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ACTIONS OF LIE SUPERALGEBRAS ON SEMIPRIME ALGEBRAS WITH CENTRAL INVARIANTS

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Abstract. Let R be a semiprime algebra over a field $\mathbb{K}$ of characteristic zero acted finitely on by a finite-dimensional Lie superalgebra $L = L_0 \oplus L_1$. It is shown that if L is nilpotent, $[L_0, L_1] = 0$ and the subalgebra of invariants R^L is central, then the action of L_0 on R is trivial and R satisfies the standard polynomial identity of degree $2 \cdot [\sqrt{2^{\dim_{\mathbb{K}} L_1}}]$. Examples of actions of nilpotent Lie superalgebras, with central invariants and with $[L_0, L_1] \neq 0$, are constructed.

1. Preliminaries. If R is an algebra over a field $\mathbb{K}$ of characteristic $\neq 2$ and σ is a $\mathbb{K}$-linear automorphism of R such that $\sigma^2 = 1$, let $D_0 = \{\delta \in \mathrm{End}_{\mathbb{K}}(R) \mid \delta(rs) = \delta(r)s + r\delta(s) \text{ and } \delta\sigma(r) = \sigma\delta(r) \text{ for all } r, s \in R\}$ and $D_1 = \{\delta \in \mathrm{End}_{\mathbb{K}}(R) \mid \delta(rs) = \delta(r)s + \sigma(r)\delta(s) \text{ and } \delta\sigma(r) = -\sigma\delta(r) \text{ for all } r, s \in R\}$. Then $D_0 \oplus D_1$ is a Lie superalgebra and the elements of D_0 and D_1 are respectively, derivations and skew derivations of R. The superbracket on $D_0 \oplus D_1$ is defined as $[\delta_1, \delta_2] = \delta_1\delta_2 - (-1)^{ij}\delta_2\delta_1$, where $\delta_1 \in D_i, \delta_2 \in D_j$ and $i, j \in \{0, 1\}$. If $L = L_0 \oplus L_1$ is a Lie superalgebra, we say that L acts on R if there is a homomorphism of Lie superalgebras $\psi : L \to D_0 \oplus D_1$, where $\psi(L_i) \subseteq D_i$, for $i = 0, 1$. Throughout the paper we will simply assume that $L \subseteq D_0 \oplus D_1$ identifying the elements of L_0 and L_1 with their images under ψ. It is well known that the homomorphism ψ induces an associative homomorphism from the universal enveloping algebra $U(L)$ to $\mathrm{End}_{\mathbb{K}}(R)$ and its image is finite dimensional if and only if the derivations and skew derivations from L_0 and L_1 are algebraic. In this case we will say that L acts **finitely** on R. Letting G be the group $\{1, \sigma\}$, we can form the skew group ring $H = U(L) * G$ and H is now a Hopf algebra acting on R. When L acts on R, we define the subalgebra of invariants R^L to be the set $\{r \in R \mid \delta(r) = 0, \text{ for all } \delta \in L\}$. Depending upon the context, the symbol [,] may represent either the superbracket on L, or the commutator map $[r, s] = rs - sr$, where r, s belong to an associative algebra. Inductively, we let $L^1 = L$ and $L^{n+1} = [L^n, L]$ and we say that L is nilpotent if there exists a positive integer N such that $L^N = 0$. If R (resp. L) is an associative algebra (resp. Lie superalgebra) we will let $\mathcal{Z}(R)$ (resp. $\mathcal{Z}(L)$) denote its centre. For an element $a \in R$, and automorphism σ of R, ad_a (resp. ∂_a) stands for the inner derivation (inner σ-derivation) adjoint to a, i.e. $\mathrm{ad}_a(x) = ax - xa$ ($\partial_a(x) = ax - \sigma(x)a$).

2. Main result. The main aim of this paper is to prove the following theorem.

THEOREM 1. *Let a finite-dimensional nilpotent Lie superalgebra $L = L_0 \oplus L_1$ acts finitely on a semiprime $\mathbb{K}$-algebra R, where $\mathbb{K}$ is a field of characteristic zero. If R^L is central and $[L_1, L_0] = 0$, then R satisfies the standard polynomial identity of degree* $2 \cdot [\sqrt{2^{\dim_{\mathbb{K}} L_1}}]$.

It generalizes a result from [**1**] concerning the actions of nilpotent Lie algebras of characteristic zero on semiprime algebras. On the other hand, in [**4**] it is proved that if a pointed Hopf algebra H acts finitely of dimension N on a semiprime algebra R and the action is such that $L^H \neq 0$ for any non-zero H-stable left ideal L of R and $R^H \subseteq \mathcal{Z}(R)$, then R satisfies PI of degree $2[\sqrt{N}]$. In Theorem 1 we prove for nilpotent Lie superalgebras with $[L_0, L_1] = 0$, that the dimension of the action of $U(L) * G$ depends only on the dimension of L_1. The key role will be played by the following easy observation: *In characteristic zero the invariants of nilpotent Lie algebras acting on central simple algebras are never proper simple central subalgebras.*

LEMMA 2. *Let R be a finite-dimensional central simple $\mathbb{F}$-algebra acted on by a nilpotent Lie $\mathbb{F}$-algebra L, where $\mathbb{F}$ is a field of characteristic zero. If R^L is a central simple $\mathbb{F}$-algebra, then $R = R^L$. In this case the action of L on R must be trivial.*

