



Lines on complex contact manifolds, II

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ABSTRACT

Let X be a complex-projective contact manifold with $b_2(X) = 1$. It has long been conjectured that X should then be rational-homogeneous, or equivalently, that there exists an embedding $X \rightarrow \mathbb{P}^n$ whose image contains lines.

We show that X is covered by a compact family of rational curves, called ‘contact lines’, that behave very much like the lines on the rational-homogeneous examples: if $x \in X$ is a general point, then all contact lines through x are smooth, no two of them share a common tangent direction at x , and the union of all contact lines through x forms a cone over an irreducible, smooth base. As a corollary, we obtain that the tangent bundle of X is stable.

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1. Introduction

Motivated by questions coming from Riemannian geometry, complex contact manifolds have received considerable attention during recent years. The link between complex and Riemannian geometry is given by the twistor space construction: twistor spaces over Riemannian manifolds with quaternion-Kähler holonomy group are complex contact manifolds. As twistor spaces are covered by rational curves, much of the research is centered about the geometry of rational curves on the contact spaces.

1.1 Setup and statement of the main result

Throughout the present paper, we maintain the assumptions and notational conventions of the first part [Keb01] of this series. In particular, we refer to [Keb01], and the references therein, for an introduction to contact manifolds and to the parameter spaces which we will use freely throughout.

In brief, we assume throughout that X is a complex-projective manifold of dimension $\dim X = 2n + 1$ which carries a contact structure. This structure is given by a vector bundle sequence

$$0 \longrightarrow F \longrightarrow T_X \xrightarrow{\theta} L \longrightarrow 0, \quad (1.1)$$

Received 13 June 2003, accepted in final form 18 November 2003, published online 1 December 2004.

2000 Mathematics Subject Classification 53C25 (primary), 14J45, 53C15 (secondary).

Keywords: complex contact manifold, Fano manifold, rational curve.

The author was supported by a Heisenberg Fellowship and by the program ‘Globale Methoden in der Komplexen Analysis’ of the Deutsche Forschungsgemeinschaft.

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where F is a sub-bundle of corank 1 and where the skew-symmetric O'Neill tensor

$$N : F \otimes F \rightarrow L,$$

which is associated with the Lie bracket, is non-degenerate at every point of X .

Because contact manifolds with $b_2(X) > 1$ were completely described in [KPSW00], we consider only the case where $b_2(X) = 1$. We will also assume that X is not isomorphic to the projective space \mathbb{P}^{2n+1} . By [Keb01, § 2.3], these assumptions imply that we can find a compact irreducible component $H \subset \text{RatCurves}^n(X)$ of the space of rational curves on X such that the intersection of L with the curves associated with H is 1. Curves that are associated with points of H are called ‘contact lines’. For a point $x \in X$, consider the varieties

$$H_x := \{\ell \in H \mid x \in \ell\}$$

and

$$\text{locus}(H_x) := \bigcup_{\ell \in H_x} \ell.$$

The main result of this paper is the following.

THEOREM 1.1. *Let X be a complex-projective contact manifold with $b_2(X) = 1$ and assume $X \not\cong \mathbb{P}^{\dim X}$. Let $H \subset \text{RatCurves}^n(X)$ be an irreducible component which parameterizes contact lines. Then $\text{locus}(H_x)$ is isomorphic to a projective cone over a smooth, irreducible base. Further,*

- (i) *all contact lines that contain x are smooth,*
- (ii) *the space H_x is irreducible,*
- (iii) *if ℓ_1 and ℓ_2 are any two contact lines through x , then $T_{\ell_1}|_x \neq T_{\ell_2}|_x$, and*
- (iv) *if ℓ_1 and ℓ_2 are any two contact lines through x , then $\ell_1 \cap \ell_2 = \{x\}$.*

The smoothness of the base of the cone guarantees that much of the theory developed by Hwang and Mok for uniruled varieties can be applied to the contact setup. We refer to [Hwa01] for an overview and mention two examples.

1.1.1 Stability of the tangent bundle. It has been conjectured for a long time that complex contact manifolds X with $b_2(X) = 1$ always carry a Kähler–Einstein metric. In particular, it is conjectured that the tangent bundle of these manifolds is stable. Using methods introduced by Hwang, stability follows as an immediate consequence of Theorem 1.1.

COROLLARY 1.2. *Let X be a complex-projective contact manifold with $b_2(X) = 1$. Then the tangent bundle T_X is stable.*

1.1.2 Continuation of analytic morphisms. The following corollary asserts that a contact manifold is determined in a strong sense by the tangent directions to contact lines. The analogous result for homogeneous manifolds appears in the work of Yamaguchi.

COROLLARY 1.3. *Let X be a complex-projective contact manifold and X' be an arbitrary Fano manifold. Assume that $b_2(X) = b_2(X') = 1$ and choose a dominating family of rational curves of minimal degree on $\mathcal{H} \subset \text{RatCurves}^n(X')$. Assume further that there exist analytic open subsets $U \subset X$, $U' \subset X'$ and a biholomorphic morphism $\phi : U \rightarrow U'$ such that the tangent map $T\phi$ maps tangents of contact lines to tangents of curves coming from \mathcal{H} , and vice versa. Then ϕ extends to a biholomorphic map $\phi : X \rightarrow X'$.*

Question 1.4. What would be the analogous statement in Riemannian geometry?

1.2 Outline of this paper

Property i of Theorem 1.1 is known from previous works (see Fact 2.3 below). After a review of known facts in § 2, properties ii–iv are shown one by one in §§ 3–5, respectively. With these results at hand, the proofs of Theorem 1.1 and Corollaries 1.2 and 1.3, which we give in § 6, are very short.

The main difficulty in this paper is the proof of property iii, which is done by a detailed analysis of the restriction of the tangent bundle T_X to pairs of contact lines that intersect tangentially. The proof relies on a number of facts on jet bundles and on deformation spaces of morphisms between polarized varieties for which the author could not find any reference. These more general results are gathered in the two appendices.

2. Known facts

The proof of Theorem 1.1 relies on a number of known facts scattered throughout the literature. For the reader’s convenience, we have gathered these results here. Full proofs were included where appropriate.

2.1 Jet bundles on contact manifolds

The O’Neill tensor yields an identification $F \cong F^\vee \otimes L$. If we dualize the contact sequence (1.1) and twist by L , we obtain a sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega_X^1 \otimes L \longrightarrow \underbrace{F}_{\cong F^\vee \otimes L} \longrightarrow 0, \tag{2.1}$$

which we would now like to compare to the dual of the first jet sequence of L ; see Appendix A.1 for more information on jets and the first jet sequence.

By [LeB95, Theorem 2.1], there exists a canonical symplectic form on the \mathbb{C}^* -principal bundle associated with L which gives rise to an identification $\text{Jet}^1(L) \cong \text{Jet}^1(L)^\vee \otimes L$. Thus, if we dualize the jet sequence and twist by L , we obtain a sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \underbrace{\text{Jet}^1(L)}_{\cong \text{Jet}^1(L)^\vee \otimes L} \longrightarrow T_X \longrightarrow 0. \tag{2.2}$$

It is known that sequence (2.1) is a sub-sequence of (2.2).

FACT 2.1 [LeB95, p. 426]. *There exists a commutative diagram with exact rows and columns*

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 (2.1) & 0 \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega_X^1 \otimes L & \longrightarrow & F \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 (2.2) & 0 \longrightarrow & \mathcal{O}_X & \longrightarrow & \text{Jet}^1(L) & \longrightarrow & T_X \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & L & \xlongequal{\quad} & L \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

where the middle column is the first jet sequence for L and the right column is the sequence (1.1) of page § 1.1 that defines the contact structure.

2.2 Contact lines

It is conjectured that a projective contact manifold X with $b_2(X) = 1$ is homogeneous. This is known to be equivalent to the conjecture that there exists an embedding $X \rightarrow \mathbb{P}^N$ that maps contact lines to lines in \mathbb{P}^N . While we cannot presently prove these conjectures, it has already been shown in the first part [Keb01] of this work that a contact line through a general point shares many features with lines in \mathbb{P}^N . Some of the following results will be strengthened in § 3.1.

FACT 2.2 [Keb01, Remark 3.3]. *Let ℓ be a contact line. Then ℓ is F -integral. In other words, if $x \in \ell$ is a smooth point, then $T_\ell|_x \subset F|_y$.*

FACT 2.3. *Let $x \in X$ be a general point and $\ell \subset X$ a contact line that contains x . Then ℓ is smooth. The splitting types of the restricted vector bundles $F|_\ell$ and $T_X|_\ell$ are:*

$$T_X|_\ell \cong \mathcal{O}_\ell(2) \oplus \mathcal{O}_\ell(1)^{\oplus n-1} \oplus \mathcal{O}_\ell^{\oplus n+1},$$

$$F|_\ell \cong \underbrace{\mathcal{O}_\ell(2) \oplus \mathcal{O}_\ell(1)^{\oplus n-1} \oplus \mathcal{O}_\ell^{\oplus n-1}}_{=: F|_\ell^{\geq 0}} \oplus \mathcal{O}_\ell(-1).$$

For all points $y \in \ell$, the vector space $F|_\ell^{\geq 0}|_y$ and the tangent space $T_\ell|_y$ are perpendicular with respect to the O’Neill tensor N : $F|_\ell^{\geq 0}|_y = T_\ell|_y^\perp$.

Proof. The fact that ℓ is smooth was shown in [Keb01, Proposition 3.3]. The splitting type of $T_X|_\ell$ is given by [Keb01, Lemma 3.5]. To find the splitting type of $F|_\ell$, recall that the contact structure yields an identification $F \cong F^\vee \otimes L$. Since $L|_\ell \cong \mathcal{O}_\ell(1)$, we can therefore find positive numbers a_i and write

$$F|_\ell \cong \bigoplus_{i=1}^n (\mathcal{O}_\ell(a_i) \oplus \mathcal{O}_\ell(1 - a_i)).$$

The precise splitting type then follows from the splitting type of $T_X|_\ell$ and from Fact 2.2 above.

