sum_j \underline{b}_j} = \prod_j \delta_{\underline{b}_j}$.*

Proof This follows immediately from the multiplicativity of Q as in Proposition 2.4 and the observation that $\mathrm{c}_i(\bigoplus_{D\in\Delta}\mathcal{O}(D)) = \Delta_i$. ∎

(3) Given a monomial exponent $\underline{b}$, denote by $\widetilde{\underline{b}}$ the monomial exponent such that $\widetilde{b}_i = \min(1,b_i)$, so that $\widetilde{\underline{b}}$ captures the support of $\underline{b}$, but $\widetilde{\underline{b}}$ is MF. For example, $\widetilde{(1,2,3)} = (1,1,1)$. Moreover, we denote by $\widehat{\underline{b}}$ the monomial exponent such that

$$\widehat{b}_i = \begin{cases} 1 & b_i = 1,\\ 0 & \text{otherwise,}\end{cases}$$

so that $\widehat{\underline{b}}$ captures the part of the support of $\underline{b}$ where $\underline{b}$ has no self intersection. For example $\widehat{(2,1,3)} = (0,1,0)$.

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(4) Given a monomial exponent $\underline{b}$, let

$$\lambda_{\underline{b}} = \sum_{k\geq 0}(-1)^{k+1}\sum_{\substack{(\underline{b}_1,\dots,\underline{b}_k)\\ \sum \underline{b}_j=\underline{b}}}\Big(\prod_{j=1}^{k}\delta_{\underline{b}_j}\Big) = \delta_{\underline{b}}\sum_{k\geq 0}(-1)^{k+1}\sum_{\substack{(\underline{b}_1,\dots,\underline{b}_k)\\ \sum \underline{b}_j=\underline{b}}} 1.$$

In the summation we consider only terms with all $|\underline{b}_j| \geq 1$ and where in the tuple $(\underline{b}_1,\dots,\underline{b}_k)$, all of $\underline{b}_j$ have disjoint support and each of $\underline{b}_1,\dots,\underline{b}_{k-1}$ are MF, so that only $\underline{b}_k$ is potentially NMF. Note that when $\underline{b}$ is MF, these last three conditions are automatic. For k sufficiently large, the inner sum is an empty sum. Under these conditions the equality between the two definitions is immediate from Proposition 2.7.

Proposition 2.8 *When $\underline{b}$ is MF, we have*

$$\sum_{k\geq 0}(-1)^{k+1}\sum_{\substack{(\underline{b}_1,\dots,\underline{b}_k)\\ \sum \underline{b}_j=\underline{b}}} 1 = (-1)^{|\underline{b}|},$$

where the sum is taken as above.

Proof Each tuple $(\underline{b}_1,\dots,\underline{b}_k)$ contributing to the above summation describes an ordered factorization of $D^{\underline{b}} = D_1\cdots D_\ell$ into k non-trivial coprime parts. Denote by $N_{k,\ell}$, the number of such length k factorizations. Using that D_ℓ is a factor of $D^{\underline{b}_j}$ for a unique j, we may uniquely associate with each length k ordered factorization of $D_1\cdots D_\ell$ an ordered factorization of $D_1\cdots D_{\ell-1}$ of either length k or length $k-1$ as follows.

- If $D^{\underline{b}_j} \neq D_\ell$, then replace $\underline{b}_j$ by $\underline{b}_j'$ where $D^{\underline{b}_j} = D^{\underline{b}_j'}D_\ell$. This gives a length k factorization.
- If $D^{\underline{b}_j} = D_\ell$, we omit $\underline{b}_j$ from the factorization entirely, and shift down the indices on $\underline{b}_i$ for $i > j$. This gives a length $k-1$ factorization.

As we run over all the ordered factorizations of $D_1\cdots D_\ell$, each length k and each length $k-1$ ordered factorization of $D_1\cdots D_{\ell-1}$ occurs exactly k times. We thus obtain a recurrence relation $N_{k,\ell} = kN_{k,\ell-1} + kN_{k-1,\ell-1}$ and a straightforward computation yields that

$$\sum_{k\geq 0}(-1)^{k+1}N_{k,\ell} = -\sum_{k\geq 0}(-1)^{k+1}N_{k,\ell-1}.$$

The claim now follows by an induction on $\ell = |\underline{b}|$. ∎

Proposition 2.9 *When $\underline{b}$ is NMF and $|\widehat{\underline{b}}| \geq 1$, then*

$$\sum_{k\geq 0}(-1)^{k+1}\sum_{\substack{(\underline{b}_1,\dots,\underline{b}_k)\\ \sum \underline{b}_j=\underline{b}}} 1 = 0,$$

where the sum is taken as above.

Proof Every ordered factorization of $D^{\underline{b}}$ into k non-trivial coprime parts, where only the last one is NMF, induces an ordered factorization of $D^{\widehat{\underline{b}}}$ into either $k-1$ non-trivial coprime parts or k non-trivial coprime parts. Each factorization of $D^{\widehat{\underline{b}}}$ arises

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in exactly two ways. It follows that

$$\sum_{k\geq 0}(-1)^{k+1}\sum_{\substack{(\underline{b}_1,\dots,\underline{b}_k)\\ \sum \underline{b}_j=\underline{b}}} 1 \;=\; \sum_{k\geq 0}(-1)^{k+1}\sum_{\substack{(\underline{b}_1,\dots,\underline{b}_k)\\ \sum \underline{b}_j=\widehat{\underline{b}}}} 1 - \sum_{k\geq 0}(-1)^{k+1}\sum_{\substack{(\underline{b}_1,\dots,\underline{b}_k)\\ \sum \underline{b}_j=\widehat{\underline{b}}}} 1 \;=\; 0,$$

which gives the desired result. ■

The constants $\lambda_{\underline{b}}$ will be used in Corollaries 4.3, 4.5, and 5.2.

As an example, by considering the different ordered decompositions of $(1,1,1)$, for instance,

$$(1,1,1),\quad (1,1,0)+(0,0,1),\quad (0,0,1)+(1,1,0),\quad (1,0,1)+(0,1,0),\quad \dots$$

including also the six permutations of $(1,0,0)+(0,1,0)+(0,1,0)$, we see that $\lambda_{(1,1,1)} = \delta_{(1,1,1)} - 6\delta_{(1,1)}\delta_{(1)} + 6\delta_{(1)}^3 = \frac{1}{8}$. We can also compute that

$$\lambda_{(0)} = -1,\quad \lambda_{(1)} = \delta_{(1)} = \frac{1}{2},\quad \lambda_{(1,1)} = \delta_{(1,1)} - 2\delta_{(1)}^2 = -\frac{1}{4},$$
$$\lambda_{(2)} = \delta_{(2)} = \frac{1}{12},\quad \lambda_{(2,1)} = \delta_{(2,1)} - \delta_{(2)}\delta_{(1)} = 0,\quad \lambda_{(3)} = \delta_{(3)} = 0.$$

Proposition 2.10 *Let X be a smooth projective variety and let $\Delta = \bigcup\{D_i\}$ be a collection of smooth divisors with simple normal crossings on X. We have a relation*

$$\mathrm{c}_i(\Omega_X^1) = \sum_j (-1)^{i-j}\mathrm{c}_j(\Omega_X^1(\log\Delta))\Delta_{i-j}.$$

Recall that Δ_k is the k-th elementary symmetric polynomial in the irreducible components of the boundary of X. This can also be expressed as

$$\mathrm{c}(\Omega_X^1) = \mathrm{c}(\Omega_X^1(\log))\prod_{D_i}(1-D_i).$$

Proof We follow essentially an argument for an analogous result from [Tsu80, Proposition 1.2]. We have the following two exact sequences:

$$0 \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1(\log\Delta) \longrightarrow \oplus\mathcal{O}_{D_i} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_X(-D_i) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0,$$

the first of which essentially defines $\Omega_{\overline{X}}^1(\log\Delta)$.