Proof. Since L acts by $\mathbb{F}$-linear transformations, any derivation from L is inner. Suppose that the action of L on R is not trivial. Then we can take a non-zero derivation $\delta = \mathrm{ad}_a \in \mathcal{Z}(L)$, where $a \in R$. For any $\mathrm{ad}_b \in L$ we have $\mathrm{ad}_{[a,b]} = [\mathrm{ad}_a, \mathrm{ad}_b] = 0$, so $[a, b] \in \mathcal{Z}(R) = \mathbb{F}$. If $[a, b] = \lambda \neq 0$, then $[a, \lambda^{-1}b] = 1$. Note that the elements a and $\lambda^{-1}b$ generate in R a subalgebra isomorphic to the Weyl algebra $\mathbb{A}_1(\mathbb{F})$, but it is impossible since R is finite dimensional. Consequently, $[a, b] = 0$ for any $\mathrm{ad}_b \in L$ and hence $a \in R^L$. In particular, ad_a acts trivially on $C_R(R^L)$, the centralizer of R^L in R. On the other hand the subalgebra R^L is simple and $\mathcal{Z}(R^L) = \mathbb{F}$, so by Theorem 2 (p. 118) in [**5**] $R \simeq R^L \otimes_{\mathbb{F}} C_R(R^L) \simeq R^L \cdot C_R(R^L)$. Consequently, $R = R^L \cdot C_R(R^L)$. It implies that ad_a acts trivially on R, a contradiction. Therefore the action of L on R is trivial. □

Suppose that a finite-dimensional nilpotent Lie superalgebra $L = L_0 \oplus L_1$ acts finitely of dimension N on an algebra R. Then by R_{L_0} we denote the largest subspace of R on which any derivation from L_0 acts nilpotently, that is

$$R_{L_0} = \{r \in R \mid \delta^N(r) = 0, \;\; \forall \delta \in L_0\}.$$

It is clear that R_{L_0} is a subalgebra of R and R_{L_0} is stable under the automorphism σ. Furthermore, it is well known that (after eventual extension of the field of scalars) the algebra R is graded (with finite support) by the dual of the Lie algebra L_0 with R_{L_0} as the identity component of the grading. Therefore, if the algebra R is semiprime (semisimple), then R_{L_0} is also semiprime (resp. semisimple). In [**3**] (Lemma 12) it is proved that

LEMMA 3. *The subalgebra R_{L_0} is L-stable. In particular, L acts on R_{L_0} by nilpotent transformations.*

In the next Proposition we consider the case of action of a nilpotent Lie superalgebra on a finite-dimensional G-simple algebra.

PROPOSITION 4. *Let a nilpotent Lie superalgebra $L = L_0 \oplus L_1$ acts on a G-simple finite-dimensional $\mathbb{K}$-algebra R, where $\mathbb{K}$ is a field of characteristic zero. If R^L is central and $[L_0, L_1] = 0$, then $L_0 = 0$.*

Proof. First, we will consider the case when L acts on R by nilpotent transformations, that is, $R = R_{L_0}$. Suppose that $L_0 \neq 0$ and take a non-zero derivation δ from the centre of L_0. Since $[L_0, L_1] = 0$, it is clear that δ is in the centre of L. Let $k > 1$ be such that $\delta^k(R) = 0$ and $V = \delta^{k-1}(R) \neq 0$. Then V is invariant under the action of L, and since L acts via nilpotent transformations it is clear that $V^L = V \cap R^L \subseteq \mathcal{Z}(R)$. On the other hand if $r, s \in R$, then the Leibniz rule gives

$$0 = \delta^k(\delta^{k-2}(r)s) = k\delta^{k-1}(r)\delta^{k-1}(s).$$

It means that $(V^L)^2 = 0$, so the centre of R contains nilpotent elements. This is impossible since R is semisimple. The obtained contradiction shows that $L_0 = 0$.

Consider the general case. The above gives us immediately that $R^{L_0} = R_{L_0}$ and consequently the algebra R^{L_0} is semisimple. Thus, any its ideal I is idempotent, i.e. $I^2 = I$. Note that if I is G-stable, then the Leibniz rule, applied to any $\partial \in L_1$, gives $\partial(I) = \partial(I^2) \subseteq \partial(I)I + \sigma(I)\partial(I) \subseteq I$. Hence any G-stable ideal I of R^{L_0} is also L-stable and $0 \neq I^L \subseteq \mathcal{Z}(R)$. Thus I contains invertible elements. Consequently, R^{L_0} is also G-simple.

We will split considerations into two cases. First, suppose that the automorphism σ is inner, and let $q \in R$ be such that $\sigma(x) = q^{-1}xq$, for $x \in R$. In this case any ideal of R is σ-stable, so R must be a simple algebra. Moreover it is easy to see that any skew derivation ∂ from L_1 must be inner. Indeed, since $\partial\sigma = -\sigma\partial$, we obtain that

$$\begin{aligned} q^{-1}\partial(x)q &= \sigma(\partial(x)) = -\partial(\sigma(x)) = -\partial(q^{-1}xq) \\ &= -\partial(q^{-1})xq - q^{-1}\partial(x)q - q^{-1}\sigma(x)\partial(q). \end{aligned}$$

Since $q\partial(q^{-1}) = -\partial(q)q^{-1}$,

$$\partial(x) = -\frac{1}{2}q\partial(q^{-1})x - \frac{1}{2}\sigma(x)\partial(q)q^{-1} = \frac{1}{2}\partial(q)q^{-1}x - \sigma(x)\frac{1}{2}\partial(q)q^{-1}.$$

