# SOME REMARKS ON THE $p$-HOMOTOPY TYPE OF $B\Sigma_{p^2}$

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**Introduction.** Let $G$ be a finite group, $H$ a copy of its $p$-Sylow subgroup, and $N$ the normalizer of $H$ in $G$. A theorem by Nishida [**10**] states the $p$-homotopy equivalence of suitable suspensions of $BN$ and $BG$ when $H$ is abelian. Recently, in [**3**] the authors proved a stronger result: let $\Omega_k H$ be the subgroup of $H$ generated by elements of order $p^k$ or less; if

$$[H, \Omega_{k+1}H] \leq \Omega_k H \quad \text{for all} \quad k \geq 0,$$

then $BN$ and $BG$ are stably $p$-homotopy equivalent. The hypothesis above is obviously verified when $H$ is abelian. In the same paper the authors recall that $H$ does not verify such condition when $p = 2$ and $G = SL_2(F_q)$ for a suitable odd prime power $q$; in this case $BG$ and $BN$ are not stably 2-homotopy equivalent.

For any $p$, there is another relevant family of groups whose $p$-Sylow subgroups do not satisfy the condition above: the symmetric groups with non-abelian $p$-Sylow subgroups. When $H$ is in fact isomorphic to an iterated wreath product, not all of its elements or order $p$ lie in the center $Z(H)$, and it is natural to ask if $BG$ and $BN$ are however stably $p$-homotopy equivalent.

It is well known that for $G = \Sigma_{p^2}$, the symmetric group on $p^2$ elements, the answer is negative when $p = 2$ (see [**8**]).

To prove that the answer is also negative for any odd prime $p$, we use Morava $K$-theories $K(n)^*(-)$, and the group-theoretical significance of the rank of $K(n)^*(BN)$. In fact we prove that

$$K(1)^*(Bi): K(1)^*(B\Sigma_{p^2}) \to K(1)^*(BN)$$

is not an isomorphism, since the rank of the latter space is bigger.

To obtain a complete stable splitting of $BN$, and estimate the role played by $B\Sigma_{p^2}$ as a stable summand of $BN$, one has to use one prime at a time the tools described in [**1**] and in [**7**]. However our results tests how this role "decreases" when $p$ grows. Notice also that we solve a purely algebraic problem (finding a suitable lower bound to the number $\chi_{1,p}$ of conjugacy classes in $N$ containing elements of order a power of $p$) by using topology; an alternative approach could be the method described in [**4**] to calculate $\chi_{1,p}$ for any group. In such an outlook one should study the lattice of abelian subgroups of $N$ (which is huge even for relatively small prime numbers), and evaluate a Moebius function defined on it on every subgroup having a non-trivial intersection with the center $Z(N)$. Our line of attack avoids such ugly calculations.

**1. Preliminaries on wreath products.** We recall in this section various facts concerning wreath products. In the old but comprehensive [**9**] the reader will find a detailed account on their basic properties.

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The minimum integer $m$ such that the group $\Sigma_m$ has a non-abelian $p$-Sylow subgroup $H$ is $p^2$. In this case $H$ is isomorphic to the wreath product $C_p \wr C_p$ which is the central term of a splitting extension

$$1 \to (C_p)^p \to H \to C_p \to 1.$$

Let $a$ denote a fixed generator for $C_p$. Identifying the elements of $H$ with the $(p+1)$-tuples

$$(a^{i_1}, a^{i_2}, \dots, a^{i_p}; a^h),$$

the group law in $H$ becomes

$$(a^{i_1}, a^{i_2}, \dots, a^{i_p}; a^h) . (a^{j_1}, a^{j_2}, \dots, a^{j_p}; a^k) = (a^{i_1 + j_{p-h+1}}, \dots, a^{i_p + j_{p-h}}; a^{h+k})$$

where indices of exponents have to be read "mod $p$". For our purposes it would be enough to show that $(C_p)^p$ is characteristic in $H$. When $p$ is odd we have actually a stronger result:

LEMMA 1.1. *Let $p$ be an odd prime. The group $H$ has only one subgroup isomorphic to $(C_p)^p$.*

*Proof.* It is an amusing exercise in group theory. Let $M$ be the subgroup of $H$ formed by elements

$$(a^{i_1}, a^{i_2}, \dots, a^{i_p}; 1),$$

and suppose there exists a subgroup $L$ isomorphic but not equal to $M$. Since

$$|L \cap M| = p^{p-1} > p = |Z(H)|,$$

the set $S = L \cap M \setminus Z(H)$ is not empty. An element

$$h = (a^{i_1}, a^{i_2}, \dots, a^{i_p}; 1) \in S$$

centralizes

$$k = (a^{j_1}, a^{j_2}, \dots, a^{jp}; a^j) \in L \setminus M$$

and all its powers, since $L$ is abelian. But this is possible only if

$$i_1 = i_2 = \dots = i_p,$$

since $j > 0$; therefore $h$ belongs to $Z(H)$, against our hypothesis. □

Let $g$ be an element in $N$, the normalizer of $H$ in $\Sigma_{p^2}$, and let $c_g$ denote the conjugation in $H$ through $g$. Related to the short exact sequence above there is a fibration of $CW$-complexes