The simple observation that every map $\mathcal{O}_\ell(2) \cong T_\ell \rightarrow L|_\ell \cong \mathcal{O}_\ell(1)$ is necessarily zero yields the fact that $F|_\ell^{\geq 0}|_y$ and $T_\ell|_y$ are perpendicular with respect to the O’Neill tensor N . \square

FACT 2.4. *Let $x \in X$ be a general point, $\ell \subset X$ a contact line that contains x and $y \in \ell$ any point. If $s \in H^0(\ell, T_X|_\ell)$ is a section such that $s(y) \in F|_y$, then s is contained in $H^0(\ell, F|_\ell)$ if and only if $T_\ell|_y$ and $s(y)$ are orthogonal with respect to the O’Neill tensor N .*

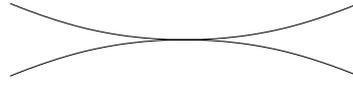
In particular, we have that if $s(y) \in F|_\ell^{\geq 0}$, then $s \in H^0(\ell, F|_\ell^{\geq 0})$.

Proof. Let $f : \mathbb{P}^1 \rightarrow X$ be a parameterization of ℓ . We know from [Kol96, Theorems II.3.11.5 and II.2.8] that the space $\text{Hom}(\mathbb{P}^1, X)$ is smooth at f . As a consequence, we can find an embedded unit disc $\Delta \subset \text{Hom}(\mathbb{P}^1, X)$, centered about f such that $s \in T_\Delta|_f$ holds; see Fact B.1 in Appendix B for a brief explanation of the tangent space to $\text{Hom}(\mathbb{P}^1, X)$. In this situation we can apply [Keb01, Proposition 3.1] to the family Δ , and the claim is shown. \square

2.3 Dubbies

In § 4 we will show that no two contact lines through a general point share a common tangent direction at x . For this, we will argue by contradiction and assume that X is covered by pairs of contact lines which intersect tangentially in at least one point. Such a pair is always dominated by a pair of smooth rational curves that intersect in one point with multiplicity exactly 2. These particularly simple pairs were called ‘dubbies’ and extensively studied in [KK03, § 3].

DEFINITION 2.5. A *dubby* is a reduced, reducible curve, isomorphic to the union of a line and a smooth conic in \mathbb{P}^2 intersecting tangentially in a single point, as shown below.



Remark 2.6. Because a dubby ℓ is a plane cubic, we have $h^1(\ell, \mathcal{O}_\ell) = 1$.

2.3.1 *Low-degree line bundles on dubbies.* It is the key observation in [KK03] that an ample line bundle on a dubby always gives a canonical identification of its two irreducible components. In the setup of § 4, where dubbies are composed of contact lines, the identification is quite apparent so that we do not need to refer to the complicated general construction of [KK03, § 3.2].

PROPOSITION 2.7. Let $\ell = \ell_1 \cup \ell_2$ be a dubby and $H \in \text{Pic}(\ell)$ be a line bundle whose restriction to both ℓ_1 and ℓ_2 is of degree 1. Then there exists a unique isomorphism $\gamma : \ell \rightarrow \mathbb{P}^1$ such that

- (i) the restriction $\gamma|_{\ell_i} : \ell_i \rightarrow \mathbb{P}^1$ to any component is isomorphic and
 - (ii) a pair of smooth points $y_1 \in \ell_1$ and $y_2 \in \ell_2$ forms a divisor for H if and only if $\gamma(y_1) = \gamma(y_2)$.
- In particular, we have that $h^0(\ell, H) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 2$.

Proof. Consider the restriction morphisms

$$r_i : H^0(\ell, H) \rightarrow H^0(\ell_i, H|_{\ell_i}) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)).$$

We claim that the morphism r_i is an isomorphism for all $i \in \{1, 2\}$. The roles of r_1 and r_2 are symmetric, so it is enough to prove the claim for r_1 . First note that $h^0(\ell, H) \geq 2$ by [KK03, Lemma 3.2]. It is then sufficient to prove that r_1 is injective. Let $s \in \ker(r_1) \subset H^0(\ell, H)$. In order to show that $s = 0$ it is enough to show that $r_2(s) = 0$. Notice that $r_2(s)$ is a section in $H^0(\ell_2, H|_{\ell_2})$ that vanishes on the scheme-theoretic intersection $\ell_1 \cap \ell_2$. The length of this intersection is 2 and any non-zero section in $H^0(\ell_2, H|_{\ell_2}) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ has a unique zero of order 1, hence $r_2(s)$ must be zero, and so r_i is indeed an isomorphism for all $i \in \{1, 2\}$.

This implies that H is generated by global sections and gives a morphism $\gamma : \ell \rightarrow \mathbb{P}^1$, whose restriction $\gamma|_{\ell_i}$ to any of the two components is an isomorphism. Property ii follows by construction. □

Notation 2.8. We call a pair of points (y_1, y_2) as in Proposition 2.7 ‘mirror points with respect to H ’.

COROLLARY 2.9. Let $\ell = \ell_1 \cup \ell_2$ be a dubby and $\text{Pic}^{(1,1)}(\ell)$ be the component of the Picard group that represents line bundles whose restriction to both ℓ_1 and ℓ_2 is of degree 1. Then the natural action of the automorphism group $\text{Aut}(\ell)$ on $\text{Pic}^{(1,1)}(\ell)$ is transitive.

Proof. Consider the open set $\Omega = \ell_2 \setminus (\ell_1 \cap \ell_2)$. By Proposition 2.7 it suffices to show that there exists a group $G \subset \text{Aut}(\ell)$ that fixes ℓ_1 pointwise and acts transitively on Ω .

For this, define a group action on the disjoint union $\ell_1 \amalg \ell_2$ as follows. Let $G \subset \text{Aut}(\ell_2)$, $G \cong \mathbb{C}$ be the isotropy group of the scheme-theoretic intersection $\ell_1 \cap \ell_2 \subset \ell_2$. Let G act trivially on ℓ_1 . It is clear that G acts freely on Ω . By construction, G acts trivially on the scheme-theoretic intersection $\ell_1 \cap \ell_2$ so that the actions on ℓ_1 and ℓ_2 glue to give a global action on ℓ . □

COROLLARY 2.10. Let ℓ and H be as in Proposition 2.7 and let

$$\text{Aut}(\ell, H) := \{g \in \text{Aut}(\ell) \mid g^*(H) \cong H\}$$

be the subgroup of automorphisms that respect the line bundle H . If $y \in \ell$ is any smooth point, then there exists a vector field, i.e. a section of the tangent sheaf

$$s \in T_{\text{Aut}(\ell, H)}|_{\text{Id}} \subset H^0(\ell, T_\ell),$$

that does not vanish at y .

Proof. Let $\sigma = \ell_1 \cap \ell_2$ be the (reduced) singular point, let $\eta : \ell_1 \amalg \ell_2 \rightarrow \ell$ be the normalization and consider the natural action of \mathbb{C} on \mathbb{P}^1 that fixes the image point $\gamma(\sigma) \in \mathbb{P}^1$. Use the isomorphisms $\gamma|_{\ell_1}$ and $\gamma|_{\ell_2}$ to define a \mathbb{C} -action on $\ell_1 \amalg \ell_2$. As before, observe that this action acts trivially on the scheme-theoretic preimage

$$\eta^{-1}(\ell_1 \cap \ell_2).$$

The \mathbb{C} -action on $\ell_1 \amalg \ell_2$ therefore descends to a \mathbb{C} -action on ℓ . To see that the associated vector field does not vanish on y , it suffices to note that the singular point σ is the only \mathbb{C} -fixed point on ℓ .

Because the action preserves γ -fibers, it follows from Proposition 2.7 that \mathbb{C} acts via a morphism

$$\mathbb{C} \rightarrow \text{Aut}(\ell, H). \quad \square$$

In § 4 we will need to consider line bundles of degree $(2, 2)$. The following remark will be useful.

LEMMA 2.11. *Let $\ell = \ell_1 \cup \ell_2$ be a dubby and let $H \in \text{Pic}(\ell)$ be a line bundle whose restriction to both ℓ_1 and ℓ_2 is of degree 2. For $i \in \{1, 2\}$ there exist sections $s_i \in H^0(\ell, H)$ which vanish identically on ℓ_i but not on the other component.*

Proof. By [KK03, Lemma 3.2], we have $h^0(\ell, H) \geq 4$. Thus, the restriction map $H^0(\ell, H) \rightarrow H^0(\ell_i, H|_{\ell_i}) \cong \mathbb{C}^3$ has a non-trivial kernel. □

2.3.2 *Vector bundles on dubbies.* Dubbies are in many ways similar to elliptic curves. While $H^1(\ell, \mathcal{O}_\ell)$ does not vanish, the higher cohomology groups of ample vector bundles are trivial.

LEMMA 2.12. *Let \mathcal{E} be a vector bundle on ℓ whose restriction to both ℓ_1 and ℓ_2 is ample. Then $H^1(\ell, \mathcal{E}) = 0$.*

Proof. Let $\bar{\sigma} := \ell_1 \cap \ell_2 \subset \ell$ be the scheme-theoretic intersection, which is a zero-dimensional subscheme of length 2. Now consider the normalization $\eta : \ell_1 \amalg \ell_2 \rightarrow \ell$ and the associated natural sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \eta_*(\eta^*\mathcal{E}) \xrightarrow{\alpha} \mathcal{E}|_{\bar{\sigma}} \longrightarrow 0, \tag{2.3}$$

where α is defined on the level of presheaves as follows. Assume we are given an open neighborhood U of the singular point $\sigma \in \ell$. By definition of $\eta_*(\eta^*\mathcal{E})$, to give a section $s \in \eta_*(\eta^*\mathcal{E})(U)$ it is equivalent to give two sections $s_1 \in (\mathcal{E}|_{\ell_1})(U \cap \ell_1)$ and $s_2 \in (\mathcal{E}|_{\ell_2})(U \cap \ell_2)$. If

$$r_i : (\mathcal{E}|_{\ell_i})(U \cap \ell_i) \rightarrow \mathcal{E}|_{\bar{\sigma}}$$

are the natural restriction morphisms, then we write α as

$$\alpha(s) = r_1(s_1) - r_2(s_2).$$

A section of the long homology sequence associated with (2.3) reads

$$H^0(\ell, \eta_*\eta^*\mathcal{E}) \xrightarrow{\beta} H^0(\bar{\sigma}, \mathcal{E}|_{\bar{\sigma}}) \longrightarrow H^1(\ell, \mathcal{E}) \longrightarrow H^1(\ell, \eta_*\eta^*\mathcal{E}),$$

where β is again the difference of the restriction morphisms. We have that

$$\begin{aligned} H^0(\ell, \eta_*\eta^*\mathcal{E}) &= H^0(\ell_1, \mathcal{E}|_{\ell_1}) \oplus H^0(\ell_2, \mathcal{E}|_{\ell_2}), \\ H^1(\ell, \eta_*\eta^*\mathcal{E}) &= H^1(\ell_1, \mathcal{E}|_{\ell_1}) \oplus H^1(\ell_2, \mathcal{E}|_{\ell_2}) = \{0\}, \end{aligned}$$

and it remains to show that β is surjective. That, however, follows from the fact that $\mathcal{E}|_{\ell_i}$ is an ample bundle on \mathbb{P}^1 that generates 1-jets so that even the single restriction

$$r_1 : H^0(\ell_1, \mathcal{E}|_{\ell_1}) \rightarrow H^0(\bar{\sigma}, \mathcal{E}|_{\bar{\sigma}})$$

alone is surjective. □

3. Irreducibility

As a first step towards the proof of Theorem 1.1, we show the irreducibility of the space of contact lines through a general point.