This immediately gives, that $\partial(x) = bx - \sigma(x)b$, where $b = \frac{1}{2}\partial(q)q^{-1}$. Consequently, any mapping from $L_0 \cup L_1$ is $\mathcal{Z}(R)$-linear. We will show that the algebra R^{L_0} is simple and the centres of R^{L_0} and R coincide. Since the automorphism σ has order 2, $q^2 \in \mathcal{Z}(R)$. Thus for any $\delta = \mathrm{ad}_a \in L_0$,

$$\delta(q) = \delta(\sigma(q)) = \sigma(\delta(q)) = q^{-1}(aq - qa)q = qa - aq = -\delta(q),$$

so $\delta(q) = 0$. This implies that $q \in R^{L_0}$, the restriction of σ to R^{L_0} is inner and hence the algebra R^{L_0} is simple. Since the action of L on R is inner, $\mathcal{Z}(R) = \mathcal{Z}(R) \cap R^{L_0} \subseteq \mathcal{Z}(R^{L_0})$. We will show that $\mathcal{Z}(R^{L_0}) \subseteq \mathcal{Z}(R)$. To this end, since $R^L \subseteq \mathcal{Z}(R)$, it suffices to show that $\mathcal{Z}(R^{L_0}) \subseteq R^{L_1}$. Take any $z \in \mathcal{Z}(R^{L_0})$, and $\partial = \partial_b \in L_1$, where $b = \frac{1}{2}\partial(q)q^{-1}$. Notice that $b \in R^{L_0}$. Indeed, by assumption $[\delta, \partial] = 0$ for any $\delta \in L_0$ and by the above $q \in R^{L_0}$, so

$$\delta(b) = \frac{1}{2}\delta(\partial(q)q^{-1}) = \frac{1}{2}\delta(\partial(q))q^{-1} + \frac{1}{2}\partial(q)\delta(q^{-1}) = \frac{1}{2}\partial(\delta(q))q^{-1} = 0.$$

It means that $b \in R^{L_0}$ and

$$\partial(z) = bz - \sigma(z)b = bz - zb = 0,$$

so $z \in R^{L_1}$. It proves that $\mathcal{Z}(R^{L_0}) = \mathcal{Z}(R)$. By Lemma 2 the action of L_0 on R must be trivial.

Finally, suppose that the automorphism σ is outer. Since R is G-simple, the algebra R must be either simple or $R = I \oplus \sigma(I)$ for some minimal ideal I. In the first case, by the Skolem–Noether theorem, σ is not an identity map on $\mathcal{Z}(R)$. In the second case $\mathcal{Z}(R) = \mathcal{Z}(I) \oplus \sigma(\mathcal{Z}(I))$. Thus in both cases σ acts non identically on $\mathcal{Z}(R)$. Now since the centre of R^{L_0} contains $\mathcal{Z}(R)$, the restriction of σ to R^{L_0} is also outer. Consequently, one can choose a non-zero element $c \in \mathcal{Z}(R)$ such that $\sigma(c) \neq c$. Then $(c - \sigma(c))^2$ is non-zero and belongs to the field $\mathcal{Z}(R)^\sigma$. Thus $c - \sigma(c)$ is invertible. Now let $\partial \in L_1$ and $x \in R$. Notice that

$$\partial(x)c + \sigma(x)\partial(c) = \partial(xc) = \partial(cx) = \partial(c)x + \sigma(c)\partial(x).$$

In particular, we have

$$\partial(x) = (c - \sigma(c))^{-1}\partial(c)x - \sigma(x)(c - \sigma(c))^{-1}\partial(c) = \partial_b(x),$$

where $b = (c - \sigma(c))^{-1}\partial(c)$. Thus L_1 acts on R via inner σ-derivations and in, particular, every mapping from L is $\mathcal{Z}(R)^\sigma$-linear. We will prove that $\mathcal{Z}(R^{L_0})^\sigma = \mathcal{Z}(R)^\sigma$. Similarly as above, it suffices to show that $\mathcal{Z}(R^{L_0})^\sigma \subseteq R^{L_1}$. Take any $\partial = \partial_b \in L_1$, where $b = (c - \sigma(c))^{-1}\partial(c)$ for some $c \in \mathcal{Z}(R)$. Since L_0 acts trivially on the centre of R, one obtains that $b \in R^{L_0}$. Now it is clear that ∂_b acts trivially on $\mathcal{Z}(R^{L_0})^\sigma$, and consequently $\mathcal{Z}(R^{L_0})^\sigma \subseteq R^{L_1}$.

Consider skew group rings $R * G$ and $R^{L_0} * G$. Since both of R and R^{L_0} are G-simple, and σ is outer on R and R^{L_0}, the rings $R * G$ and $R^{L_0} * G$ are simple. Moreover it is clear that $\mathcal{Z}(R * G) = \mathcal{Z}(R)^\sigma$ and $\mathcal{Z}(R^{L_0} * G) = \mathcal{Z}(R^{L_0})^\sigma$. Thus $R * G$ and $R^{L_0} * G$ are central simple $\mathcal{Z}(R)^\sigma$-algebras. Notice that the action of L_0 on R can be extended to an action on $R * G$, via the formula $\delta(a + b\sigma) = \delta(a) + \delta(b)\sigma$. In that case $(R * G)^{L_0} = R^{L_0} * G$ Again applying Lemma 2 we obtain that L_0 must act trivially on R and the proof is complete. □