By the multiplicativity of the total Chern class we obtain

$$\mathrm{c}(\Omega_X^1) = \mathrm{c}(\Omega_X^1(\log\Delta))\prod_{D_i}(1-D_i).$$ ■

Proposition 2.11 *Logarithmic Chern classes restrict to the boundary. That is, let X be a smooth projective variety and $\Delta = \bigcup\{D_i\}$ be a collection of smooth divisors with simple normal crossings on X. Suppose $D \in \Delta$ is a fixed irreducible divisor, then*

$$\mathrm{c}^{\alpha}(\Omega_X^1(\log\Delta))\cdot D = \mathrm{c}^{\alpha}(\Omega_D^1(\log\Delta')).$$

This equality should be interpreted as an equality on D.

Proof The result is analogous to [Tsu80, Lemma 5.1]; this proof was suggested by the referee. By Proposition 2.10 we have

$$\mathrm{c}(\Omega^1_X(\log\Delta)) = \mathrm{c}(\Omega^1_X)\frac{1}{(1-D)}\prod_{D_i\neq D}\frac{1}{(1-D_i)}.$$

As $\mathrm{c}(\Omega^1_X)\frac{1}{(1-D)}$ restricts to $\mathrm{c}(\Omega^1_D)$ on D, the right-hand side of the above expression restricts to

$$\mathrm{c}(\Omega^1_D)\prod_{D_i\neq D}\frac{1}{(1-D'_i)},$$

which in turn equals $\mathrm{c}(\Omega^1_D(\log\Delta'))$ by Proposition 2.10. As the Chern classes agree, so to do their products. ■

Notation 2.12 Consider $\pi\colon X\to Y$, a ramified covering. For $Z\subset X$ irreducible, we will denote by e_Z the ramification degree of π at Z as it is defined in [Ful98, Example 4.3.4].

We note that in the context of smooth varieties [Ful98, Proposition 7.1], we can compute the ramification degree as $e_Z = \operatorname{length}(\mathcal{O}_{X,Z}\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y,\pi(Z)}/J_{\pi(Z)})$. In the expression above, $J_{\pi(Z)}$ is the ideal associated with $\pi(Z)$ and the length is that of the ring as a module over itself.

The following proposition is well known [Ful98, Example 4.3.7]. Although we will not make direct use of it, the statement motivates our understanding of ramification.

Proposition 2.13 *Let X and Y be smooth projective varieties. Consider $\pi\colon X\to Y$, a potentially ramified finite covering of degree μ. For any $Z'\subset Y$ irreducible, if we decompose $\pi^{-1}(Z') = \bigcup_i Z_i$ into irreducible components, then $\sum_i \mu_{Z_i}e_{Z_i} = \mu$. where μ_{Z_i} is the degree of $\pi|_{Z_i}$.*

Notation 2.14 Fix a ramified covering $\pi\colon X\to Y$ of smooth projective varieties of dimension n.

The collection of reduced irreducible components of the branch locus will be denoted $\Delta(B)$, and we will denote monomial exponents for the branch locus by $\underline{b}$ and write $B^{\underline{b}}$ for the associated equivalence class of a cycle.

The collection of reduced irreducible components of the ramification locus will be denoted $\Delta(R)$, and we denote monomial exponents for the ramification locus by $\underline{a}$ and write $R^{\underline{a}}$ for the associated equivalence class of a cycle. Recall that $\Delta(R) = \pi^{-1}(\Delta(B))$ includes all components R_j in $\pi^{-1}(B_i)$ even those that may not themselves be ramified.

For an irreducible component R_i it is then clear $\pi(R_i) = B_j$ for a unique j. Given a pair of monomial exponents $\underline{a}$ and $\underline{b}$, we say $\pi(\underline{a}) = \underline{b}$ if for each j we have

$$b_j = \sum_{\pi(R_i)=B_j} a_i.$$

We denote by $E_{R^{\underline{a}}} = \prod_i (e_{R_i})^{a_i}$ the product of the ramification degrees. This notation is justified by Lemma 2.16, which says that if $\underline{a}$ is MF, then $E_{R^{\underline{a}}}$ is the ramification degree of each irreducible component of $R^{\underline{a}}$.

Lemma 2.15 *Consider a potentially ramified finite map $\pi: X \to Y$ between smooth projective varieties. Suppose D_1 and D_2 are two (reduced irreducible) divisors on X that meet with simple normal crossings and that $\pi(D_1) = \pi(D_2) = D$ is smooth. Let Z be a (reduced) irreducible component of $D_1 \cap D_2$. Then there is a component R of the ramification locus of π such that $Z \subset R$ and $Z \notin \{D_1, D_2\}$. In particular, the collection $\{D_1, D_2, R\}$ does not have simple normal crossings.*

Proof We will consider the completed local rings at Z and $\pi(Z)$. By the Cohen structure theorem, these are power series rings over the coordinate ring (for regular complete local rings see [Sta17, Tag 0323 Lemma 10.154.10]). We can thus write them in the form $K(Z)[[s_1, s_2]]$ and $K(\pi(Z))[[t_1, t_2]]$, where s_i is the local coordinate defining D_i on X near Z, the coordinate t_1 defines D on Y near $\pi(Z)$, and t_2 is any other local coordinate defining a divisor that meets D transversely at $\pi(Z)$.

By the assumption that $\pi(D_1) = \pi(D_2) = D$, we can choose our coordinates s_1 and s_2 so that $\pi^*(t_1) = us_1^{a_1}s_2^{a_2}$ with $a_1, a_2 \geq 1$ and $u \in K(Z)^\times$. By the assumption that the map is finite, we have that $s_1, s_2 \nmid \pi^*(t_2)$. Moreover $\pi^*(t_2)$ vanishes at Z and thus $\pi^*(t_2)$ has trivial constant term. It follows that

$$\pi^*(t_2) = v_1 s_1^{b_1} + v_2 s_2^{b_2} + (\text{terms not including those monomials})$$

with $b_1, b_2 \geq 1$ and $v_1, v_2 \in K(Z)^\times$.