THEOREM 3.1. *If $x \in X$ is a general point, then the subset $H_x \subset H$ of contact lines through x is irreducible. In particular, $\text{locus}(H_x)$ is irreducible.*

The proof of Theorem 3.1, which is given in § 3.2 below, requires a strengthening of Fact 2.3, which we give in the following section.

3.1 Contact lines with special splitting type

We adopt the notation of [Hwa01, § 1.2] and call a contact line $\ell \subset X$ ‘standard’ if

$$\eta^*(T_X) \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n+1},$$

where $\eta : \mathbb{P}^1 \rightarrow \ell$ is the normalization. It is known that the set of standard curves is Zariski-open in H (see again [Hwa01, § 1.2]). We can therefore consider the subvariety

$$H' := \{\ell \in H \mid \ell \text{ not standard}\}.$$

The proof of Theorem 3.1 is based on the observation that there is only a small set in X whose points are not contained in a standard contact line. For a proper formulation, set

$$D := \text{locus}(H') = \bigcup_{\ell \in H'} \ell.$$

It follows immediately from Fact 2.3 that D is a proper subset of X .

PROPOSITION 3.2. *If $D^0 \subset D$ is any irreducible component with $\text{codim}_X D^0 = 1$, $x \in D^0$ a general point, and $H_x^0 \subset H_x$ any irreducible component, then there exists a curve $\ell \in H_x^0$ which is not contained in D and therefore free.*

The proof of Proposition 3.2 is a variation of the argumentation in [Keb01, ch. 4]. While we work here in a more delicate setup, moving only components of $\text{locus}(H_x)$ along the divisor D^0 , parts of the proof were taken almost verbatim from [Keb01].

Proof of Proposition 3.2.

Step 1 (Setup). Assume to the contrary, i.e. assume that there exists a divisor $D^0 \subset D$ such that for a general point $x \in D^0$ there exists a component of H_x whose associated curves are all contained in D^0 . Since by [Keb01, Proposition 4.1] for all $y \in X$, the space H_y is of pure dimension $n - 1$, we can find a closed, proper subvariety $H^0 \subset H$ with $\text{locus}(H^0) = D^0$ such that, for all points $y \in D^0$,

$$H_y^0 = \{\ell \in H^0 \mid y \in \ell\}$$

is the union of irreducible components of H_y . In particular, we have that for all $y \in D^0$, $\dim \text{locus}(H_y^0) = n$.

Step 2 (Incidence variety). In analogy to [Keb01, Notation 4.2], define the incidence variety

$$V := \{(x', x'') \in D^0 \times D^0 \mid x'' \in \text{locus}(H_{x'}^0)\} \subset D^0 \times X.$$

Let $\pi_1, \pi_2 : V \rightarrow D^0$ be the natural projections. We have seen in Step 1 above that for every point $y \in D^0$, $\pi_1^{-1}(y)$ is a subscheme of X of dimension $\dim \pi_1^{-1}(y) = n$. In particular, V is a well-defined family of cycles in X in the sense of [Kol96, § I.3.10]. The universal property of the Chow variety therefore yields a map

$$\Phi : D^0 \rightarrow \text{Chow}(X).$$

Since $\dim \text{locus}(H_y^0) = n < \dim D^0$, this map is not constant. On the other hand, since $D^0 \subset X$ is ample, the Lefschetz hyperplane section theorem [BS95, Theorem 2.3.1] asserts that $b_2(D^0) = 1$. As a consequence, we obtain that the map Φ is finite: for any given point $y \in D^0$ there are at most finitely many points y_i such that $\text{locus}(H_y^0) = \text{locus}(H_{y_i}^0)$. In analogy to [Keb01, Lemma 4.3] we conclude the following.

LEMMA 3.3. *Let Δ be a unit disc and $\gamma : \Delta \rightarrow D^0$ an embedding. Then there exists a Euclidean open set $V^0 \subset \pi_1^{-1}(\gamma(\Delta))$ such that $\pi_2(V^0) \subset X$ is a submanifold of dimension*

$$\dim \pi_2(V^0) = n + 1.$$

In particular, $\pi_2(V^0)$ is not F -integral.

Step 3 (Conclusion). We shall now produce a map $\gamma : \Delta \rightarrow D^0$ to which Lemma 3.3 can be applied. For that, recall that D^0 cannot be F -integral. Thus, if $y \in D^0$ is a general smooth point of D^0 , then

$$F_{D^0,y} := F|_y \cap T_{D^0}|_y$$

is a proper hyperplane in $F|_y$, and the set $F_{D^0,y}^\perp$ of tangent vectors that are orthogonal to $F_{D^0,y}$ with respect to the O’Neill tensor is a line that is contained in $F_{D^0,y}$. The $F_{D^0,y}^\perp$ give a (singular) one-dimensional foliation on D^0 which is regular in a neighborhood of the general point y . Let $\gamma : \Delta \rightarrow D^0$ be an embedding of the unit disk that is an integral curve of this foliation, i.e. a curve such that for all points $y' \in \gamma(\Delta)$ we have that

$$T_{\gamma(\Delta)}|_{y'} = F_{D^0,y'}^\perp. \tag{3.1}$$

Now let $\mathcal{H} \subset (\text{Hom}_{\text{bir}}(\mathbb{P}^1, X))_{\text{red}}$ be the family of generically injective morphisms parameterizing the curves associated with H^0 . Fix a point $0 \in \mathbb{P}^1$ and set

$$\mathcal{H}_\Delta := \{f \in \mathcal{H} \mid f(0) \in \gamma(\Delta)\}.$$

If $\mu : \mathcal{H}_\Delta \times \mathbb{P}^1 \rightarrow X$ is the universal morphism, then it follows by construction that

$$\mu(\mathcal{H}_\Delta \times \mathbb{P}^1) = \pi_2(\pi_1^{-1}(\gamma(\Delta))) \supset \pi_2(V^0),$$

where V^0 comes from Lemma 3.3. In particular, since $\pi_2(V^0)$ is not F -integral, there exists a smooth point $(f, p) \in \mathcal{H}_\Delta \times \mathbb{P}^1$ with $f(p) \in \pi_2(V^0)$ and there exists a tangent vector $\vec{w} \in T_{\mathcal{H}_\Delta \times \mathbb{P}^1}|_{(f,p)}$ such that the image of the tangent map is not in F :

$$T\mu_\Delta(\vec{w}) \notin \mu^*(F).$$

Decompose $\vec{w} = \vec{w}' + \vec{w}''$, where $\vec{w}' \in T_{\mathbb{P}^1}|_p$ and $\vec{w}'' \in T_{\mathcal{H}_\Delta}|_f$. Then, since $f(\mathbb{P}^1)$ is F -integral, it follows that $T\mu(\vec{w}') \in \mu^*(F)$ and therefore

$$T\mu(\vec{w}'') \notin \mu^*(F). \tag{3.2}$$

As a next step, since \mathcal{H}_Δ is smooth at f , we can choose an immersion

$$\begin{aligned} \beta : \Delta &\rightarrow \mathcal{H}_\Delta \\ t &\mapsto \beta_t \end{aligned}$$

such that $\beta_0 = f$ and such that

$$T\beta \left(\frac{\partial}{\partial t} \Big|_{t=0} \right) = \vec{w}''.$$

In particular, if $s \in H^0(\mathbb{P}^1, f^*(T_X))$ is the section associated with $\vec{w}'' = T\beta((\partial/\partial t)|_{t=0})$, and $s' := f^*(\theta)(s) \in H^0(\mathbb{P}^1, f^*(L))$, then the following hold:

- (i) It follows from (3.2) and from [Kol96, Proposition II.3.4] that s' is not identically zero.
- (ii) At $0 \in \mathbb{P}^1$, the section s satisfies $s(0) \in f^*(T_{\gamma(\Delta)}) \subset f^*(F)$. In particular, $s'(0) = 0$.
- (iii) If z is a local coordinate on \mathbb{P}^1 about 0, then it follows from (3.1) that $(\partial/\partial z)|_0 \in f^*(F)$ and $s(0) \in f^*(F')$ are perpendicular with respect to the non-degenerate form N .

Items ii and iii ensure that we can apply [Keb01, Proposition 3.1] to the family β_t . Since the section s' does not vanish completely, the proposition states that s' has a zero of order at least 2 at 0. But s' is an element of $H^0(\mathbb{P}^1, f^*(L))$, and $f^*(L)$ is a line bundle of degree 1. We have thus reached a contradiction, and the proof of Proposition 3.2 is finished. □

3.2 Proof of Theorem 3.1

Let $\pi : U \rightarrow H$ be the restriction of the universal \mathbb{P}^1 -bundle $\text{Univ}^{\text{rc}}(X)$ to H and let $\iota : U \rightarrow X$ be the universal morphism. Consider the Stein factorization of ι .

$$\begin{array}{ccccc}
 & & \iota = \alpha \circ \beta & & \\
 & & \curvearrowright & & \\
 U & \xrightarrow{\alpha} & X' & \xrightarrow{\beta} & X \\
 \text{connected fibers} & & & \text{finite} & \\
 \downarrow \text{\mathbb{P}^1\text{-bundle } \pi} & & & & \\
 H & & & &
 \end{array}$$

Let $T \subset X$ be the union of the following subvarieties of X :

- a) the components $D_i \subset D$ which have $\text{codim}_X D_i \geq 2$, where $D \subset X$ is the subvariety defined in § 3.1;
- b) for every divisorial component $D_i \subset D$, the Zariski-closed set of points $y \in D_i$ for which there exists an irreducible component $H_y^0 \subset H_y$ such that none of the associated curves in X are free;
- c) the image $\beta(X'_{\text{Sing}})$ of the singular set of X' .

It follows immediately from Proposition 3.2 that $\text{codim}_X T \geq 2$.