Corollary 5. *Let a nilpotent Lie superalgebra $L = L_0 \oplus L_1$ acts on a G-simple finite-dimensional $\mathbb{K}$-algebra R with centre $\mathcal{Z}$, where char $\mathbb{K} = 0$. If $R^L \subseteq \mathcal{Z}$ and $[L_0, L_1] = 0$, then $\dim_{\mathcal{Z}^G} R \leq [\mathcal{Z} : \mathcal{Z}^G] \cdot 2^{\dim_{\mathbb{K}} L_1}$. Moreover, in this case R satisfies the standard polynomial identity of degree $2 \cdot [\sqrt{2^{\dim_{\mathbb{K}} L_1}}]$.*

Proof. By Proposition 4, $L_0 = 0$. Thus L is spanned by a family $\{\partial_1, \ldots, \partial_n\}$ of inner skew derivations such that $\partial_j^2 = 0$ and $\partial_i\partial_j + \partial_j\partial_i = 0$. It is clear that every ∂_j is $\mathcal{Z}^G$-linear. Let us consider a chain

$$V_0 = R \supseteq V_1 \supseteq \cdots \supseteq V_n$$

of subspaces of R, where $V_j = \ker \partial_1 \cap \cdots \cap \ker \partial_j$ for $j = 1, \ldots, n$. Then $V_n \subseteq R^L \subseteq \mathcal{Z}$ and ∂_j maps V_{j-1} into V_j. Moreover, it is clear that $\dim_{\mathcal{Z}^G} V_{j-1} = \dim_{\mathcal{Z}^G}(\ker \partial_j \cap V_{j-1}) + \dim_{\mathcal{Z}^G} \partial_j(V_{j-1}) \leq 2 \cdot \dim_{\mathcal{Z}^G} V_j$. Thus

$$\dim_{\mathcal{Z}^G} R \leq 2^n \cdot \dim_{\mathcal{Z}^G} V_n \leq [\mathcal{Z} : \mathcal{Z}^G] \cdot 2^{\dim_{\mathbb{K}} L_1}.$$

Since R is G-simple, the algebra R must be either simple or $R = I \oplus \sigma(I)$ for a minimal ideal I of R. Then I is certainly a simple algebra. The above inequality implies that

$\dim_{\mathcal{Z}} R \leq 2^{\dim_{\mathbb{K}} L_1}$ in the first case, and $\dim_{\mathcal{Z}(I)} I \leq 2^{\dim_{\mathbb{K}} L_1}$ in the second case. The result follows now by the Amitsur–Levitzki theorem. □

If R is semiprime we let $Q = Q(R)$ to denote the symmetric Martindale quotient ring. Its centre, known as the extended centroid of R, we denote by C. The following properties of Q in the case when R is acted on by a Hopf algebra are proved in [**3**].

LEMMA 6. *Let R be a semiprime H-module algebra such that the H-action on R extends to an H-action on Q and any non-zero H-stable ideal of R contains non-trivial invariants. Then*

(*1*) *the ring $C^H = C \cap Q^H$ is von Neumann regular and selfinjective.*
(*2*) *If a non-empty subset $S \subseteq C^H \setminus \{0\}$ is closed under a multiplication, then the localization Q_S of Q at S is semiprime and $\mathcal{Z}(Q_S) = C_S$.*
(*3*) *If H acts finitely on Q and $S = C^H \setminus M$, where M is a maximal ideal of C^H, then the H-action on Q extends to an H-action on Q_S and $(Q^H)_S = (Q_S)^H$, $(C^H)_S = (C_S)^H = C_S \cap (Q_S)^H$ is a field contained in the centre of Q_S.*

We can now prove the main result of the paper.

Proof of Theorem 1. Let $H = U(L) * G$. By ([**2**], Corollary 6) every H-invariant non-nilpotent subalgebra of R contains non-zero invariants. Thus we can apply the results from [**4**]. Let M be a maximal ideal of $C^H = C \cap Q^H$ and put $S = C^H \setminus M$. By the above lemma and [**4**] it follows that $(C_S)^H$ is a field and Q_S is a finite dimensional, G-simple $(C_S)^H$-algebra. Using Corollary 5 we obtain that Q_M satisfies the standard polynomial identity of degree $2 \cdot [\sqrt{2^{\dim_{\mathbb{K}} L_1}}]$. Since it holds for any maximal ideal M of C^H, the ring Q, and consequently R, satisfies the standard polynomial identity of degree $2 \cdot [\sqrt{2^{\dim_{\mathbb{K}} L_1}}]$. □

3. Examples. In this section, we construct examples of actions of nilpotent Lie superalgebras with central invariants and with $[L_0, L_1] \neq 0$. We start with general properties of inner derivations and skew derivations of an algebra R with an automorphism σ of order 2. Then $R = R_0 \oplus R_1$ is $\mathbb{Z}_2$-graded, where $R_0 = \{x \in R \mid \sigma(x) = x\}$ and $R_1 = \{x \in R \mid \sigma(x) = -x\}$. For any inner derivation δ of R, the condition $\delta\sigma = \sigma\delta$ is equivalent to that δ is induced by some $a \in R_0$. To see that, we let δ be induced by $a = a_0 + a_1 \in R$. Then

$$\delta(x) = ax - xa = (a_0x - xa_0) + (a_1x - xa_1). \tag{1}$$

This immediately implies that

$$\delta(\sigma(x)) = (a_0\sigma(x) - \sigma(x)a_0) + (a_1\sigma(x) - \sigma(x)a_1)$$

and

$$\sigma(\delta(x)) = (a_0\sigma(x) - \sigma(x)a_0) - (a_1\sigma(x) - \sigma(x)a_1).$$

Since δ and σ commute, the previous equations imply that $a_1\sigma(x) - \sigma(x)a_1 = 0$. Replacing x by $\sigma(x)$ yields $a_1x - xa_1 = 0$. Equation (1) now becomes

$$\delta(x) = a_0x - xa_0 = \mathrm{ad}_{a_0}(x).$$

In the same manner we can show that for any inner skew derivation ∂ of R, the condition $\partial\sigma = -\sigma\partial$ is equivalent to that $\partial = \partial_b$ for some $b \in R_1$.