$$(BC_p)^p \to BH \to BC_p.$$

COROLLARY 1.2. *The homeomorphism $Bc_g: BH \to BH$ is a fiber preserving map.*

*Proof.* When $p$ is odd the thesis is an immediate consequence of Lemma 1.1. When $p = 2$ use the fact that in each copy of $C_2 \wr C_2$ in $\Sigma_4$ there are only two 2-cycles. □

The structure of $K(n)^*(BH)$ is detected by studying the spectral sequence

$$E_2 = H^*(BC_p; K(n)^*((BC_p)^p)) \Rightarrow K(n)^*(BH).$$

The group $\pi_1(BC_p) \cong C_p$ acts on

$$K(n)^*((BC_p)^p) = K(n)^*[u_1, u_2, \dots, u_p]/(u_1^{p^n}, \dots, u_p^{p^n})$$

permuting generators. As a module over $C_p$

$$K(n)^*((BC_p)^p) = F \oplus T$$

where $F$ is a free $C_p$-module and $T$ has trivial $C_p$-action. We have (see [5]):

$$K(n)^*(BH) \cong H^0(BC_p; F) \oplus (T \otimes K(n)^*(BC_p))$$

We call the elements belonging to the first summand "elements of type I", and "elements of type II" those which belong to the second one. Elements of type I have a basis formed by elements

$$\sum_{i=0}^{p-1} a^i(u_1^{i_1} \dots u_p^{i_p}),$$

with $i_h \neq i_k$ for some $h$ and $k$. Elements of type II, which are in

$$T \otimes K(n)^*(BC_p) \cong T \otimes K(n)^*[x]/(x^{p^n})$$

have instead the form $u_1^h \dots u_p^h \otimes x^k$.

**2. A lower bound for the rank of $K(n)^*(BN)$.** It was proved in [6] that the rank of $K(1)^*(BG)$ as $K(1)^*$-module is given by the number $\chi_{1,p}(G)$ of conjugacy classes in $G$ represented by elements of order a power of $p$. It is easily seen that $\chi_{1,p}(\Sigma_{p^2})$ is $p+2$: the $r$-th conjugacy class contains those elements that can be written as a product of $r-1$ disjoint $p$-cycles; the last class contains all the $p^2$-cycles.

PROPOSITION 2.1. *Let $N$ be the normalizer of a $p$-Sylow subgroup $H$ of $\Sigma_{p^2}$. The $K(1)^*$-rank of $K(1)^*(BN)$ is strictly bigger than $p+2$.*

*Proof.* Let $S(p, G)$ be the set of conjugacy classes in $G$ of elements of $p$-power order. If $H$ is a $p$-Sylow subgroup of $G$, then the inclusions of $H$ in $N$, and of $N$ in $G$ induce the following maps:

$$\theta_{H\to N}: S(p, H) \to S(p, N) \quad \text{and} \quad \theta_{N\to G}: S(p, N) \to S(p, G).$$

Notice that the composition of the two maps is surjective by one of Sylow's thoerems, therefore the second of them is surjective. Let $G = \Sigma_{p^2}$. It will be enough to show that in the case at hand the map $\theta_{N-\Sigma_{p2}}$ is not injective. Following notations introduced in the previous section, consider the following elements of $H \subseteq N$

$$(a, a, \dots, a; 1) \quad \textit{and} \quad (1, 1, \dots, 1; a).$$

They are represented in $\Sigma_{p^2}$ by two products of $p$ disjoint $p$-cycles, therefore they are conjugate through an element $\sigma \in \Sigma_{p^2}$. A direct analysis shows that the subgroup of $H$ isomorphic to $(C_p)^p$ is not mapped onto itself by conjugation through $\sigma$. By Lemma 1.1, this is sufficient to prove that $\sigma$ does not belong to $N$. □

As a consequence of Proposition 2.1 the map

$$K(1)^*(Bi): K(1)^*(B\Sigma_{p^2}) \to K(1)^*(BN)$$

has to be a strict monomorphism, hence $B\Sigma_{p^2}$ and $BN$ are not even stably homotopy equivalent. As before, let $H$ denote the wreath product $C_p \wr C_p$, and $W$ the group $N/H$. The group $N$ is the semi-direct product of $H$ and $W$, since these two groups have coprime order. This time the group $W \cong \pi_1(BW)$ acts on $BH$ and then on $K(n)^*(BH)$.