CLAIM 3.4. *The morphism β is unbranched away from T , i.e. the restricted morphism*

$$\beta|_{X \setminus T} : \beta^{-1}(X \setminus T) \rightarrow X \setminus T$$

is smooth.

Proof. Let $y \in \beta^{-1}(X \setminus T)$ be any point. To show that β has maximal range at y , it suffices to find a point $z \in \alpha^{-1}(y)$ such that

- a) z is a smooth point of U and
- b) ι is smooth at z .

By [Kol96, ch. II, Theorems 1.7, 2.15 and Corollary 3.5.4], both requirements are satisfied if $\pi(z) \in H$ is a point that corresponds to a free curve. The existence of a free curve in the component $\pi(\alpha^{-1}(y))$, however, is guaranteed by choice of T . □

Since X is Fano, it is simply connected. Because $T \subset X$ is not a divisor, its complement $X \setminus T$ is also simply connected. Claim 3.4 therefore implies either that X' is reducible, or that the general β -fiber is a single point. But X' is irreducible by construction, and it follows that the general fiber of ι must be connected. By Seidenberg's classical theorem [BS95, Theorem 1.7.1], the general fiber $\iota^{-1}(x)$ is then irreducible, and so is its image

$$H_x = \pi(\iota^{-1}(x)).$$

This ends the proof of Theorem 3.1.

4. Contact lines sharing a common tangent direction

The aim of the present section is to give a proof of part iii of Theorem 1.1. More precisely, we show the following.

THEOREM 4.1. *If $x \in X$ is a general point, then all contact lines through x are smooth, and no two of them share a common tangent at x .*

The proof is at its core a repeat performance of [KK03, § 4.1] where the global assumptions of [KK03, Theorem 1.3] are replaced by a careful study of the restriction of the tangent bundle T_X to a pair of rational curves with non-transversal intersection.

4.1 Setup

We will argue by contradiction and assume throughout the rest of this section to the contrary. More precisely, we stick to the following.

Assumption 4.2. Assume that for a general point $x \in X$, we can find a pair $\ell' = \ell'_1 \cup \ell'_2 \subset X$ of distinct contact lines $\ell'_i \in H$ that intersect tangentially at x .

The pair ℓ' is then dominated by a dubby $\ell = \ell_1 \cup \ell_2$ whose singular point $\sigma = \ell_1 \cap \ell_2$ maps to x . For the remainder of this section we fix a generically injective morphism $f : \ell \rightarrow \ell'$ such that $f(\sigma) = x$. We also fix the line bundle $H := f^*(L) \in \text{Pic}^{(1,1)}(\ell)$.

4.2 Proof of Theorem 4.1

Assumption 4.2 implies that for a fixed point x , there is a positive-dimensional family of pairs of curves which contain x and have a point of non-transversal intersection. Loosely speaking, we will move the point of intersection to obtain a positive-dimensional family of dubbies that all contain the point x (see Figure 1).

To formulate more precisely, consider the quasi-projective reduced subvariety

$$\mathcal{H} \subset (\text{Hom}(\ell, X))_{\text{red}}$$

of morphisms $g \in \text{Hom}(\ell, X)$ such that $g^*(L) \cong H$. Note that such a morphism will always be generically injective on each irreducible component of ℓ . Consider the diagram

$$\begin{array}{ccc} \mathcal{H} \times \ell & \xrightarrow[\text{universal morphism}]{\mu} & X \\ \text{projection } \pi \downarrow & & \\ \mathcal{H} & & \end{array}$$

and conclude from Corollary 2.9 that the restricted universal morphism $\mu|_{\mathcal{H} \times \{\sigma\}}$ is dominant. By general choice of f , there exists a unique positive-dimensional irreducible component

$$\mathcal{H}_x \subset \pi(\mu^{-1}(x))$$

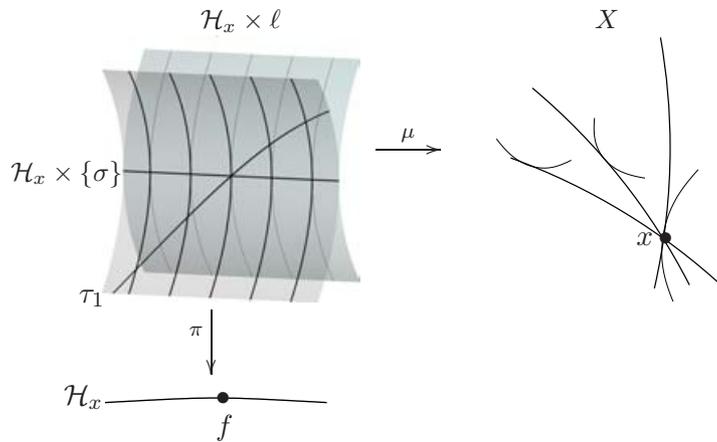


FIGURE 1. Proof of Theorem 4.1.

which contains f and which is smooth at f . It is clear that for a general point $g \in \mathcal{H}_x$, the point x is a smooth point of the pair of curves $g(\ell)$. This implies the following decomposition lemma.

LEMMA 4.3. *The preimage of x decomposes as*

$$\mu^{-1}(x) \cap \pi^{-1}(\mathcal{H}_x) = \tau_1 \cup \bigcup_{i=1}^N \eta_i,$$

where $\tau_1 \subset \mathcal{H}_x \times \ell$ is a section that intersects $\mathcal{H}_x \times \{\sigma\}$ over f , but is not contained in $\mathcal{H}_x \times \{\sigma\}$, and where the η_i are finitely many lower-dimensional components, $\dim \eta_i < \dim H_x$.

Proof. Since all curves in X that are associated with points of \mathcal{H}_x contain x , it is clear that there exists a component $\tau_1 \subset \mu^{-1}(x) \cap \pi^{-1}(\mathcal{H}_x)$ that surjects onto \mathcal{H}_x .

We have seen above that, for $g \in \mathcal{H}_x$ general, x is a smooth point of the pair of curves $g(\ell)$, i.e. that the scheme-theoretic intersection $\mu^{-1}(x) \cap \pi^{-1}(g)$ is a single closed point that is not equal to σ . Since $\mu^{-1}(x) \cap \pi^{-1}(g)$ is necessarily discrete for all $g \in \mathcal{H}_x$, it follows that τ_1 is a section that is not contained in $\mathcal{H}_x \times \{\sigma\}$. It follows further that no other component η_i of $\mu^{-1}(x) \cap \pi^{-1}(\mathcal{H}_x)$ dominates \mathcal{H}_x . In particular, $\dim \eta_i < \dim \tau_1$.

To see that $(f, \sigma) \in \tau_1$, we first note that $f(\sigma) = 0$, so that (f, σ) is contained in the preimage, $(f, \sigma) \in \mu^{-1}(x) \cap \pi^{-1}(\mathcal{H}_x)$. On the other hand, Fact 2.3 asserts that both $f(\ell_1)$ and $f(\ell_2)$ are smooth so that $\sigma = f^{-1}(x)$ and $(f, \sigma) = \mu^{-1}(x) \cap \pi^{-1}(f)$. This ends the proof. \square

After renaming ℓ_1 and ℓ_2 , if necessary, we assume without loss of generality that $\tau_1 \subset \mathcal{H}_x \times \ell_1$. By Proposition 2.7, the line bundle $H \in \text{Pic}(\ell)$ yields an identification morphism $\gamma : \ell \rightarrow \mathbb{P}^1$. Let

$$(\text{Id} \times \gamma) : \mathcal{H}_x \times \ell \rightarrow \mathcal{H}_x \times \mathbb{P}^1$$

be the associated morphism of bundles and consider the mirror section

$$\tau_2 := ((\text{Id} \times \gamma)|_{\mathcal{H}_x \times \ell_2})^{-1}(\text{Id} \times \gamma)(\tau_1).$$

CLAIM 4.4. *The universal morphism μ contracts τ_2 to a point: $\mu(\tau_2) = (*)$.*

Proof. The proof of Claim 4.4 makes use of Proposition 4.11 which is shown independently in §§ 4.3–4.4 below.

For the proof, we pick a general smooth point $z \in \tau_2$, and an arbitrary tangent vector $\vec{v} \in T_{\tau_2}|_z$. It suffices to show that \vec{v} is mapped to zero,

$$T\mu(\vec{v}) = 0 \in \mu^*(T_X)|_z.$$

Since τ_2 is a section over \mathcal{H}_x , and since \mathcal{H}_x is smooth at $\pi(z)$, we can find a small embedded unit disc $\Delta \subset \mathcal{H}_x$ with coordinate t such that $T\pi(\vec{v}) = \pi^*(\partial/\partial t)|_z$. For the remainder of the proof, it is convenient to introduce new bundle coordinates on the restricted bundle $\Delta \times \ell$. It follows from Corollary 2.10 that, after perhaps shrinking Δ , we can find a holomorphic map

$$\alpha : \Delta \rightarrow \text{Aut}^0(\ell, H)$$

with associated coordinate change diagram

$$\begin{array}{ccc}
 \Delta \times \ell & \xrightarrow[\substack{\text{coord. change } \kappa \\ (t,y) \rightarrow (t, \alpha(t) \cdot y)}]{\tilde{\mu} := \mu \circ \kappa} & \mathcal{H} \times \ell \xrightarrow[\text{universal morphism}]{\mu} X \\
 \downarrow \text{projection } \pi_1 & & \\
 \Delta & \xrightarrow{\alpha} & \text{Aut}(\ell, H)
 \end{array}$$

such that $\kappa^{-1}(\tau_1) \cup \kappa^{-1}(\tau_2)$ is a fiber of the projection $\pi_2 : \Delta \times \ell \rightarrow \ell$.

Let $\tau'_i := \kappa^{-1}(\tau_i)$, $z' := \kappa^{-1}(z)$, and let $\vec{v}' \in T_{\tau'_2}|_{z'}$ be the preimage of \vec{v} , i.e. the unique tangent vector that satisfies $T\kappa(\vec{v}') = \kappa^{-1}(\vec{v})$. The new coordinates make it easy to write down an extension of the tangent vector \vec{v}' to a global vector field, i.e. to a section $s \in H^0(\Delta \times \ell, T_{\Delta \times \ell})$ of the tangent sheaf. Indeed, if we use the product structure to decompose

$$T_{\Delta \times \ell} \cong \pi_1^*(T_\Delta) \oplus \pi_2^*(T_\ell),$$

then the ‘horizontal vector field’ $s := \pi_1^*(\partial/\partial t)$ will already satisfy $s(z') = \vec{v}'$.