LEMMA 7. *Let R be an algebra over a field $\mathbb{K}$ of characteristic $\neq 2$ and σ be a $\mathbb{K}$-linear automorphism of R of order 2. Let $u \in R$ be invertible and $\sigma(u) = -u$. Let $\widetilde{R}$ be the $\mathbb{K}$-algebra $M_2(R)$, the 2×2 matrices over R. Then the map $\varphi\colon R \to \widetilde{R}$ given by*

$$\varphi(x) = \begin{pmatrix} x & 0 \\ 0 & u^{-1}\sigma(x)u \end{pmatrix}$$

is an injective homomorphism of algebras, satisfying $\widetilde{\sigma}\varphi = \varphi\sigma$ (where $\widetilde{\sigma}$ is a componentwise extension of σ to $\widetilde{R}$).

If a Lie superalgebra $L = L_0 \oplus L_1$ acts on R by inner derivations and inner σ-derivations with $R^L = \mathbb{K}$, then L acts on $\widetilde{R}$ by inner derivations and inner $\widetilde{\sigma}$-derivations with

$$\widetilde{R}^L = \left\{ \begin{pmatrix} \alpha & \beta u \\ \gamma u^{-1} & \lambda \end{pmatrix} \in \widetilde{R} \mid \alpha, \beta, \gamma, \lambda \in \mathbb{K} \right\}.$$

Proof. Notice that

$$(\widetilde{\sigma}\varphi)(x) = \widetilde{\sigma}\left(\begin{pmatrix} x & 0 \\ 0 & u^{-1}\sigma(x)u \end{pmatrix}\right) = \begin{pmatrix} \sigma(x) & 0 \\ 0 & u^{-1}xu \end{pmatrix} = (\varphi\sigma)(x).$$

In order to prove the second part, observe that for all inner derivation $\mathrm{ad}_a \in L_0$ and the inner skew derivation $\partial_b \in L_1$ of R and for every matrix $\widetilde{x} = \left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right) \in \widetilde{R}$ the following equations hold

$$\mathrm{ad}_{\varphi(a)}(\widetilde{x}) = \begin{pmatrix} a & 0 \\ 0 & u^{-1}au \end{pmatrix} \cdot \widetilde{x} - \widetilde{x} \cdot \begin{pmatrix} a & 0 \\ 0 & u^{-1}au \end{pmatrix} = \begin{pmatrix} \mathrm{ad}_a(x_{11}) & \mathrm{ad}_a(x_{12}u^{-1})u \\ u^{-1}\,\mathrm{ad}_a(ux_{21}) & u^{-1}\,\mathrm{ad}_a(ux_{22}u^{-1})u \end{pmatrix}$$

and

$$\begin{aligned} \partial_{\varphi(b)}(\widetilde{x}) &= \begin{pmatrix} b & 0 \\ 0 & -u^{-1}bu \end{pmatrix} \cdot \widetilde{x} - \widetilde{\sigma}(\widetilde{x}) \cdot \begin{pmatrix} b & 0 \\ 0 & -u^{-1}bu \end{pmatrix} \\ &= \begin{pmatrix} \partial_b(x_{11}) & \partial_b(x_{12}u^{-1})u \\ \sigma(u^{-1})\partial_b(ux_{21}) & \sigma(u^{-1})\partial_b(ux_{22}u^{-1})u \end{pmatrix}. \end{aligned}$$

From the above equations it follows that $\widetilde{x} \in \widetilde{R}^L$ if and only if the elements $x_{11}, x_{12}u^{-1}$, ux_{21} and $ux_{22}u^{-1}$ belong to R^L. Under the assumption that $R^L = \mathbb{K}$, we now easily obtain the assertion of the lemma. □

We start our construction from the algebra $R = M_2(\mathbb{K})$ of 2×2 matrices over a field $\mathbb{K}$ of characteristic 0. Let σ be the inner automorphism of order 2 of R induced by the diagonal matrix diag$(1, -1)$ and let ∂_{b_1} and ∂_{b_2} be the inner σ-derivations of R induced by

$$b_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in R_1 \text{ and } b_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in R_1,$$

respectively. It can be easily checked that

$${b_1}^2 = -{b_2}^2 = 1 \text{ and } b_1 b_2 + b_2 b_1 = 0.$$

As a result, the skew derivations ∂_{b_1} and ∂_{b_2} span an Abelian Lie superalgebra $L = L_0 \oplus L_1$ where $L_0 = 0$ and $L_1 = \mathrm{Span}_{\mathbb{K}}\{\partial_{b_1}, \partial_{b_2}\}$. From the explicit formulas for ∂_{b_1} and ∂_{b_2}

$$\partial_{b_1}\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \begin{pmatrix} x_{21} + x_{12} & x_{22} - x_{11} \\ x_{11} - x_{22} & x_{21} + x_{12} \end{pmatrix}$$

and

$$\partial_{b_2}\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \begin{pmatrix} x_{21} - x_{12} & x_{22} - x_{11} \\ x_{22} - x_{11} & x_{21} - x_{12} \end{pmatrix},$$

it follows immediately that $R^L = \mathbb{K}$.