We can understand the ramification locus near Z by way of the Jacobian condition. The Jacobian is precisely

$$a_1 u s_1^{a_1-1} s_2^{a_2}\Big(b_2 v_2 s_2^{b_2-1} + \frac{\partial(\text{other terms})}{\partial s_2}\Big) + a_2 u s_1^{a_1} s_2^{a_2-1}\Big(b_1 v_1 s_1^{b_1-1} + \frac{\partial(\text{other terms})}{\partial s_1}\Big).$$

As the expressions

$$s_2\frac{\partial(s_1^{\ell_1}s_2^{\ell_2})}{\partial s_2} \quad \text{and} \quad s_1\frac{\partial(s_1^{\ell_1}s_2^{\ell_2})}{\partial s_1}$$

both have the same monomial type, namely $s_1^{\ell_1}s_2^{\ell_2}$, as the starting monomial, we find that we may rewrite the Jacobian above as

$$us_1^{a_1-1}s_2^{a_2-1}\big(a_2 b_1 v_1 s_1^{b_1} + a_1 b_2 v_2 s_2^{b_2} + (\text{terms not including those monomials})\big).$$

The term $(a_2 b_1 v_1 s_1^{b_1} + a_1 b_2 v_2 s_2^{b_2} + (\text{other terms}))$ vanishes at Z and is not divisible by s_1 or s_2 and thus defines at least one component of the ramification locus that passes through Z which is not equal to D_1 or D_2. ∎

Lemma 2.16 *Consider a potentially ramified finite map $\pi: X \to Y$ between smooth projective varieties. Let $\Delta(B)$ be the collection of irreducible components of the branch locus (on Y), and let $\Delta(R) = \pi^{-1}(\Delta(B))$ be the collection of (reduced) irreducible components of the ramification locus (on X). Suppose $\Delta(B)$ and $\Delta(R)$ have simple normal crossings. If $R_1, \dots, R_\ell \in \Delta(R)$ are distinct and if Z is a (reduced) irreducible component of $\bigcap_i R_i$, then $e_Z = \prod_i e_{R_i}$.*

Proof We will consider the completed local ring at Z and $\pi(Z)$. The completed local rings at generic points are of the form $K(Z)[[s_1, \dots, s_\ell]]$ and $K(\pi(Z))[[t_1, \dots, t_\ell]]$, where s_i is a local parameter defining R_i and t_i is a local parameter defining $B_i =$

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$\pi(R_i)$. That $\pi(R_i)$ are all distinct follows from Lemma 2.15. It follows from this setup that we can choose the local coordinate s_i so that $\pi^*(t_i) = u_i s_i^{a_i}$ with $a_i \geq 1$ and $u_i \in K(Z)^\times$. The claim now follows from a direct computations of lengths. In particular $e_{R_i} = a_i$ and $e_Z = \prod_i a_i$. ∎

Lemma 2.17 *Consider a potentially ramified finite map $\pi: X \to Y$ between smooth projective varieties. Let $\Delta(B)$ be the collection of irreducible components of the branch locus (on Y) and $\Delta(R) = \pi^{-1}(\Delta(B))$ be the collection of (reduced) irreducible components of the ramification locus (on X). Suppose $\Delta(B)$ and $\Delta(R)$ have simple normal crossings.*

(i) *If $\pi(\underline{a}) = \underline{b}$ and the monomial types of $\underline{a}$ and $\underline{b}$ are not the same, then $R^{\underline{a}} = 0$.*
(ii) *If $\pi(\underline{a}) = \underline{b}$ and the monomial types of $\underline{a}$ and $\underline{b}$ are the same, then in the formal expansion*

$$\pi^*(B^{\underline{b}}) = \prod_i \pi^*(B_i)^{b_i} = \prod_i \Big(\sum_{\pi(R_j)=B_i} e_{R_j} R_j \Big)^{b_i} = \sum_{\pi(\underline{a})=\underline{b}} x_{\underline{a}} R^{\underline{a}},$$

the coefficient $x_{\underline{a}}$ of $R^{\underline{b}}$ is $E_{R^{\underline{a}}}$.
(iii) *In the Chow ring, we have the identity $\pi^*(B^{\underline{b}}) = \sum_{\pi(\underline{a})=\underline{b}} E_{R^{\underline{a}}} R^{\underline{a}}$.*

Proof The first statement follows immediately from Lemma 2.15. In particular, if the monomial exponents are not the same, then the expression $R^{\underline{a}}$ involves intersecting two components that map to the same B_i. If these two components do not have trivial intersection, then the ramification locus does not have simple normal crossings.

The second statement is a straightforward check and, indeed, is a basic property of multinomial coefficients.

The third statement then combines the previous two by observing that $R^{\underline{a}} = 0$ whenever the coefficient of $R^{\underline{a}}$ is not $E_{R^{\underline{a}}}$. ∎

Theorem 2.18 *Logarithmic Chern classes respect pullbacks through ramified covers. That is, let X and Y be smooth projective varieties of dimension n. Consider a potentially ramified finite covering $\pi: X \to Y$. Let $\Delta(B)$ be the collection of irreducible components of the branch locus (on Y), and let $\Delta(R) = \pi^{-1}(\Delta(B))$ be the collection of (reduced) irreducible components of the ramification locus (on X). Suppose that $\Delta(R)$ and $\Delta(B)$ consist of simple normal crossing divisors. Then $\pi^*\big(\Omega_Y(\log \Delta(B)) = \Omega_X(\log \Delta(R))\big)$. Recall that $\Delta(R)$ includes even those irreducible components of $\pi^{-1}(B)$ that are not themselves ramified.*

Proof The claim can be checked locally on Y. Suppose $x_1, \dots, x_n$ is a local system of coordinates at some point $\underline{x}$ of X, and $y_1, \dots, y_n$ is a local system of coordinates near $\underline{y} = \pi(\underline{x})$. We can suppose that $y_1, \dots, y_\ell$ defines the branch locus of π near $\underline{y}$ and further that $\pi^*(y_i) = x_i^{a_i}$, so that $x_1, \dots, x_\ell$ defines the ramification locus of π near $\underline{x}$ (see the proof of Lemma 2.16). Set $\epsilon_i = 1$ if $i \leq \ell$, and 0 otherwise. Then the bundle $\Omega_Y(\log \Delta(R))$ has a basis of sections near $\underline{y}$ given by

$$\frac{dy_1}{y_1^{\epsilon_1}}, \dots, \frac{dy_n}{y_n^{\epsilon_n}}.$$

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By the choice of ϵ_i we find that for all i,

$$\frac{d(\pi^* y)}{(\pi^* y)^{\epsilon_i}} = \frac{d(x_i^{a_i})}{x_i^{a_i \epsilon_i}} = a_i \frac{dx_i}{x_i^{\epsilon_i}}.$$

We find that $\pi^*(\Omega_Y(\log \Delta(R)))$ has a basis of sections near $\underline{x}$:

$$\frac{dx_1}{x_1^{\epsilon_1}}, \dots, \frac{dx_n}{x_n^{\epsilon_n}}.$$

This agrees precisely with the bundle $\Omega_X(\log \Delta(B))$ near $\underline{x}$. ∎

3 The Logarithmic Euler Characteristic

Aside from its present application to a Riemann–Hurwitz formula, the following definition is motivated in part by its appearance in [Mum77, Corollary 3.5].

Definition 3.1 Let X be a smooth projective variety and Δ be a collection of smooth divisors with simple normal crossings on X. We define the logarithmic Euler characteristic of a sheaf $\mathcal{F}$ on X with respect to the boundary Δ to be

$$\chi(X, \Delta, \mathcal{F}) = Q_n\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega^1_{\overline{X}}(\log \Delta)), \dots, c_n(\Omega^1_{\overline{X}}(\log \Delta))\big).$$

Although it is not *a priori* clear what use this definition can have, the following theorem shows that in some sense it behaves better than the standard Euler characteristic.