In this setup, it follows from the definition of \mathcal{H} and Theorem B.2 that the section $T\tilde{\mu}(s) \in H^0(\Delta \times \ell, \tilde{\mu}^*(T_X))$ is in the image of the map

$$H^0(\Delta \times \ell, \tilde{\mu}^*(\text{Jet}^1(L)^\vee \otimes L)) \rightarrow H^0(\Delta \times \ell, \tilde{\mu}^*(T_X))$$

that comes from the dualized and twisted jet sequence (2.1).

To end the proof of Claim 4.4, let $\bar{z}' \in \{\pi_1(z)\} \times \ell$ be the mirror point with respect to the line bundle H . Since the coordinate change respects the line bundle H , Proposition 2.7 asserts that $\bar{z}' \in \tau'_1$. In particular, we have that $s(\bar{z}') \in T_{\tau'_1}|_{\bar{z}'}$ and therefore, since τ'_1 is contracted, $T\tilde{\mu}(s(\bar{z}')) = 0$. Proposition 4.11 implies that $T\tilde{\mu}(s(z')) = 0$, too. This shows that μ contracts τ_2 to a point. The proof of Claim 4.4 is finished. \square

Using Claim 4.4, we will derive a contradiction, showing that Assumption 4.2 is absurd. The proof of Theorem 4.1 will then be finished.

For this, observe that $\tau_1 \cap \pi^{-1}(f) = \{f\} \times \{\sigma\}$. The sections τ_1 and τ_2 are therefore not disjoint. In this setup, Claim 4.4 implies that $\mu(\tau_2) = \{x\}$, so that $\tau_2 \subset \mu^{-1}(x)$. That violates the decomposition Lemma 4.3 from above.

4.3 Sub-bundles in the pull-back of F and T_X

We will now lay the ground for the proof of Proposition 4.11 in the next section. Our line of argumentation is based on the following fact, which is an immediate consequence of Assumption 4.2 and the infinitesimal description of the universal morphism μ .

FACT 4.5 ([Kol96, Proposition II.3.4] and Fact B.1). *In the setup of § 4.1, let $\sigma \in \ell$ be the singular point. Then the restriction morphism*

$$H^0(\ell, f^*(T_X)) \rightarrow f^*(T_X)|_\sigma$$

is surjective. In other words, the vector space $f^(T_X)|_\sigma$ is generated by global sections.*

Recall from Fact 2.3 that the non-negative part of the restriction of the vector bundle F to one of the smooth contact lines ℓ_i was denoted by $F|_{\ell_i}^{\geq 0}$. We use Fact 4.5 to show that the two vector bundles $F|_{\ell_1}^{\geq 0}$ on ℓ_1 and $F|_{\ell_2}^{\geq 0}$ on ℓ_2 together give a global vector bundle on ℓ .

LEMMA 4.6. *There exists a vector bundle $f^*(F)^{\geq 0} \subset f^*(F)$ on ℓ whose restriction to any of the irreducible components ℓ_i equals $F|_{\ell_i}^{\geq 0} \subset f^*(F)$. If $y \in \ell$ is a general point, then the restriction morphism*

$$H^0(\ell, f^*(F)^{\geq 0}) \rightarrow f^*(F)^{\geq 0}|_y$$

is surjective.

Proof. By Fact 4.5, we can find sections $s_1, \dots, s_{2n-1} \in H^0(\ell, f^*(T_X))$ that span

$$Tf(T_{\ell_1}|_\sigma)^\perp = Tf(T_{\ell_2}|_\sigma)^\perp \subset f^*(F)|_\sigma$$

where $\sigma \in (\ell_1 \cap \ell_2)_{\text{red}}$ is the singular point of ℓ , and where \perp means: ‘perpendicular with respect to the O’Neill tensor N ’. Note that the sections s_1, \dots, s_{2n-1} become linearly dependent only at smooth points of the curve ℓ . Thus, the double dual of the sheaf generated by s_1, \dots, s_{2n-1} is a locally free subsheaf of $f^*(T_X)$.

It follows from Fact 2.4 that $s_1|_{\ell_i}, \dots, s_{2n-1}|_{\ell_i}$ are in fact sections of $f^*(F)$ that generate $F|_{\ell_i}^{\geq 0}$ on an open set of ℓ_i . The claim follows. □

COROLLARY 4.7. *There exists a vector sub-bundle $T \subset f^*(F)$ whose restriction to any component ℓ_i is exactly the image of the tangent map*

$$T|_{\ell_i} = \text{Image}(Tf : T_{\ell_i} \rightarrow f^*(T_X)|_{\ell_i}).$$

Since $f|_{\ell_i}$ is an embedding, $T|_{\ell_i}$ is of degree 2.

Proof. By Fact 2.3, we can set $T := (f^*(F)^{\geq 0})^\perp$. □

The vector bundle $f^*(F)^{\geq 0}$ is a sub-bundle of both $f^*(F)$ and $f^*(T_X)$. As a matter of fact, it appears as a direct summand in these bundles.

LEMMA 4.8. *The vector bundle sequences on ℓ*

$$0 \rightarrow f^*(F)^{\geq 0} \rightarrow f^*(F) \rightarrow f^*(F)/f^*(F)^{\geq 0} \rightarrow 0 \tag{4.1}$$

and

$$0 \rightarrow f^*(F)^{\geq 0} \rightarrow f^*(T_X) \rightarrow f^*(T_X)/f^*(F)^{\geq 0} \rightarrow 0 \tag{4.2}$$

are both split. We have $f^(T_X)/f^*(F)^{\geq 0} \cong \mathcal{O}_\ell \oplus \mathcal{O}_\ell$.*

Proof. In order to show that sequence (4.1) splits, we show that the obstruction group

$$\text{Ext}_\ell^1(f^*(F)/f^*(F)^{\geq 0}, f^*(F)^{\geq 0}) = H^1(\ell, \underbrace{(f^*(F)/f^*(F)^{\geq 0})^\vee \otimes f^*(F)^{\geq 0}}_{=:\mathcal{E}})$$

vanishes. If $\ell_i \subset \ell$ is any component, it follows immediately from Fact 2.3 that

$$(f^*(F)/f^*(F)^{\geq 0})|_{\ell_i} \cong \mathcal{O}_{\ell_i}(-1)$$

and

$$\mathcal{E}|_{\ell_i} \cong \mathcal{O}_{\ell_i}(3) \oplus \mathcal{O}_{\ell_i}(2)^{\oplus n-1} \oplus \mathcal{O}_{\ell_i}(1)^{\oplus n-1}.$$

By Lemma 2.12, $H^1(\ell, \mathcal{E}) = 0$. That shows the splitting of the sequence (4.1).

As a next step, we will show that the quotient $f^*(T_X)/f^*(F)^{\geq 0}$ is trivial. By Fact 4.5, we can find two sections $s_1, s_2 \in H^0(\ell, f^*(T_X))$ such that the induced sections $s'_1, s'_2 \in H^0(\ell, f^*(T_X)/f^*(F)^{\geq 0})$ generate the quotient $f^*(T_X)/f^*(F)^{\geq 0}|_{\sigma}$ at the singular point $\sigma \in \ell$. Restricting these sections to ℓ_i , it follows that the sections

$$s'_1|_{\ell_i}, s'_2|_{\ell_i} \in H^0(\ell_i, \underbrace{f^*(T_X)/f^*(F)^{\geq 0}|_{\ell_i}}_{\cong \mathcal{O}_{\ell_i} \oplus \mathcal{O}_{\ell_i} \text{ by Fact 2.3}})$$

do not vanish anywhere and are everywhere linearly independent. As a consequence, the induced morphism of sheaves on ℓ

$$\begin{aligned} \mathcal{O}_{\ell} \oplus \mathcal{O}_{\ell} &\rightarrow f^*(T_X)/f^*(F)^{\geq 0} \\ (g, h) &\mapsto g \cdot s'_1 + h \cdot s'_2 \end{aligned}$$

is an isomorphism, and the map

$$\begin{aligned} \mathcal{O}_{\ell} \oplus \mathcal{O}_{\ell} &\rightarrow f^*(T_X) \\ (g, h) &\mapsto g \cdot s_1 + h \cdot s_2 \end{aligned}$$

splits the sequence (4.2). □

COROLLARY 4.9. *The natural morphism*

$$H^0(\ell, f^*(\Omega_X^1)) \rightarrow H^0(\ell, f^*(F)^{\vee}),$$

which comes from the dual of the contact sequence (1.1), is an isomorphism.

Proof. The morphism is part of the long exact sequence

$$0 \rightarrow H^0(\ell, f^*(L)^{\vee}) \rightarrow H^0(\ell, f^*(\Omega_X^1)) \rightarrow H^0(\ell, f^*(F)^{\vee}) \rightarrow \dots$$

Since $f^*(L)^{\vee}$ is a line bundle whose restriction to any irreducible component $\ell_i \subset \ell$ is of degree -1 , there are no sections to it: $h^0(\ell, f^*(L)^{\vee}) = 0$. It remains to show that $h^0(\ell, f^*(\Omega_X^1)) = h^0(\ell, f^*(F)^{\vee})$.

The direct sum decomposition of Lemma 4.8 yields

$$\begin{aligned} h^0(\ell, f^*(F)^{\vee}) &= h^0(\ell, (f^*(F)^{\geq 0})^{\vee}) + \underbrace{h^0(\ell, (f^*(F)/f^*(F)^{\geq 0})^{\vee})}_{=2 \text{ by Fact 2.3 and Proposition 2.7}}, \\ h^0(\ell, f^*(\Omega_X^1)) &= h^0(\ell, (f^*(F)^{\geq 0})^{\vee}) + \underbrace{h^0(\ell, (f^*(T_X)/f^*(F)^{\geq 0})^{\vee})}_{=h^0(\mathcal{O}_{\ell} \oplus \mathcal{O}_{\ell})=2 \text{ by Lemma 4.8}}. \end{aligned}$$

The corollary follows. □

4.4 The vanishing locus of sections in the pull-back of T_X

Using Corollary 4.9, we can now establish a criterion, Proposition 4.11, that guarantees that certain sections in $f^*(T_X)$ that vanish at a point $y \in \ell$ will also vanish at the mirror point. The following lemma is a first precursor.