Using Lemma 7, applied to the invertible element $u = b_2$, we have an embedding of R into $\widetilde{R} = M_2(R)$, according to the formula

$$\varphi(x) = \begin{pmatrix} x & 0 \\ 0 & {b_2}^{-1}\sigma(x) b_2 \end{pmatrix}.$$

Put

$$\widetilde{b_1} = \varphi(b_1) = \begin{pmatrix} b_1 & 0 \\ 0 & b_1 \end{pmatrix} \in \widetilde{R}_1 \text{ and } \widetilde{b_2} = \varphi(b_2) = \begin{pmatrix} b_2 & 0 \\ 0 & -b_2 \end{pmatrix} \in \widetilde{R}_1,$$

and consider the additional matrices

$$\widetilde{b_3} = \begin{pmatrix} 0 & b_2 \\ -b_2 & 0 \end{pmatrix} \in \widetilde{R}_1 \text{ and } \widetilde{b_4} = \begin{pmatrix} 0 & b_2 \\ b_2 & 0 \end{pmatrix} \in \widetilde{R}_1.$$

It is not hard to check that

$$\widetilde{b_1}^2 = -\widetilde{b_2}^2 = \widetilde{b_3}^2 = -\widetilde{b_4}^2 = 1 \text{ and } \widetilde{b_i}\widetilde{b_j} + \widetilde{b_j}\widetilde{b_i} = 0$$

for all $i \neq j$. As before, the inner skew derivations $\partial_{\widetilde{b_1}}, \partial_{\widetilde{b_2}}, \partial_{\widetilde{b_3}}$ and $\partial_{\widetilde{b_4}}$ span an Abelian Lie superalgebra $\widetilde{L} = \widetilde{L}_0 \oplus \widetilde{L}_1$, where $\widetilde{L}_0 = 0$ and $\widetilde{L}_1 = \mathrm{span}_{\mathbb{K}}\{\partial_{\widetilde{b_1}}, \partial_{\widetilde{b_2}}, \partial_{\widetilde{b_3}}, \partial_{\widetilde{b_4}}\}$. Lemma 7 says that the subalgebra of invariants $\widetilde{R}^L$ under the action of L consists of all matrices of the form $\begin{pmatrix} \alpha & \beta b_2 \\ \gamma b_2 & \lambda \end{pmatrix}$ where $\alpha, \beta, \gamma, \lambda \in \mathbb{K}$. Furthermore, a simple calculation shows that

$$\partial_{\widetilde{b_3}}\left(\begin{pmatrix} \alpha & \beta b_2 \\ \gamma b_2 & \lambda \end{pmatrix}\right) = \begin{pmatrix} \beta - \gamma & (\lambda - \alpha) b_2 \\ (\lambda - \alpha) b_2 & \beta - \gamma \end{pmatrix}$$

and

$$\partial_{\widetilde{b_4}}\left(\begin{pmatrix}\alpha & \beta b_2\\ \gamma b_2 & \lambda\end{pmatrix}\right)=\begin{pmatrix}-\beta-\gamma & (\lambda-\alpha)b_2\\ (\alpha-\lambda)b_2 & -\beta-\gamma\end{pmatrix}.$$

This immediately implies that $\widetilde{R}^{\widetilde{L}}=\mathbb{K}$.

Applying Lemma 7 for the invertible element $u=\widetilde{b_4}$ we have the next embedding of $\widetilde{R}$ into the algebra $\boldsymbol{R}=M_2(\widetilde{R})$, the 2×2 matrices over $\widetilde{R}$ according to the formula

$$\varphi(\widetilde{x})=\begin{pmatrix}\widetilde{x} & 0\\ 0 & \widetilde{b_4}^{-1}\widetilde{\sigma}(\widetilde{x})\widetilde{b_4}\end{pmatrix}.$$

Put

$$B_i=\varphi(\widetilde{b_i})=\begin{pmatrix}\widetilde{b_i} & 0\\ 0 & \widetilde{b_i}\end{pmatrix}\in\boldsymbol{R}_1 \text{ and } B_4=\varphi(\widetilde{b_4})=\begin{pmatrix}\widetilde{b_4} & 0\\ 0 & -\widetilde{b_4}\end{pmatrix}\in\boldsymbol{R}_1$$