Theorem 3.2 *Let X and Y be smooth projective varieties. Consider a potentially ramified finite covering $\pi: X \to Y$ of degree μ. Let $\Delta(B)$ be the collection of irreducible components of the branch locus (on Y) and let $\Delta(R) = \pi^{-1}(\Delta(B))$ be the collection of (reduced) irreducible components of the ramification locus (on X). Suppose $\Delta(B)$ and $\Delta(R)$ have simple normal crossings. Let $\mathcal{F}$ be any coherent sheaf on Y.*

Then $\chi(X, \Delta(R), \pi^*(\mathcal{F})) = \mu \cdot \chi(Y, \Delta(B), \mathcal{F})$.

Proof By Theorem 2.18 (and functoriality) we have that

$$\pi^*\big(\operatorname{ch}(\mathcal{F})\operatorname{Todd}(\Omega^1_Y(\log \Delta(B)))\big) = \operatorname{ch}(\pi^*(\mathcal{F}))\operatorname{Todd}\big(\Omega^1_X(\log \Delta(R))\big).$$

The result then follows by recalling that the effect of pullback on the degree of a class is to multiply by μ. ∎

4 The Logarithmic Euler Characteristic vs the Euler Characteristic

The key to obtaining our results is the following comparison between the usual Euler characteristic and the logarithmic Euler characteristic we have just defined.

Theorem 4.1 *Let X be a smooth projective variety and let $\mathcal{F}$ be any coherent sheaf on X. Suppose Δ is a collection of smooth divisors with simple normal crossings on X. Then*

$$\begin{aligned}\chi(X,\mathcal{F}) &- \chi(X,\Delta,\mathcal{F})\\ &= \sum_{|\underline{b}|\geq 1}(-1)^{|\underline{b}|}\delta_{\underline{b}}D^{\underline{b}}Q_{n-|\underline{b}|}\big(\operatorname{ch}(\mathcal{F})\,;\,\mathrm{c}_1(\Omega^1_X(\log\Delta)),\dots,\mathrm{c}_{n-|\underline{b}|}(\Omega^1_X(\log\Delta))\big).\end{aligned}$$

(The notation $D^{\underline{b}}$ is from Notation 2.1 (9), the polynomial Q was defined in Theorem 2.2, and the constants $\delta_{\underline{b}}$ are from Notation 2.6 (1).)

Proof Recall that by Proposition 2.4 we have

$$Q_n(\operatorname{ch}(\underline{x})\,;\,y_1,\dots,y_n) = \sum_{\ell+m=n} Q_\ell(\operatorname{ch}(\underline{x})\,;\,u_1,\dots,u_\ell)Q_m(1\,;\,v_1,\dots,v_m).$$

In this context, if we set $x_i = \mathrm{c}_i(\mathcal{F})$, $u_i = \mathrm{c}_i(\Omega^1_X(\log\Delta))$, and $v_i = (-1)^i\Delta_i$, then, by Proposition 2.10, we have, in the setting of Proposition 2.4, that $y_i = \mathrm{c}_i(\Omega^1_X)$, and it follows that we can rewrite $Q_n(\operatorname{ch}(\mathcal{F})\,;\,\mathrm{c}_1(\Omega^1_X),\dots,\mathrm{c}_n(\Omega^1_X))$ as

$$\begin{aligned}\sum_{\ell+m=n} Q_\ell\big(\operatorname{ch}(\mathcal{F})\,;\,\mathrm{c}_1(\Omega^1_X(\log\Delta)),\dots,\mathrm{c}_\ell(\Omega^1_X(\log\Delta))\big)&\\ \times Q_m\big(1\,;\,(-1)^1\Delta_1,(-1)^2\Delta_2,\dots,(-1)^m\Delta_m\big)&.\end{aligned}$$

The result then follows from the observation that

$$Q_m\big(1\,;\,(-1)^1\Delta_1,(-1)^2\Delta_2,\dots,(-1)^m\Delta_m\big) = \sum_{|\underline{b}|=m}(-1)^{|\underline{b}|}\delta_{\underline{b}}D^{\underline{b}}. \qquad\blacksquare$$

Notation 4.2 If $\underline{a}$ is multiplicity free, so that $D^{\underline{a}}$ has no self intersections, then we may write $D^{\underline{a}} = \sum_j x_j[C_j]$, where $\big(\bigcap_{a_i\neq 0} D_i\big)_{\mathrm{red}} = \bigcup_j C_j$. In this setting we interpret $\chi(D^{\underline{a}},\Delta',\mathcal{F}|_{D^{\underline{a}}})$ to mean $\chi(D^{\underline{a}},\Delta',\mathcal{F}|_{D^{\underline{a}}}) = \sum_i m_i\chi(C_i,\Delta',\mathcal{F}|_{C_i})$, the weighted sum of the logarithmic Euler characteristics of the irreducible components of $D^{\underline{a}}$, the weights being precisely the intersection multiplicities. We interpret $\chi(D^{\underline{a}},\mathcal{F}|_{D^{\underline{a}}})$ similarly. Both of these expressions live most naturally on the disjoint unions of irreducible components of $D^{\underline{a}}$. Note that in the context of simple normal crossings, the intersection will already be reduced and the multiplicities m_i will all be 1.

When $\underline{a}$ is MF, by Proposition 2.11 we have that

$$\chi(D^{\underline{a}},\Delta',\mathcal{F}|_{D^{\underline{a}}}) = D^{\underline{a}}Q_{n-|\alpha|}\big(\operatorname{ch}(\mathcal{F})\,;\,\mathrm{c}_1(\Omega^1_X(\log\Delta)),\dots,\mathrm{c}_{n-|\alpha|}(\Omega^1_X(\log\Delta))\big),$$

when this expression is viewed as an equality on the disjoint union of the irreducible components of $D^{\underline{a}}$.

By an abuse of notation, we will extend this to the case where there may be self intersections, and let

$$\chi(D^{\underline{a}},\Delta',\mathcal{F}|_{D^{\underline{a}}}) = D^{\underline{a}}Q_{n-|\alpha|}\big(\operatorname{ch}(\mathcal{F})\,;\,\mathrm{c}_1(\Omega^1_X(\log\Delta)),\dots,\mathrm{c}_{n-|\alpha|}(\Omega^1_X(\log\Delta))\big),$$

even when a_i are potentially greater than 1, so that we may interpret $\chi(D^{\underline{a}},\Delta',\mathcal{F}|_{D^{\underline{a}}})$ as an object on X. This interpretation is compatible with the interpretation as a push-forward whenever $\underline{a}$ is MF.

Corollary 4.3 *With the same notation as in Theorem 4.1, if the irreducible components of Δ have trivial self intersection, then*

$$\chi(X,\mathcal{F})-\chi(X,\Delta,\mathcal{F})=\sum_{|\underline{b}|\geq 1}(-1)^{|\underline{b}|}\lambda_{\underline{b}}\chi(D^{\underline{b}},\mathcal{F}|_{D^{\underline{b}}}).$$

(The notation $D^{\underline{b}}$ is from 2.1 (9), the constants $\lambda_{\underline{b}}$ are from 2.6 (4).)