PROPOSITION 2.2. *$K(n)^*(BN)$ can be identified with the subring of $K(n)^*(BH)$ of the invariants under the action of $W$.*

*Proof.* We look at the $E_2$ term of the spectral sequence

$$E_2^{s,t} = H^s(BW; K(n)^t(BH)) \Rightarrow K(n)^{s+t}(BN).$$

We have

$$E_2^{s,t} = \begin{cases} (K(n)^s(BH))^W & \text{if } s = 0 \\ 0 & \text{otherwise,} \end{cases}$$

since the order of $W$ is prime to $p$. □

To proceed in the description of $K(n)^*(BN)$, we have to understand how an element $w \in W$ acts on $K(n)^*(BH)$. Lemma 1.1 states that conjugation by $w$ has to map the subgroup $(C_p)^p$ onto itself. If we denote by $a_j$ the generator of the $j$-th copy of $C_p$ in the cartesian product, and simply by $a$ the generator of $H/(C_p)^p$, the action of $c_w$ on the $a_j$'s and on $a$ determines $c_w(g)$ for any other $g \in N$. Since a $p$-cycle in $\Sigma_{p^2}$ goes to another $p$-cycle under conjugation, the restriction of $c_w$ on $(C_p)^p$ can be seen as an element of $\mathrm{Aut}(C_p) \wr \Sigma_p$, and for any $j$,

$$c_w(a_j) = a_{\sigma(j)}^{k_j},$$

where $\sigma$ is an element in $\Sigma_p$ and $k_j \not\equiv 0 (\bmod p)$.

PROPOSITION 2.3. *Let $w$ be an element in $W$. Suppose that*

$$c_w(a_j) = a_{\sigma(j)}^{k_j} \quad \text{and} \quad c_w(a) = a^k,$$

*then $(Bc_w)^*$ acts on the two types of elements in $K(n)^*(BH)$ as follows*

$$\sum_{i=0}^{p-1} a^i(u_1^{i_1} \dots u_p^{i_p}) \to \sum_{i=0}^{p-1} a^i((k_1 u_{\sigma(1)})^{i_1} \dots (k_p u_{\sigma(p)})^{i_p})$$

*and*

$$u_1^j u_2^j \dots u_p^j \otimes x^h \to (k_1 \dots k_p)^j k^h (u_1^j u_2^j \dots u_p^j \otimes x^h).$$

*Proof.* The key-point is that an element of $\mathrm{Aut}(C_p)$

$$f : a \in C_p \to a^k \in C_p$$

induces in Morava $K$-theories the following automorphism

$$(Bf)^* : x \in K(n)^*[x]/(x^{p^n}) \mapsto [k]_F x \in K(n)^*[x]/(x^{p^n}),$$

and it is shown in [2] that $[k]_F x = kx \in K(n)^*(BC_p)$ whenever $k \not\equiv 0 (\text{mod } p)$. $\square$

COROLLARY 2.4. *In $K(n)^*(BN)$ there is a subalgebra $A_n$ generated by elements*

$$\sum_{i=0}^{p-1} a^i . (u_1^{l_1} \dots u_p^{l_p}) \quad \text{and} \quad u_1^m u_2^m \dots u_p^m \otimes x^h$$

*where*

$$l_j, m, h \equiv 0(\text{mod}(p-1))$$

*for every j.*

The reader could ask if the subalgebra $A_n$ spans $K(n)^*(BN)$. The answer is in general negative: take $p = 3$, the element

$$u_1^2 u_2 u_3 + u_1 u_2^2 u_3 + u_1 u_2 u_3^2 \in K(n)^*(B(C_3 \wr C_3))$$

is invariant under the action of $W$, but is not in $A_n$. The rank of $K(n)^*(BN)$ grows exponentially with $n$, and we can state the following.

COROLLARY 2.5. *For every $n$, the rank of $K(n)^*(BN)$ is greater than*

$$\frac{2^p - 2}{p} + 3.$$

*Therefore N has at least $(2^p - 2)/p + 4$ conjugacy classes represented by elements having order a power of p.*

*Proof.* The maximal number of independent elements of type I in $A_1$ is

$$\frac{1}{p}\left(\binom{p}{1} + \binom{p}{2} + \dots + \binom{p}{p-1}\right) = \frac{2^p - 2}{p},$$

and we find also four independent elements of type II:

$$1, \qquad x^{p-1}, \qquad 1 \otimes u_1^{p-1} \dots u_p^{p-1}, \quad \text{and} \quad x^{p-1} \otimes u_1^{p-1} \dots u_p^{p-1}. \qquad \square$$

As a final remark we notice that $C_2 \wr C_2$ is isomorphic to its normalizer in $\Sigma_4$ and $\chi_{1,2}(C_2 \wr C_2) = 5$, therefore the number $(2^p - 2)/p + 3$ is the best possible lower bound for $\chi_{1,p}(N)$ which holds for any $p$.

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