for $i=1,2,3$ and consider the additional matrices

- $A_1=\left(\begin{smallmatrix}0 & \widetilde{a_1}\\ -\widetilde{a_1} & 0\end{smallmatrix}\right)\in\boldsymbol{R}_0$ and $C_1=\left(\begin{smallmatrix}0 & \widetilde{a_1}\\ 0 & 0\end{smallmatrix}\right)\in\boldsymbol{R}_0$, where $\widetilde{a_1}=\left(\begin{smallmatrix}1 & 1\\ -1 & -1\end{smallmatrix}\right)\in\widetilde{R}_0$,
- $A_2=\left(\begin{smallmatrix}0 & \widetilde{a_2}+1\\ -\widetilde{a_2}+1 & 0\end{smallmatrix}\right)\in\boldsymbol{R}_0$ and $C_2=\left(\begin{smallmatrix}0 & \widetilde{a_2}+1\\ 0 & 0\end{smallmatrix}\right)\in\boldsymbol{R}_0$, where $\widetilde{a_2}=\left(\begin{smallmatrix}0 & b_1b_2\\ b_1b_2 & 0\end{smallmatrix}\right)\in\widetilde{R}_0$,
- $A_3=\left(\begin{smallmatrix}\widetilde{a_3}-\widetilde{a_1} & 0\\ 0 & \widetilde{a_3}+\widetilde{a_1}\end{smallmatrix}\right)\in\boldsymbol{R}_0$, where $\widetilde{a_3}=\left(\begin{smallmatrix}b_1b_2 & b_1b_2\\ -b_1b_2 & -b_1b_2\end{smallmatrix}\right)\in\widetilde{R}_0$,
- $B_5=\left(\begin{smallmatrix}0 & \widetilde{d_5}\\ \widetilde{b_5} & 0\end{smallmatrix}\right)\in\boldsymbol{R}_1$, $B_6=\left(\begin{smallmatrix}0 & \widetilde{b_4}\\ -\widetilde{b_4} & 0\end{smallmatrix}\right)\in\boldsymbol{R}_1$ and $B_7=\left(\begin{smallmatrix}0 & \widetilde{b_4}\\ \widetilde{b_4} & 0\end{smallmatrix}\right)\in\boldsymbol{R}_1$, where $\widetilde{d_5}=\left(\begin{smallmatrix}b_1+b_2 & b_1+b_2\\ -b_1-b_2 & -b_1-b_2\end{smallmatrix}\right)$, $\widetilde{b_5}=\left(\begin{smallmatrix}-b_1+b_2 & -b_1+b_2\\ b_1-b_2 & b_1-b_2\end{smallmatrix}\right)\in\widetilde{R}_1$,
- $D_5=\left(\begin{smallmatrix}0 & \widetilde{d_5}\\ 0 & 0\end{smallmatrix}\right)+B_7\in\boldsymbol{R}_1$ and $D_6=\left(\begin{smallmatrix}0 & \widetilde{b_4}\\ 0 & 0\end{smallmatrix}\right)\in\boldsymbol{R}_1$.

Notice that if $\boldsymbol{N}_0=\text{span}_{\mathbb{K}}\{\text{ad}_{C_1},\text{ad}_{C_2},\text{ad}_{A_3}\}$ and $\boldsymbol{N}_1=\text{span}_{\mathbb{K}}\{\partial_{B_1},\partial_{B_2},\partial_{B_3},\partial_{B_4},\partial_{D_5},\partial_{D_6}\}$, then $\boldsymbol{N}=\boldsymbol{N}_0\oplus\boldsymbol{N}_1$ is a nine-dimensional Lie superalgebra of nilpotency class 4 (see Table 1). Lemma 7 asserts that the subalgebra of invariants $\boldsymbol{R}^{\widetilde{L}}$ under the action of $\widetilde{L}$ consists of all matrices of the form $\left(\begin{smallmatrix}\alpha & \beta\widetilde{b_4}\\ \gamma\widetilde{b_4} & \lambda\end{smallmatrix}\right)$, where $\alpha,\beta,\gamma,\lambda\in\mathbb{K}$. Moreover,

$$\begin{aligned}\partial_{D_5}\left(\begin{pmatrix}\alpha & \beta\widetilde{b_4}\\ \gamma\widetilde{b_4} & \lambda\end{pmatrix}\right)&=\begin{pmatrix}\gamma\widetilde{d_5}\widetilde{b_4}-\beta-\gamma & (\lambda-\alpha)(\widetilde{b_4}+\widetilde{d_5})\\ (\alpha-\lambda)\widetilde{b_4} & \gamma\widetilde{b_4}\widetilde{d_5}-\beta-\gamma\end{pmatrix}\\ &=\begin{pmatrix}\gamma(\widetilde{a_3}-\widetilde{a_1})-\beta-\gamma & (\lambda-\alpha)(\widetilde{b_4}+\widetilde{d_5})\\ (\alpha-\lambda)\widetilde{b_4} & \gamma(\widetilde{a_3}+\widetilde{a_1})-\beta-\gamma\end{pmatrix}.\end{aligned}$$

As a result we obtain that $\boldsymbol{R}^{\boldsymbol{N}}=\mathbb{K}$.

Table 1. Operation table of $\boldsymbol{N}$

$[\cdot,\cdot]$	ad_{C_1}	ad_{C_2}	ad_{A_3}	∂_{B_1}	∂_{B_2}	∂_{B_3}	∂_{B_4}	∂_{D_5}	∂_{D_6}
ad_{C_1}	0	0	0	0	$-2\partial_{D_6}$	$2\partial_{D_6}$	0	$\partial_{B_2+B_3}$	0
ad_{C_2}	0	0	0	$2\partial_{D_6}$	0	0	$2\partial_{D_6}$	$-\partial_{B_1-B_4}$	0
ad_{A_3}	0	0	0	$-2\partial_{B_2+B_3}$	$-2\partial_{B_1-B_4}$	$2\partial_{B_1-B_4}$	$-2\partial_{B_2+B_3}$	0	0
∂_{B_1}	0	$-2\partial_{D_6}$	$2\partial_{B_2+B_3}$	0	0	0	0	$2\,\mathrm{ad}_{C_1}$	0
∂_{B_2}	$2\partial_{D_6}$	0	$2\partial_{B_1-B_4}$	0	0	0	0	$-2\,\mathrm{ad}_{C_2}$	0
∂_{B_3}	$-2\partial_{D_6}$	0	$-2\partial_{B_1-B_4}$	0	0	0	0	$2\,\mathrm{ad}_{C_2}$	0
∂_{B_4}	0	$-2\partial_{D_6}$	$2\partial_{B_2+B_3}$	0	0	0	0	$2\,\mathrm{ad}_{C_1}$	0
∂_{D_5}	$-\partial_{B_2+B_3}$	$\partial_{B_1-B_4}$	0	$2\,\mathrm{ad}_{C_1}$	$-2\,\mathrm{ad}_{C_2}$	$2\,\mathrm{ad}_{C_2}$	$2\,\mathrm{ad}_{C_1}$	$2\,\mathrm{ad}_{A_3}$	0
∂_{D_6}	0	0	0	0	0	0	0	0	0