LEMMA 4.10. *In the setup of § 4.1, let $y \in \ell$ be a general point and let $s \in H^0(\ell, f^*(T_X))$ be a section that vanishes at y . Then the associated section $s' \in H^0(\ell, f^*(T_X)/T)$ vanishes at the mirror point \bar{y} . Here T is the vector bundle from Corollary 4.7.*

Proof. We claim that $s \in H^0(\ell, f^*(F))$. The proof of this claim is a twofold application of Fact 2.4. If we assume without loss of generality that $y \in \ell_1$, then a direct application of Fact 2.4 shows that $s|_{\ell_1} \in H^0(\ell_1, F|_{\ell_1}^{\geq 0})$. If $\sigma = (\ell_1 \cap \ell_2)_{\text{red}}$ is the singular point, this implies that $s(\sigma) \in (F|_{\ell_1}^{\geq 0})|_{\sigma} = (F|_{\ell_2}^{\geq 0})|_{\sigma}$. Another application of Fact 2.4 then shows the claim.

As a consequence, in order to show Lemma 4.10 it suffices to show that the associated section $s'' \in H^0(\ell, f^*(F)/T)$ vanishes at \bar{y} . We assume to the contrary.

Since $T^\perp = f^*(F)^{\geq 0}$, the non-degenerate O’Neill tensor gives an identification

$$f^*(F)^{\geq 0}|_y \cong ((f^*(F)/T)^\vee \otimes f^*(L))|_y.$$

By Lemma 4.6, we can therefore find a section $t \in H^0(\ell, f^*(F)^{\geq 0})$ such that s and t pair to give a section

$$N(s, t) \in H^0(\ell, f^*(L))$$

that vanishes at y , but does not vanish on \bar{y} . That is a contradiction to Proposition 2.7. □

In Lemma 4.10, it is generally *not* true that the section s vanishes at \bar{y} ; to a given section s , we can always add a vector field on ℓ that stabilizes y , but does not stabilize the mirror point \bar{y} . However, the statement becomes true if we restrict ourselves to sections s that come from L -jets.

PROPOSITION 4.11. *In the setup of § 4.1, let $y \in \ell$ be a general point and $s \in H^0(\ell, f^*(T_X))$ be a section that vanishes on y . If s is in the image of the map*

$$H^0(\ell, f^*(\text{Jet}^1(L)^\vee \otimes L)) \rightarrow H^0(\ell, f^*(T_X))$$

that comes from the dualized and twisted jet sequence (2.2), then s vanishes also at the mirror point \bar{y} .

The proof of Proposition 4.11 requires the following lemma, which we state and prove first.

LEMMA 4.12. *Let $s \in H^0(\ell, f^*(F))$ be a section and let $D \in |f^*(L)|$ be an effective divisor that is supported on the smooth locus of ℓ . If s vanishes on D , then s is in the image of the map*

$$H^0(\ell, f^*(\text{Jet}^1(L)^\vee \otimes L)) \rightarrow H^0(\ell, f^*(T_X))$$

that comes from the dualized and twisted jet sequence (2.2).

Proof. In view of Fact 2.1, we need to show that s is in the image of the map

$$\alpha : H^0(\ell, f^*(\Omega_X^1 \otimes L)) \rightarrow H^0(\ell, f^*(F))$$

that comes from the dualized and twisted contact sequence (2.1). For that, let $t \in H^0(\ell, f^*(L))$ be a non-zero section that vanishes on D . Using the O’Neill tensor N to identify F with $F^\vee \otimes L$, we can view s as a section that lies in the image

$$H^0(\ell, f^*(F^\vee)) \xrightarrow{\cdot t} H^0(\ell, f^*(F^\vee \otimes L)).$$

The claim then follows from Corollary 4.9, and the commutativity of the following diagram.

$$\begin{array}{ccc} H^0(\ell, f^*(\Omega_X^1)) & \xrightarrow[\text{by Corollary 4.9}]{\text{surjective}} & H^0(\ell, f^*(f^\vee)) \\ \cdot t \downarrow & & \downarrow \cdot t \\ H^0(\ell, f^*(\Omega_X^1 \otimes L)) & \longrightarrow & H^0(\ell, f^*(F^\vee \otimes L)) \end{array} \quad \square$$

Proof of Proposition 4.11. Since $s \in H^0(\ell, f^*(F))$, Fact 2.1 implies that s is in the image of the map α from the long exact sequence associated with the dualized and twisted contact sequence (2.1)

$$0 \rightarrow \underbrace{H^0(\ell, f^*(\mathcal{O}_X))}_{\cong \mathbb{C}} \rightarrow H^0(\ell, f^*(\Omega_X^1 \otimes L)) \xrightarrow{\alpha} H^0(\ell, f^*(F)) \rightarrow \underbrace{H^1(\ell, f^*(\mathcal{O}_X))}_{\cong \mathbb{C} \text{ by Remark 2.6}} \rightarrow \dots \quad (4.3)$$

By Lemma 4.10, the vector space

$$H_{y\bar{y}} := \{\tau \in H^0(\ell, f^*(F)) \mid \tau(y) = 0, \tau(\bar{y}) = 0\}$$

is a linear hyperplane in

$$H_y := \{\tau \in H^0(\ell, f^*(F)) \mid \tau(y) = 0\}.$$

Because $\mathcal{O}_\ell(y + \bar{y}) \cong f^*(L)$, Lemma 4.12 implies that

$$H_{y\bar{y}} \subset \text{Image}(\alpha) \cap H_y.$$

But $\text{codim}_{H^0(\ell, f^*(F))} \text{Image}(\alpha) \leq 1$, so that there are only two possibilities here:

- (i) $H_y \subseteq \text{Image}(\alpha)$ and $\text{Image}(\alpha) \cap H_y = H_y$,
- (ii) $\text{Image}(\alpha) \cap H_y = H_{y\bar{y}}$.

Observe that Proposition 4.11 is shown if we rule out possibility i. For that, it suffices to show that there exists a section $t \in H_y$ which is not in the image of α .

To this end, let $\theta \in H^0(X, \Omega_X^1 \otimes L)$ be the nowhere-vanishing L -valued 1-form that defines the contact structure in sequence (1.1). The beginning part of sequence (4.3) says that its pull-back $f^*(\theta) \in H^0(\ell, f^*(\Omega_X^1 \otimes L))$ is, up to a multiple, the unique section that is in the kernel of α . If we fix $i \in \{0, 1\}$, then the analogous sequence for $f|_{\ell_i}$ tells us that $f^*(\theta)|_{\ell_i}$ is the unique (again up to a multiple) section in $H^0(\ell_i, (f|_{\ell_i})^*(\Omega_X^1 \otimes L))$ which is in the kernel of $\alpha|_{\ell_i}$. As a consequence, there exists no section $u \in H^0(\ell, f^*(\Omega_X^1 \otimes L))$ such that $\alpha(u)$ vanishes on one component of $\ell = \ell_1 \cup \ell_2$, but not on the other.

By Lemma 2.11, however, there exists a section $t \in H^0(\ell, T) \subset H^0(\ell, f^*(F))$ that vanishes on the component of y and not on the other. The section t is therefore contained in H_y but not in $\text{Image}(\alpha)$. This ends the proof of Proposition 4.11. □

5. Contact lines sharing more than one point

As a last step before the proof of the main theorem, we show property iv from the list of Theorem 1.1.

THEOREM 5.1. *Let $x \in X$ be a general point and let ℓ_1, ℓ_2 be two distinct contact lines through x . Then ℓ_1 and ℓ_2 intersect in x only, $\ell_1 \cap \ell_2 = \{x\}$.*

The proof is really a corollary to the results of the previous section. In analogy to Definition 2.5, we name the simplest arrangement of rational curves that intersect in two points.

DEFINITION 5.2. A pair with proper double intersection is a reduced, reducible curve, isomorphic to the union of a line and a smooth conic in \mathbb{P}^2 intersecting transversally in two points.



Proof of Theorem 5.1. We argue by contradiction and assume that for a general point x there is a pair of contact lines ℓ_1, ℓ_2 through x which meet in at least one further point. The pair $\ell_1 \cup \ell_2$ will then be dominated by a pair with proper double intersection $\ell = \ell'_1 \cup \ell'_2$. More precisely, there exists a generically injective morphism $f : \ell \rightarrow \ell_1 \cup \ell_2$ which maps ℓ'_i to ℓ_i and which maps one of the two singular points of ℓ to x . Let $y \in \ell$ be that point.

Since x is assumed to be a general point, there exists an irreducible component of the reduced Hom scheme

$$\mathcal{H} \subset [\text{Hom}(\ell, X)]_{\text{red}}$$

with universal morphism $\mu : \mathcal{H} \times \ell \rightarrow X$ such that the restriction

$$\mu' = \mu|_{\mathcal{H} \times \{y\}} : \mathcal{H} \rightarrow X$$

is dominant. We can further assume that f is a smooth point of \mathcal{H} and that the tangent map $T\mu'$ has maximal rank $2n + 1$ at f .

By Fact 2.3 and Theorem 4.1, the tangent spaces $T_{\ell_1}|_x \subset F|_x$ and $T_{\ell_2}|_x \subset F|_x$ are both one-dimensional and distinct. We can thus find a tangent vector $\vec{v} \in F|_x$ which is perpendicular (with respect to the non-degenerate O'Neill tensor) to $T_{\ell_1}|_x$ but not to $T_{\ell_2}|_x$. Since the tangent map

$$T\mu'|_f : T_{\mathcal{H}}|_f \rightarrow T_X|_x$$

has maximal rank, we can find a tangent vector $s \in T_{\mathcal{H}}|_f$ such that $T\mu'(s) = \vec{v}$. By [Kol96, Proposition II.3.4] that means that we can find a section

$$s \in T_{\mathcal{H}}|_f = H^0(\ell, f^*(T_X))$$

with $T\mu'(s(y)) = \vec{v} \in f^*(T_X)$.

Now let $\theta : T_X \rightarrow L$ be the L -valued 1-form that defines the contact structure in sequence (1.1). We need to consider the section $s' := f^*(\theta)(s) \in H^0(\ell, f^*(L))$. Recall Fact 2.4 which asserts that s' vanishes identically on ℓ'_1 , but does not vanish identically on ℓ'_2 . In particular, $\ell'_1 \cap \ell'_2$ is contained in the zero-locus of $s'|_{\ell'_2}$ and we have

$$\text{deg } f^*(L)|_{\ell'_2} \geq \#(\ell'_1 \cap \ell'_2) = 2.$$

But ℓ_2 is a contact line and $f^*(L)|_{\ell_2}$ is a line bundle of degree 1, a contradiction. □

6. Proof of the main results

6.1 Proof of Theorem 1.1

In view of Theorem 3.1, to prove Theorem 1.1, it only remains to show that $\text{locus}(H_x)$ is a cone. This will turn out to be a corollary to Theorems 4.1 and 5.1.