Proof In the above notation, Theorem 4.1 gives us that

$$\chi(X,\mathcal{F})-\chi(X,\Delta,\mathcal{F})=\sum_{|\underline{b}|\geq 1}(-1)^{|\underline{b}|}\delta_{\underline{b}}\chi(D^{\underline{b}},\Delta',\mathcal{F}|_{D^{\underline{b}}}).$$

As the same result allows us to compute $\chi(D^{\underline{b}},\Delta',\mathcal{F}|_{D^{\underline{b}}})-\chi(D^{\underline{b}},\mathcal{F}|_{D^{\underline{b}}})$ whenever $\underline{b}$ is MF, a recursive process will allow us to write

$$\chi(X,\mathcal{F})-\chi(X,\Delta,\mathcal{F})=\sum_{|\underline{b}|\geq 1}e_{\underline{b}}\chi(D^{\underline{b}},\mathcal{F}|_{D^{\underline{b}}}).$$

We must only show that $e_{\underline{b}}=(-1)^{\underline{b}}\lambda_{\underline{b}}$

The coefficient of $\chi(D^{\underline{b}},\mathcal{F}|_{D^{\underline{b}}})$ can be computed by explicitly writing out the result of the recursive process. The process will yield a sequence of formulas, indexed by ℓ, of the form

$$\begin{aligned}\chi(X,\mathcal{F})-\chi(X,\Delta,\mathcal{F})=&\sum_{k=1}^{\ell-1}(-1)^{k+1}\sum_{(\underline{b}_1,\ldots,\underline{b}_k)}\Big(\prod_{j=1}^{k}(-1)^{|\underline{b}_j|}\delta_{\underline{b}_j}\Big)\chi\Big(D^{\sum_{j=1}^{k}\underline{b}_j},\mathcal{F}|_{D^{\sum_{j=1}^{k}\underline{b}_j}}\Big)\\&+(-1)^{\ell+1}\sum_{(\underline{b}_1,\ldots,\underline{b}_\ell)}\Big(\prod_{j=1}^{\ell}(-1)^{|\underline{b}_j|}\delta_{\underline{b}_j}\Big)\chi\Big(D^{\sum_{j=1}^{k}\underline{b}_j},\Delta',\mathcal{F}|_{D^{\sum_{j=1}^{\ell+1}\underline{b}_j}}\Big)\end{aligned}$$

In the summations the elements of the tuples $(\underline{b}_1,\ldots,\underline{b}_k)$ always have disjoint support and $|\underline{b}_j|\geq 1$. We note that in the context of this corollary we need never consider any terms where $\underline{b}=\sum_{j=1}^{\ell}\underline{b}_j$ is NMF, since $D^{\underline{b}}$ vanishes for each such term.

The base case of the induction, case $\ell=1$, is precisely the statement of Theorem 4.1.

The formula for $\ell+1$ is obtained from that for ℓ by simply expanding every term

$$\begin{aligned}&\chi\Big(D^{\sum_{j=1}^{k}\underline{b}_j},\Delta',\mathcal{F}|_{D^{\sum_{j=1}^{\ell}\underline{b}_j}}\Big)\\&\quad=\chi\Big(D^{\sum_{j=1}^{\ell}\underline{b}_j},\mathcal{F}|_{D^{\sum_{j=1}^{\ell}\underline{b}_j}}\Big)-\sum_{\underline{c}}(-1)^{|\underline{c}|}\delta_{\underline{c}}\chi\Big(D^{\underline{c}+\sum_{j=1}^{k}\underline{b}_j},\Delta',\mathcal{F}|_{D^{\underline{c}+\sum_{j=1}^{\ell+1}\underline{b}_j}}\Big)\end{aligned}$$

with each term $\underline{c}$ in the summation, avoiding the support of $\sum_{j=1}^{k}\underline{b}_j$. This recursion terminates as soon as $\ell>n$, because then $D^{\underline{b}}$ is an intersection of more than n divisors, hence empty.

By regrouping terms on $\chi(D^{\underline{b}},\mathcal{F}|_{\underline{b}})$, we find that the coefficient of this term is precisely

$$(-1)^{|\underline{b}|}\lambda_{\underline{b}}=(-1)^{|\underline{b}|}\sum_{k\geq 0}\sum_{\substack{(\underline{b}_1,\ldots,\underline{b}_k)\\ \sum\underline{b}_j=\underline{b}}}\Big(\prod_{j=1}^{k}\delta_{\underline{b}_j}\Big).$$

In the summation we consider only terms with all $|\underline{b}_j| \geq 1$ and where in the tuple $(\underline{b}_1, \dots, \underline{b}_k)$ all of $\underline{b}_j$ have disjoint support. For k sufficiently large, the inner sum is an empty sum. ∎

Remark 4.4 The proofs of Theorem 4.1 and Corollary 4.3 work formally when we replace $Q(x_1, \dots, x_n\ ;\ y_1, \dots, y_n)$ by any other polynomial that is a multiplicative sequence in the y_i with respect to products of varieties and such that the x_j are functorial with respect to restriction.

We should also note that in light of Propositions 2.7 and 2.8, the coefficient $(-1)^{|\underline{b}|}\lambda_{\underline{b}}$ can be rewritten as $\delta^{|\underline{b}|}_{(1)}$ whenever $\underline{b}$ is MF (as in the Corollary 4.3 or 4.5). Also, by Proposition 2.9 the constants $\lambda_{\underline{b}}$ in Corollary 4.5 are typically 0 when $\underline{b}$ is NMF.

Corollary 4.5 *With the same notation as in Theorem 4.1, we have*

$$\chi(X, \mathcal{F}) - \chi(X, \Delta, \mathcal{F}) = \sum_{\substack{\underline{b}\mathrm{MF} \\ |\underline{b}|\geq 1}} (-1)^{|\underline{b}|}\lambda_{\underline{b}}\chi(D^{\underline{b}}, \mathcal{F}|_{D^{\underline{b}}}) + \sum_{\underline{b}\ \mathrm{NMF}} (-1)^{|\underline{b}|}\lambda_{\underline{b}}\chi(D^{\underline{b}}, \Delta', \mathcal{F}|_{D^{\underline{b}}}).$$

(The notation $D^{\underline{b}}$ is from 2.1 (9), the terminology MF and NMF are from Notation 2.6 (2), and the constants $\lambda_{\underline{b}}$ were introduced in Notation 2.6 (4).)

Proof The argument is the same as above, except that, rather than being able to completely ignore any NMF term that may appear, we simply include their contribution separately. The constant $\lambda_{\underline{b}}$ is defined precisely so as to count the appropriate weighted count of the number of possible factorizations of $D^{\underline{b}}$ in which terms have disjoint support and only the final term is potentially NMF. ∎

5 The Riemann–Hurwitz Theorem

In this section we establish our main result.