Table 2. Operation table of $\boldsymbol{L}$

$[\cdot,\cdot]$	ad_{A_1}	ad_{A_2}	ad_{A_3}	∂_{B_1}	∂_{B_2}	∂_{B_3}	∂_{B_4}	∂_{B_5}	∂_{B_6}	∂_{B_7}
ad_{A_1}	0	$-2\,\mathrm{ad}_{A_3}$	0	0	$-2\partial_{B_6}$	$2\partial_{B_6}$	0	0	$-2\partial_{B_2+B_3}$	0
ad_{A_2}	$2\,\mathrm{ad}_{A_3}$	0	0	$2\partial_{B_6}$	0	0	$2\partial_{B_6}$	0	$2\partial_{B_1-B_4}$	0
ad_{A_3}	0	0	0	$-2\partial_{B_2+B_3}$	$-2\partial_{B_1-B_4}$	$2\partial_{B_1-B_4}$	$-2\partial_{B_2+B_3}$	0	0	0
∂_{B_1}	0	$-2\partial_{B_6}$	$2\partial_{B_2+B_3}$	0	0	0	0	$2\,\mathrm{ad}_{A_1}$	0	0
∂_{B_2}	$2\partial_{B_6}$	0	$2\partial_{B_1-B_4}$	0	0	0	0	$-2\,\mathrm{ad}_{A_2}$	0	0
∂_{B_3}	$-2\partial_{B_6}$	0	$-2\partial_{B_1-B_4}$	0	0	0	0	$2\,\mathrm{ad}_{A_2}$	0	0
∂_{B_4}	0	$-2\partial_{B_6}$	$2\partial_{B_2+B_3}$	0	0	0	0	$2\,\mathrm{ad}_{A_1}$	0	0
∂_{B_5}	0	0	0	$2\,\mathrm{ad}_{A_1}$	$-2\,\mathrm{ad}_{A_2}$	$2\,\mathrm{ad}_{A_2}$	$2\,\mathrm{ad}_{A_1}$	0	$-2\,\mathrm{ad}_{A_3}$	0
∂_{B_6}	$2\partial_{B_2+B_3}$	$-2\partial_{B_1-B_4}$	0	0	0	0	0	$-2\,\mathrm{ad}_{A_3}$	0	0
∂_{B_7}	0	0	0	0	0	0	0	0	0	0

Notice also that if $\boldsymbol{M}_0 = \mathrm{span}_{\mathbb{K}}\{\mathrm{ad}_{A_1}, \mathrm{ad}_{A_2}, \mathrm{ad}_{A_3}\}$ and $\boldsymbol{M}_1 = \mathrm{span}_{\mathbb{K}}\{\partial_{B_1}, \partial_{B_2}, \partial_{B_3}, \partial_{B_4}, \partial_{B_5+B_7}, \partial_{B_6}\}$, then $\boldsymbol{M} = \boldsymbol{M}_0 \oplus \boldsymbol{M}_1$ is a nilpotent Lie superalgebra of nilpotency class 6 (see Table 2). We have

$$\partial_{B_5+B_7}\left(\begin{pmatrix} \alpha & \beta\widetilde{b_4} \\ \gamma\widetilde{b_4} & \lambda \end{pmatrix}\right) = \begin{pmatrix} \gamma\widetilde{d_5}\widetilde{b_4} + \beta\widetilde{b_4}\widetilde{b_5} - \beta - \gamma & (\lambda-\alpha)(\widetilde{b_4}+\widetilde{d_5}) \\ (\alpha-\lambda)(\widetilde{b_4}+\widetilde{b_5}) & \beta\widetilde{b_5}\widetilde{b_4} + \gamma\widetilde{b_4}\widetilde{d_5} - \beta - \gamma \end{pmatrix}$$
$$= \begin{pmatrix} (\gamma-\beta)(\widetilde{a_3}-\widetilde{a_1}) - \beta - \gamma & (\lambda-\alpha)(\widetilde{b_4}+\widetilde{d_5}) \\ (\alpha-\lambda)(\widetilde{b_4}+\widetilde{b_5}) & (\gamma-\beta)(\widetilde{a_3}+\widetilde{a_1}) - \beta - \gamma \end{pmatrix}.$$

This implies immediately that $\boldsymbol{R}^{\boldsymbol{M}} = \mathbb{K}$.

Finally, observe also that $\boldsymbol{M}$ is an subalgebra of a nilpotent Lie superalgebra $\boldsymbol{L} = \boldsymbol{L}_0 \oplus \boldsymbol{L}_1$ of nilpotency class 6, where $\boldsymbol{L}_0 = [\boldsymbol{L}_1, \boldsymbol{L}_1] = \mathrm{span}_{\mathbb{K}}\{\mathrm{ad}_{A_1}, \mathrm{ad}_{A_2}, \mathrm{ad}_{A_3}\}$ and $\boldsymbol{L}_1 = \mathrm{span}_{\mathbb{K}}\{\partial_{B_1}, \partial_{B_2}, \partial_{B_3}, \partial_{B_4}, \partial_{B_5}, \partial_{B_6}, \partial_{B_7}\}$ (see Table 2). Obviously, $\boldsymbol{R}^{\boldsymbol{L}} = \mathbb{K}$. Starting with the algebra $\boldsymbol{R}$, the invertible element $u = B_7$ and the Lie superalgebra $\boldsymbol{L}$, and again applying the above procedure, we can produce successive examples.

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