Let \tilde{H}_x be the normalization of the subspace $H_x \subset H$ of the contact line through x . Since all contact lines through x are free, it follows from [Kol96, ch. II, Proposition 3.10 and Corollary 3.11.5] that \tilde{H}_x is smooth. We have a diagram

$$\begin{array}{ccc}
 & & \hat{X} \\
 & \hat{\iota} = \beta^{-1} \circ \iota & \dashrightarrow \\
 \tilde{U}_x & \xrightarrow{\iota} & X \\
 \downarrow \pi & \text{evaluation morphism} & \downarrow \beta \text{ blow-up of } x \\
 \tilde{H}_x & &
 \end{array}$$

where \tilde{U}_x is the pull-back of the universal \mathbb{P}^1 -bundle $\text{Univ}^{\text{rc}}(X)$, ι the natural evaluation morphism, and $\hat{X} = \text{BlowUp}(X, x)$ the blow-up of x with exceptional divisor E . Since all contact lines through x are smooth, the scheme-theoretic fiber $\iota^{-1}(x)$ is a Cartier-divisor in \tilde{U}_x , and it follows from the universal property [Har77, ch. II, Proposition 7.14] of the blow-up that $\hat{\iota} = \beta^{-1} \circ \iota$ is actually a morphism.

To show that $\text{locus}(H_x) = \text{Image}(\iota)$ really is a cone in the sense of [BS95, § 1.1.8], it suffices to show that $\hat{\iota}$ is an embedding, i.e. that $\hat{\iota}$ is injective and immersive.

- (i) *Injective.* Let $y \in \text{Image}(\hat{\iota})$ be any point. If $y \in E$, Theorem 4.1 asserts that $\#\hat{\iota}^{-1}(y) = 1$. If $y \notin E$, the same is guaranteed by Theorem 5.1.
- (ii) *Immersive.* Fact 2.3 implies that for every π -fiber $\ell \cong \mathbb{P}^1$, we have

$$\hat{\iota}^*(T_{\hat{X}})|_{\ell} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n+1}.$$

Under this condition, [Kol96, ch. II, Proposition 3.4] shows that $\hat{\iota}$ is immersive as required.

This ends the proof of Theorem 1.1.

6.2 Proof of Corollary 1.2

Once Theorem 1.1 is shown, the proof of [Hwa01, Theorem 2.11] applies nearly verbatim to contact manifolds. Note, however, that the estimate of [Hwa01, Theorem 2.11] is not optimal. For the reader’s convenience, we recall the argumentation here.

Assume that the tangent bundle T_X is not stable. By [Hwa98, Proposition 4], this implies that we can find a subsheaf $\mathcal{G} \subset T_X$ of positive rank with the following intersection property. If $x \in X$ is a general point, $\mathcal{C}_x \subset \mathbb{P}(T_X|_x^\vee)$ the projective tangent cone of $\text{locus}(H_x)$, $y \in \mathcal{C}_x$ a general point and $T \subset \mathbb{P}(T_X|_x^\vee)$ the projective tangent space to \mathcal{C}_x at y , then

$$\dim(T \cap \mathbb{P}(\mathcal{G}|_x^\vee)) \geq \frac{\text{rank}(\mathcal{G})}{\dim X}(n + 1) - 1. \tag{6.1}$$

We will show that this leads to a contradiction. Let

$$\psi : \mathbb{P}(T_X|_x^\vee) \setminus \mathbb{P}(\mathcal{G}|_x^\vee) \rightarrow \mathbb{P}^{\dim X - \text{rank}(\mathcal{G}) - 1}$$

be the projection from $\mathbb{P}(\mathcal{G}|_x^\vee)$ to a complementary linear space, and let q be the generic fiber dimension of $\psi|_{\mathcal{C}_x}$. We will give two estimates for q .

Estimate 1. Since a tangent vector in $T_{\mathcal{C}_x}|_y$ is in the kernel of the tangent map $T(\psi|_{\mathcal{C}_x})$ if the associated line in T intersects $\mathbb{P}(\mathcal{G}|_x^\vee)$, Equation (6.1) implies that the kernel of $T(\psi|_{\mathcal{C}_x})$ is of dimension

$$\dim \ker(T(\psi|_{\mathcal{C}_x})) \geq \frac{\text{rank}(\mathcal{G})}{\dim X}(n + 1).$$

As a consequence, we have

$$q \geq \frac{\text{rank}(\mathcal{G})}{\dim X}(n + 1). \tag{6.2}$$

Estimate 2. Let $T' \subset \mathbb{P}^{\dim X - \text{rank}(\mathcal{G}) - 1}$ be the projective tangent space to the (smooth) point $\psi(y)$ of the image of ψ . Then $\psi^{-1}(T')$ is a linear projective subspace of dimension

$$\dim \psi^{-1}(T') = \dim T + \text{rank}(\mathcal{G}) = (\dim \mathcal{C}_x - q) + \text{rank}(\mathcal{G}).$$

This linear space is tangent to \mathcal{C}_x along the fiber of $\psi|_{\mathcal{C}_x}$ through y . Since \mathcal{C}_x is smooth by Theorem 1.1, Zak’s theorem on tangencies [Zak93] (see also [Hwa01, Theorem 2.7]) asserts that

$$\begin{aligned} \dim(\text{fiber of } \psi|_{\mathcal{C}_x} \text{ through } \alpha) &\leq \dim(\psi^{-1}(T')) - \dim \mathcal{C}_x \\ q &\leq (\dim \mathcal{C}_x - q + \text{rank}(\mathcal{G})) - \dim \mathcal{C}_x \\ q &\leq \frac{\text{rank}(\mathcal{G})}{2}. \end{aligned}$$

Application of the Estimates. Combining Estimate 2 with Estimate 1 in (6.2), we obtain

$$\frac{\text{rank}(\mathcal{G})}{\dim X}(n + 1) \leq \frac{\text{rank}(\mathcal{G})}{2}$$

$$2(n + 1) \leq \dim X.$$

But we have $\dim X = 2n + 1$, a contradiction. Corollary 1.2 is thus shown.

6.3 Proof of Corollary 1.3

This corollary follows directly from Theorem 1.1 and [Hwa01, Theorem 3.2].

ACKNOWLEDGEMENTS

The main ideas for this paper were perceived while the author visited the Korea Institute for Advanced Study in 2002. Details were worked out during a visit to the University of Washington, Seattle, and while the author was Professeur Invité at the Université Louis Pasteur in Strasbourg. The author wishes to thank his hosts, Olivier Debarre, Jun-Muk Hwang and Sándor Kovács, for the invitations, and for many discussions.

The two appendices would never have appeared without the help of many people. The author is particularly grateful to Brendan Hassett and Charles Walter, who suggested to use jets in order to describe the tangent map TP , and to Hubert Flenner and János Kollár for patiently answering questions by e-mail.

Appendix A. A description of the jet sequence

A.1 The first jet sequence

Let X be a complex manifold (not necessarily projective or compact) and $L \in \text{Pic}(X)$ a line bundle. Throughout the present paper, we use the definition for the first jet bundle $\text{Jet}^1(L)$ that was introduced by Kumpera and Spencer [KS72, ch. 2] and now seems to be standard in algebraic geometry (see also [BS95, Ch. 1.6.3]). One basic feature of the first jet bundle of L is the existence of a certain sequence of vector bundles, the first jet sequence of L ,

$$0 \longrightarrow \Omega_X^1 \otimes L \xrightarrow{\gamma} \text{Jet}^1(L) \xrightarrow{\delta} L \longrightarrow 0. \tag{A.1}$$

There exists a morphism of sheaves,

$$\text{Prolong} : L \rightarrow \text{Jet}^1(L),$$

called the ‘prolongation’, which makes (A.1) a split sequence of sheaves. The first jet sequence is, however, generally *not* split as a sequence of vector bundles, and the prolongation morphism is definitely not \mathcal{O}_X -linear. In fact, an elementary computation using the definition of $\text{Jet}^1(L)$ from [KS72] and the construction of differentials from [Mat89, ch. 25] yields that, for any open set $U \subset X$, any section $\sigma \in L(U)$ and function $g \in \mathcal{O}_X(U)$, we have

$$\text{Prolong}(g \cdot \sigma) = g \cdot \text{Prolong}(\sigma) + \gamma(dg \otimes \sigma).$$

A.2 Jets and logarithmic differentials

The definitions of [KS72] are well suited for algebraic computations. If we are to apply jets to deformation-theoretic problems, however, it seems more appropriate to follow an approach similar to that of Atiyah [Ati57], and to describe jets in terms of logarithmic tangents and differentials on the (projectivized) total space of the line bundle. We refer to [KPSW00, § 2.1] for a brief review

of Atiyah’s definitions. While the relation between [KS72] and our construction here is probably understood by experts, the author could not find any reference. A detailed description is therefore included here.

Set $Y := \mathbb{P}(L \oplus \mathcal{O}_X)$. We denote the natural \mathbb{P}^1 -bundle structure by $\pi : Y \rightarrow X$ and let $\Sigma = \Sigma_0 \cup \Sigma_\infty \subset Y$ be the union of the two disjoint sections that correspond to the direct sum decomposition. By convention, let Σ_∞ be the section whose complement $Y \setminus \Sigma_\infty$ is canonically isomorphic to the total space of the line bundle L .

Let $\Omega_Y^1(\log \Sigma)$ be the locally free sheaf of differentials with logarithmic poles along Σ . This sheaf, which contains Ω_Y^1 as a subsheaf, is defined and thoroughly discussed in [Del70, § II.3]. In particular, it is shown in [Del70, § II.3.3] that the sequence of relative differentials

$$0 \longrightarrow \pi^* \Omega_X^1 \longrightarrow \Omega_Y^1 \longrightarrow \Omega_{Y|X}^1 \longrightarrow 0$$

restricts to an exact vector bundle sequence of logarithmic tangents

$$0 \longrightarrow \pi^* \Omega_X^1 \longrightarrow \Omega_Y^1(\log \Sigma) \longrightarrow \Omega_{Y|X}^1(\log \Sigma) \longrightarrow 0.$$

Since $R^1 \pi_*(\pi^*(\Omega_X^1)) = 0$, we can push down to X , twist by L and obtain a short exact sequence as follows:

$$0 \longrightarrow \Omega_X^1 \otimes L \xrightarrow{\beta} \pi_* \Omega_Y^1(\log \Sigma) \otimes L \longrightarrow \pi_* \Omega_{Y|X}^1(\log \Sigma) \otimes L \longrightarrow 0. \tag{A.2}$$

We will show that sequence (A.2) is canonically isomorphic to the first jet sequence (A.1) of L .