Theorem 5.1 *Consider a potentially ramified finite covering $\pi\colon X \to Y$ of degree μ between smooth projective varieties of dimension n. Let $\Delta(B)$ be the collection of irreducible components of the branch locus (on Y) and let$\Delta(R) = \pi^{-1}(\Delta(B))$ be the collection of (reduced) irreducible components of the ramification locus (on X). Let $\mathcal{F}$ be any coherent sheaf on Y. Suppose that $\Delta(R)$ and $\Delta(B)$ consist of simple normal crossing divisors. Then*

$$\chi(X, \pi^*(\mathcal{F})) - \mu \cdot \chi(Y, \mathcal{F}) = \sum_{\underline{a}} (-1)^{|\underline{a}|}\delta_{\underline{a}}(E_{R^{\underline{a}}} - 1)\chi\big(R^{\underline{a}}, \Delta(R)', \pi^*(\mathcal{F})\big).$$

(The notation $D^{\underline{b}}$ is from Notation 2.1 (9), the constants $\delta_{\underline{b}}$ are from Notation 2.6 (1), the notation $E_{R^{\underline{a}}}$ is from Notation 2.14, and the notation $\chi(R^{\underline{a}}, \Delta(R)', \pi^*(\mathcal{F}))$ is from Notation 4.2.)

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Proof First, by Theorem 4.1 we have

$$\chi(X,\pi^*(\mathcal{F}))-\mu\cdot\chi(Y,\mathcal{F})=\sum_{|\underline{a}|\geq 0}(-1)^{|\underline{a}|}\delta_{\underline{a}}\chi(R^{\underline{a}},\Delta(R)',\pi^*(\mathcal{F})|_{R^{\underline{a}}})$$
$$-\mu\Big(\sum_{|\underline{b}|\geq 0}(-1)^{|\underline{b}|}\delta_{\underline{b}}\chi(B^{\underline{b}},\Delta',\mathcal{F}|_{B^{\underline{b}}})\Big).$$

Next, by Theorem 3.2 we have that $\chi(X,\Delta(R),\pi^*(\mathcal{F}))=\mu\cdot\chi(Y,\Delta(B),\mathcal{F})$, so that these terms cancel out in the above expression.

With the remaining terms we can naturally group together those terms involving $\underline{a}$ and those with $\pi(\underline{a})=\underline{b}$ in the summation above. The error term arising from $\underline{a}$ in the expansion is

$$(-1)^{|\underline{a}|}(\mu\delta_{\underline{b}}\chi(B^{\underline{b}},\Delta(B)',\mathcal{F}))-\sum_{\pi(\underline{a})=\underline{b}}\delta_{\underline{a}}\chi(R^{\underline{a}},\Delta(R)',\pi^*(\mathcal{F})).$$

Next we observe that

$$\begin{aligned}
&\mu\cdot\chi(B^{\underline{b}},\Delta(B)',\mathcal{F}))\\
&\quad=\mu(B^{\underline{b}})Q_n\big(\operatorname{ch}(\mathcal{F});\mathrm{c}_1(\Omega_Y^i(\log\Delta(B))),\dots,\mathrm{c}_n(\Omega_Y^i(\log\Delta(B)))\big)\\
&\quad=\pi^*(B^{\underline{b}})Q_n\big(\operatorname{ch}(\mathcal{F});\mathrm{c}_1(\Omega_X^i(\log\Delta(B))),\dots,\mathrm{c}_n(\Omega_X^i(\log\Delta(B)))\big)\\
&\quad=\pi^*(B^{\underline{b}})Q_n\big(\operatorname{ch}(\pi^*(\mathcal{F}));\mathrm{c}_1(\Omega_X^i(\log\Delta(R))),\dots,\mathrm{c}_n(\Omega_X^i(\log\Delta(R)))\big).
\end{aligned}$$

By Lemma 2.17 we have that $\pi^*(B^{\underline{b}})=\sum_{\pi(\underline{a})=\underline{b}}E_{R^{\underline{a}}}R^{\underline{a}}$ and so we obtain

$$\mu\cdot\chi(B^{\underline{b}},\Delta(B)',\mathcal{F}))=\sum_{\pi(\underline{a})=\underline{b}}E_{R^{\underline{a}}}\chi(R^{\underline{a}},\Delta(R)',\pi^*(\mathcal{F})).$$

Grouping the terms on $\underline{a}$, we now immediately see that the contribution from the $\underline{a}$ terms is $(-1)^{|\underline{a}|}\delta_{\underline{a}}(E_{R^{\underline{a}}}-1)\chi(R^{\underline{a}},\Delta(R)',\pi^*(\mathcal{F})))$. Collecting these over all $\underline{a}$, we obtain the theorem. ∎

Using Propositions 5.3 and 5.4, the coefficients in the following corollary can be rewritten.

Corollary 5.2 *Consider a potentially ramified finite covering $\pi: X\to Y$ of degree μ between smooth projective varieties of dimension n. Let $\Delta(B)$ be the collection of irreducible components of the branch locus (on Y) and let $\Delta(R)=\pi^{-1}(\Delta(B))$ be the collection of (reduced) irreducible components of the ramification locus (on X). Let $\mathcal{F}$ be any coherent sheaf on Y. Suppose that $\Delta(R)$ and $\Delta(B)$ consist of simple normal crossing divisors. Then the difference $\chi(X,\pi^*(\mathcal{F})))-\mu\cdot\chi(Y,\mathcal{F})$ is equal to*

$$\sum_{\underline{a}\ \mathrm{MF}}(-1)^{|\underline{a}|}\Bigg(\sum_{\substack{\underline{a}'\leq\underline{a}\\|\underline{a}'|\geq 1}}(-\lambda_{\underline{a}-\underline{a}'}\delta_{\underline{a}'})(E_{R^{\underline{a}'}}-1)\Bigg)\chi(R^{\underline{a}},\pi^*(\mathcal{F})))$$
$$+\sum_{\underline{a}\ \mathrm{NMF}}(-1)^{|\underline{a}|}\Bigg(\delta_{\underline{a}}(E_{R^{\underline{a}}}-1)+\sum_{\substack{\underline{a}'\leq\underline{a}\\|\underline{a}'|\geq 1}}(-\lambda_{\underline{a}-\underline{a}'}\delta_{\underline{a}'})(E_{R^{\underline{a}'}}-1)\Bigg)\chi(R^{\underline{a}},\Delta(R)',\pi^*(\mathcal{F}))).$$

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(The notation $D^{\underline{b}}$ is from Notation 2.1 (9), the constants $\delta_{\underline{b}}$ are from Notation 2.6 (1), the terminology MF and NMF are from Notation 2.6 (2), the constants $\lambda_{\underline{b}}$ were introduced in Notation 2.6 (4), the notation $E_{R^{\underline{a}}}$ is from Notation 2.14, and the notation $\chi(R^{\underline{a}}, \Delta(R)', \pi^*(\mathcal{F}))$ is from Notation 4.2.)

Proof The proof is the same as that for Corollary 4.5. The terms $-\lambda_{\underline{a}-\underline{a}'}\delta_{\underline{a}'}(E_{R^{\underline{a}'}}-1)$ account for the contribution to the coefficient of $\chi(R^{\underline{a}}, \pi^*(\mathcal{F})))$ from the expansion of the terms $\chi(R^{\underline{a}'}, \Delta', \pi^*(\mathcal{F})))$ where $\underline{a}'$ is MF. The term $\delta_{\underline{a}}(E_{R^{\underline{a}}}-1)$ in the NMF case accounts for the contribution of the term which already appears in Theorem 5.1. ∎

Proposition 5.3 *If $\underline{a}$ is* MF *and $R^{\underline{a}} = R_1 \cdots R_k$, then*

$$(-1)^{|\underline{a}|} \sum_{\substack{\underline{a}' \leq \underline{a} \\ |\underline{a}'| \geq 1}} (-\lambda_{\underline{a}-\underline{a}'}\delta_{\underline{a}'})(E_{R^{\underline{a}'}} - 1) = \delta_{\underline{a}} \prod_{i=1}^{k} (1 - e_{R_i}).$$

Proof When $\underline{a}$ is MF, we have by Proposition 2.8 that $(-1)^{|\underline{a}|}\lambda_{\underline{a}-\underline{a}'}\delta_{\underline{a}'} = (-1)^{\underline{a}'}\delta_{\underline{a}}$. It follows that

$$(-1)^{|\underline{a}|} \sum_{\substack{\underline{a}' \leq \underline{a} \\ |\underline{a}'| \geq 1}} (-\lambda_{\underline{a}-\underline{a}'}\delta_{\underline{a}'})(E_{R^{\underline{a}'}} - 1) = \delta_{\underline{a}} \sum_{\substack{\underline{a}' \leq \underline{a} \\ |\underline{a}'| \geq 1}} (-1)^{|\underline{a}'|}(1 - E_{R^{\underline{a}'}}).$$

A direct computation yields that

$$\sum_{\substack{\underline{a}' \leq \underline{a} \\ |\underline{a}'| \geq 1}} (-1)^{\underline{a}'}(1 - E_{R^{\underline{a}'}}) = \prod_{i=1}^{k} (1 - e_{R_i})$$

from which the result follows. ∎

Proposition 5.4 *If $\underline{a}$ is* NMF *and $|\widehat{\underline{a}}| \geq 1$, then*

$$\delta_{\underline{a}}(E_{R^{\underline{a}}} - 1) + \sum_{\substack{\underline{a}' \leq \underline{a} \\ |\underline{a}'| \geq 1}} (-\lambda_{\underline{a}-\underline{a}'}\delta_{\underline{a}'})(E_{R^{\underline{a}'}} - 1) = \delta_{\underline{a}}(E_{R^{\underline{a}}} - E_{R^{\widehat{\underline{a}}}}).$$

Proof When $\underline{a}$ is NMF and $|\widehat{\underline{a}}| \geq 1$, the same will be true for $\underline{a} - \underline{a}'$ for all choices of $\underline{a}'$ except $\underline{a}' = \widehat{\underline{a}}$. We thus have by Proposition 2.9 that

$$\delta_{\underline{a}}(E_{R^{\underline{a}}} - 1) + \sum_{\substack{\underline{a}' \leq \widehat{\underline{a}} \\ |\underline{a}'| \geq 1}} (-\lambda_{\underline{a}-\underline{a}'}\delta_{\underline{a}'})(E_{R^{\underline{a}'}} - 1) = \delta_{\underline{a}}(E_{R^{\underline{a}}} - 1) - \lambda_{\underline{a}-\widehat{\underline{a}}}\delta_{\widehat{\underline{a}}}(E_{R^{\widehat{\underline{a}}}} - 1).$$

By noting that $\lambda_{\underline{a}-\widehat{\underline{a}}}\delta_{\widehat{\underline{a}}} = \delta_{\underline{a}}$, the result now follows immediately. ∎

6 Handling Self Intersections

The purpose of this section is to describe a method for interpreting the logarithmic Euler characteristic when there are self intersection terms. In particular we will show that these can be viewed as a weighted sum of the Euler characteristics of the components of the self intersection. The expressions one obtains are non-canonical, but may be amenable to computation depending on the context.

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In order to carry out the procedure outlined here, one needs to have a good understanding of the Chow ring of the variety X. In particular, the process may require a large number of relations consisting entirely of elements with simple normal crossings. The reason we need an alternate approach is that, although ideally we would be able to write $D^\ell c_i(\Omega_X(\log D)) = c_i(\Omega_{D^\ell})$, this is simply not true if $\ell > 1$. In order to handle this, we must have at least enough information to compute D^ℓ. In particular, we will need to make use of relations $D \sim \sum_i u_i E_i$ with the E_i not being equal to any other divisor already in use, and with the total collection E_i, D, and every other divisor in use having simple normal crossings.

Lemma 6.1 *Let X be a smooth projective variety and let $\mathcal{F}$ be any coherent sheaf on X. Suppose Δ is a collection of smooth divisors with simple normal crossings on X. Fix $D \in \Delta$ and a relation $D \sim \sum_{i\in I} u_i E_i$ with simple normal crossings as above. We can rewrite*

$$D^{\underline{a}} D^\ell Q_m\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta)),\dots,c_m(\Omega_X(\log\Delta))\big)$$

as

$$\begin{aligned} &D^{\underline{a}} D^{\ell-1}\sum_{k=1}^{m}(-1)^{k-1}\delta_{(k-1)}\\ &\quad\times\sum_i u_i E_i^k Q_{m-k+1}\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta\cdot E_i)),\dots,c_{m-k+1}(\Omega_X(\log\Delta\cdot E_i))\big). \end{aligned}$$

(The constant $\delta_{(k-1)}$ is from Notation 2.6.(1).)

Proof This follows immediately by a comparison between

$$Q_m\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta)),\dots,c_m(\Omega_X(\log\Delta))\big)$$

and

$$Q_m\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta\cdot E_i)),\dots,c_m(\Omega_X(\log\Delta\cdot E_i))\big)$$

as in Theorem 4.1. ■

Lemma 6.2 *Let X be a smooth projective variety and let $\mathcal{F}$ be any coherent sheaf on X. Let Δ be a collection of smooth divisors with simple normal crossings on X.*

Suppose we are given sufficiently many rules in the Chow ring of X of the form

$$D_j \sim \sum_{i\in I_{ja}} u_i E_i \quad\text{and}\quad E_j \sim \sum_{i\in I_{jb}} u_i E_i$$

expressed with respect to a collection of divisors E_i indexed by $I = \bigsqcup I_{ja}$, a universal family of shared indices, and such that the total collection of divisors D_i, E_j has simple normal crossings. Then we can rewrite

$$D^{\underline{a}} D^\ell Q_m\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta)),\dots,c_m(\Omega_X(\log\Delta))\big)$$

as a weighted sum of terms

$$D^{\underline{\widetilde{a}}} E^{\underline{b}} Q_{n-|\underline{\widetilde{a}}|-|\underline{b}|}\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta\cdot E^{\underline{b}})),\dots,c_{n-|\underline{\widetilde{a}}|-|\underline{b}|}(\Omega_X(\log\Delta\cdot E^{\underline{b}}))\big)$$

with $b_i \le 1$.

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Proof The key is to inductively apply the previous lemma. We observe that at each application of the lemma we produce new terms of the form $D^{\underline{a}'}E^{\underline{b}'}Q_{n-|\underline{a}'|-|\underline{b}'|}$. However, each new term introduced either satisfies that the number of self intersections has been decreased, or that the subscript on Q_m has decreased. It follows that the inductive process terminates provided we have enough rules to carry it out. ∎

Proposition 6.3 *In the setting of the lemma, the coefficient of*

$$D^{\widetilde{\underline{a}}}E^{\underline{b}}Q_m\big(\operatorname{ch}(\mathcal{F})\,;\,\mathrm{c}_1(\Omega_X(\log\Delta\cdot E^{\underline{b}})),\dots,\mathrm{c}_m(\Omega_X(\log\Delta\cdot E^{\underline{b}}))\big)$$

in the formal expansion of $D^{\underline{a}}Q_m\big(\operatorname{ch}(\mathcal{F});\mathrm{c}_1(\Omega_X(\log\Delta)),\dots,\mathrm{c}_m(\Omega_X(\log\Delta))\big)$ *is*

$$\prod_i (u_i^{b_i}\delta_{(y_i)}),$$

where $y_j = |\bigcup_z I_{jz}\cap\underline{b}|$, *that is,* y_j *is the number of rules that must be used in the expansion of* E_j.

Proof The appearance of the $\prod_i u_i^{b_i}\delta_{(y_i)}$ is apparent from the lemma, as these are precisely the terms that appear when we apply it. The only remaining question is the computation of y_i based on the shape of $\underline{b}$. One readily checks the given formula. ∎

The only information we still lack about our expansion is which $E^{\underline{b}}$ actually appear. This depends on choices made during the inductive process. However, if one orders the rules, one can obtain a systematic result. The following proposition is an immediate consequence of the inductive process.

Proposition 6.4 *Carrying out the inductive procedure as above, if the rules*

$$\text{(a)}\quad D_j\sim\sum_{i\in I_{ja}}u_iE_i\quad\textit{and}\quad\text{(b)}\quad E_j\sim\sum_{i\in I_{jb}}u_iE_i$$

are ordered by (a) *and* (b) *and we always select the first rule that does not conflict with choices already made, then the collection* $E^{\underline{b}}$ *appearing in the expansion is precisely that which satisfies the following.*

(i) $|\underline{b}\cap I_{jc}| = 0,1$.
(ii) *For each* D_j, *the number of* a *for which* $|\underline{b}\cap I_{ja}| = 1$ *is* $a_j - 1$.
(iii) $|\underline{b}\cap I_{jc}| = 1$ *and* $c>0$ *implies* $|\underline{b}\cap I_{jc-1}| = 1$.
(iv) $|\underline{b}|\le n-|\widetilde{\underline{a}}|$.

Remark 6.5 Because

$$D^{\widetilde{\underline{a}}}E^{\underline{b}}Q_m\big(\operatorname{ch}(\mathcal{F});\mathrm{c}_1(\Omega_X(\log\Delta\cdot E^{\underline{b}})),\dots,\mathrm{c}_m(\Omega_X(\log\Delta\cdot E^{\underline{b}}))\big)$$

is computing a logarithmic Euler characteristic on $D^{\underline{a}}E^{\underline{b}}$, the above expansion gives a weighted sum of the logarithmic Euler characteristics for some representative cycles for various $D^{\underline{x}}$. We note that $\prod_i u_i^{b_i}$ is somehow related to the coefficient that would have appeared had we been computing the self intersection, whereas the coefficient $\prod_i\delta_{(y_i)}$ is universal. Nonetheless, we note that this process involves a number of non-canonical choices.

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It is worth noting that by performing a further induction, as in Corollary 4.3, we could simply replace the logarithmic Euler characteristics with the actual Euler characteristics of the same components of the self intersections with different weights.

Example 6.6 Suppose we have relations

$$D \sim E_1 + E_2, \quad E_1 \sim E_3 \sim E_4, \quad E_2 \sim E_5 \sim E_6, \quad E_3 \sim E_7, \quad E_5 \sim E_8.$$

(Note that the implied relation $E_3 \sim E_4$ (respectively $E_5 \sim E_6$) is not being viewed as a rule for E_3 (respectively E_5). Then we may carry out the procedure above as follows.

$$\begin{aligned}
D^2 Q_2\big(&\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta)), c_2(\Omega_X(\log\Delta))\big)\\
&= DE_1 Q_2\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta\cdot E_1)), c_2(\Omega_X(\log\Delta\cdot E_1))\big)\\
&\quad + DE_1^2\delta_{(1)}Q_1\big(\operatorname{ch}(\mathcal{F}); c_1(\Omega_X(\log\Delta\cdot E_1))\big) + DE_1^3\delta_{(2)}Q_0(\operatorname{ch}(\mathcal{F})\,;\,\cdot)\\
&\quad + DE_2 Q_1\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta\cdot E_2)), c_2(\Omega_X(\log\Delta\cdot E_2))\big)\\
&\quad + DE_2^2\delta_{(1)}Q_1\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta\cdot E_2))\big) + DE_2^3\delta_{(2)}Q_0(\operatorname{ch}(\mathcal{F})\cdot)\\
&= DE_1 Q_2\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta\cdot E_1)), c_2(\Omega_X(\log\Delta\cdot E_1))\big)\\
&\quad + DE_1E_3\delta_{(1)}Q_1\big(\operatorname{ch}(\mathcal{F}); c_1(\Omega_X(\log\Delta\cdot E_1E_3))\big)\\
&\quad + DE_1E_3^2\delta_{(1)}\delta_{(1)}Q_0(\operatorname{ch}(\mathcal{F})\,;\,\cdot) + DE_1E_3E_4\delta_{(2)}Q_0(\operatorname{ch}(\mathcal{F})\,;\,\cdot)\\
&\quad + DE_2 Q_1\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta\cdot E_2)), c_2(\Omega_X(\log\Delta\cdot E_2))\big)\\
&\quad + DE_2E_5\delta_{(1)}Q_1\big(\operatorname{ch}(\mathcal{F})\,;\, c_1(\Omega_X(\log\Delta\cdot E_2E_5))\big)\\
&\quad + DE_2E_5^2\delta_{(1)}\delta_{(1)}Q_0(\operatorname{ch}(\mathcal{F})\,;\,\cdot) + DE_2E_5E_6\delta_{(2)}Q_0(\operatorname{ch}(\mathcal{F})\,;\,\cdot),
\end{aligned}$$

which we can ultimately express as

$$\begin{aligned}
\chi(&DE_1, \Delta', \mathcal{F}|_{DE_1}) + \delta_{(1)}\chi(DE_1E_3, \Delta', \mathcal{F}|_{DE_1E_3})\\
&+ \delta_{(1)}\delta_{(1)}\chi(DE_1E_3E_7, \Delta', \mathcal{F}|_{DE_1E_3E_7}) + \delta_{(2)}\chi(DE_1E_3E_4, \Delta', \mathcal{F}|_{DE_1E_3E_4})\\
&+ \chi(DE_2, \Delta', \mathcal{F}|_{DE_2}) + \delta_{(1)}\chi(DE_2E_5, \Delta', \mathcal{F}|_{DE_2E_5})\\
&+ \delta_{(1)}\delta_{(1)}\chi(DE_2E_5E_8, \Delta', \mathcal{F}|_{DE_2E_5E_8}) + \delta_{(2)}\chi(DE_2E_5E_6, \Delta', \mathcal{F}|_{DE_2E_5E_6}).
\end{aligned}$$

In particular we can express the result purely as a sum of logarithmic Euler characteristics.

7 Conclusions and Further Questions

We have obtained a natural generalization of the Riemann–Hurwitz results to the algebraic Euler characteristic. The formulas given are certainly more complicated than for the standard Euler characteristic.

It is natural to ask to what extent any of the results here can be generalized outside the context in which we are able to prove them.

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