THEOREM A.1. *With the notation from above, there exists an isomorphism of vector bundles*

$$\alpha : \pi_* \Omega_Y^1(\log \Sigma) \otimes L \rightarrow \text{Jet}^1(L)$$

such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_X^1 \otimes L & \xrightarrow{\beta} & \pi_* \Omega_Y^1(\log \Sigma) \otimes L & \longrightarrow & \pi_* \Omega_{Y|X}^1(\log \Sigma) \otimes L \longrightarrow 0 \\ & & \text{Identity} \downarrow & & \downarrow \alpha & & \\ 0 & \longrightarrow & \Omega_X^1 \otimes L & \xrightarrow{\gamma} & \text{Jet}^1(L) & \xleftarrow[\text{Prolong}]{\delta} & L \longrightarrow 0 \end{array} \tag{A.3}$$

commutes, i.e. $\gamma = \alpha \circ \beta$.

In Appendix B, where deformations of morphisms are discussed, we will need to consider tangents rather than differentials. For that reason, we state a ‘dualized and twisted’ version of Theorem A.1. Recall from [Del70, § II.3] that the dual of $\Omega_Y^1(\log \Sigma)$ is the locally free sheaf $T_Y(-\log \Sigma)$ of vector fields on Y which are tangent to Σ .

COROLLARY A.2. *There exists an isomorphism A of vector bundles such that the diagram*

$$\begin{array}{ccc} \text{Jet}^1(L)^\vee \otimes L & \xrightarrow{\gamma^\vee \otimes \text{Id}_L} & T_X \\ \downarrow A & & \downarrow \text{Identity} \\ \pi_* T_Y(-\log \Sigma) & \xrightarrow{\text{tangent map } T\pi} & T_X \end{array}$$

commutes.

Informally speaking, we can say the following.

Summary A.3. A vector field on the manifold X comes from L -jets if and only if it lifts to a vector field on Y whose flow stabilizes Σ_0 and Σ_∞ .

Proof of Theorem A.1.

Setup. Let $U \subset X$ be an open set and $\sigma \in L(U)$ a nowhere-vanishing section. We will construct the isomorphism α locally at first by defining an \mathcal{O}_X -linear morphism

$$\alpha_{U,\sigma} : [\pi_*\Omega_Y^1(\log \Sigma) \otimes L](U) \rightarrow [\text{Jet}^1(L)](U),$$

which we will later show not to depend on the choice of the section σ . It will then follow trivially from the construction that the various $\alpha_{U,\sigma}$ glue together to give a morphism of vector bundles.

Throughout the proof of Theorem A.1, we constantly identify sections $[\pi_*\Omega_Y^1(\log \Sigma) \otimes L](U)$ with $[\Omega_Y^1(\log \Sigma) \otimes \pi^*(L)](\pi^{-1}(U))$. Likewise, we will use the letter σ to denote the subvariety of $\mathbb{P}(L \oplus \mathcal{O}_X)|_U$ that is associated with the section.

Definition of $\alpha_{U,\sigma}$. In order to define $\alpha_{U,\sigma}$, use the nowhere-vanishing section σ to introduce a bundle coordinate on $\pi^{-1}(U)$, which we can view as a meromorphic function z on $\pi^{-1}(U)$ with a single zero along Σ_0 and a single pole along Σ_∞ such that

$$\pi \times z : \pi^{-1}(U) \rightarrow U \times \mathbb{P}^1$$

is an isomorphism with $z|_\sigma \equiv 1$. The coordinate z immediately gives a differential form

$$d \log z := \frac{1}{z} dz \in [\Omega_Y^1(\log \Sigma)](\pi^{-1}(U))$$

with logarithmic poles along both components of Σ . Note that $d \log z$ yields a nowhere-vanishing section of the line bundle $\Omega_{Y|X}^1(\log \Sigma)$ of relative logarithmic differentials. As a consequence, there exists a relative vector field

$$\vec{v}_z \in [T_{Y|X}(-\log \Sigma)](\pi^{-1}(U))$$

with zeros along Σ which is dual to $d \log z$, i.e. $(d \log z)(\vec{v}_z) = 1$. In the literature, \vec{v}_z is sometimes denoted by $z \partial / \partial z$, but we will not use this notation here.

With these notations, if $\omega \in [\Omega_Y^1(\log \Sigma) \otimes \pi^*(L)](\pi^{-1}(U))$ is a $\pi^*(L)$ -valued logarithmic form, set

$$\alpha_{U,\sigma}(\omega) := \underbrace{\gamma(\omega - d \log z \otimes \omega(\vec{v}_z))}_{=: \theta} + z \circ \omega(\vec{v}_z) \cdot \text{Prolong}(\sigma).$$

As an explanation, we point out that $\omega(\vec{v}_z)$ is a section of $[\pi^*(L)](\pi^{-1}(U))$ so that we can regard $z \circ \omega(\vec{v}_z)$ as a function. It is an elementary calculation in coordinates to see that θ is a regular L -valued 1-form on $\pi^{-1}(U)$ that vanishes on relative tangents. We can therefore see θ as the pull-back of a uniquely determined L -valued 1-form on U . In particular, $\gamma(\theta)$ is a well-defined 1-jet in $[\text{Jet}^1(L)](U)$.

Injectivity. It follows immediately from the definition that $\alpha_{U,\sigma}$ is injective. Namely, if $\alpha_{U,\sigma}(\omega) = 0$, then the exactness of the second row of diagram (A.3) implies that

$$\theta = 0, \text{ i.e. that } \omega = \omega(\vec{v}_z) d \log(z)$$

and

$$z \circ \omega(\vec{v}_z) = 0, \text{ i.e. that } \omega(\vec{v}_z) = 0.$$

Together this implies that $\omega = 0$.

Coordinate change. Let $\tau \in L(U)$ be another nowhere-vanishing section, $\tau = g \cdot \sigma$ with $g \in \mathcal{O}_X^*(U)$. The section τ gives rise to a new bundle coordinate z' . We have

$$z' = \frac{1}{g} \cdot z, \quad d \log z' = d \log z - d \log g$$

and therefore

$$\vec{v}_{z'} = \vec{v}_z.$$

Using these equalities, it is a short computation to see that $\alpha_{U,\sigma}$ and $\alpha_{U,\tau}$ agree:

$$\begin{aligned} \alpha_{U,\tau}(\omega) &= \gamma(\omega - d \log z' \otimes \omega(\vec{v}_{z'})) - z' \circ \omega(\vec{v}_{z'}) \cdot \text{Prolong}(\tau) \\ &= \gamma(\omega - [d \log z - d \log g] \otimes \omega(\vec{v}_z)) \\ &\quad - \frac{z}{g} \circ \omega(\vec{v}_z) \cdot [g \cdot \text{Prolong}(\sigma) + \gamma(dg \otimes \sigma)] \\ &= \gamma(\omega - d \log z \otimes \omega(\vec{v}_z)) - z \circ \omega(\vec{v}_z) \cdot \text{Prolong}(\sigma) \\ &\quad + \gamma(d \log g \otimes \omega(\vec{v}_z)) - \frac{z}{g} \circ \omega(\vec{v}_z) \gamma(dg \otimes \sigma) \\ &= \alpha_{U,\sigma}(\omega) + \gamma\left(\underbrace{\left[d \log g - \frac{1}{g} dg \right]}_{=0} \otimes \omega(\vec{v}_z)\right). \end{aligned}$$

We have thus constructed an injective morphism of sheaves. We will later see that α is an isomorphism.

Commutativity of diagram (A.3). Let $\theta \in [\Omega_X^1 \otimes L](U)$. The image $\beta(\theta)$ is nothing but the pull-back of θ to $\pi^{-1}(U)$. In particular, if z is any bundle coordinate, we have that $\beta(\theta)(\vec{v}_z) \equiv 0$. Therefore

$$\begin{aligned} \alpha_{U,\sigma} \circ \beta(\theta) &= \gamma\left(\beta(\theta) - d \log z \otimes \underbrace{\beta(\theta)(\vec{v}_z)}_{=0}\right) - z \circ \underbrace{\beta(\theta)(\vec{v}_z)}_{=0} \cdot \text{Prolong}(\sigma) \\ &= \gamma(\beta(\theta)), \end{aligned}$$

where we again identify a form θ with its pull-back.

End of proof. It remains to show that the sheaf-morphism α is isomorphic, i.e. surjective. Because diagram (A.3) is commutative, to show that α is surjective, it suffices that $\delta \circ \alpha$ is surjective. Let $\sigma \in L(U)$ again be a nowhere-vanishing section and let $\tau \in L(U)$ be any section, $\tau = g \cdot \sigma$, where $g \in \mathcal{O}_X(U)$. We show that τ is in the image of $\delta \circ \alpha_{U,\sigma}$.

For this, let z be the bundle coordinate that is associated with σ and set

$$\omega := d \log z \otimes (g \cdot \sigma).$$

We have

$$\begin{aligned} \delta \circ \alpha_{U,\sigma}(\omega) &= \delta\left(\underbrace{\gamma(\cdots)}_{=0} + z \circ \omega(\vec{v}_z) \cdot \text{Prolong}(\sigma)\right) \\ &= \delta\left(\underbrace{z \circ \sigma}_{\equiv 1} \cdot g \cdot \underbrace{d \log z(\vec{v}_z)}_{=1} \cdot \text{Prolong}(\sigma)\right) \\ &= g \cdot \delta(\text{Prolong}(\sigma)) = g \cdot \sigma = \tau. \end{aligned}$$

The proof of Theorem A.1 is thus finished. □

Appendix B. Morphisms between polarized varieties

B.1 The tangent space to the Hom scheme

Let X be a complex-projective manifold, ℓ a projective variety and $f : \ell \rightarrow X$ a morphism. It is well known that there exists a scheme $\text{Hom}(\ell, X)$ that represents morphisms $\ell \rightarrow X$ (see e.g. [Kol96, ch. I]). In particular, there exists a functorial one-to-one correspondence between closed points of $\text{Hom}(\ell, X)$ and actual morphisms. As a consequence we have a ‘universal morphism’ $\text{Hom}(\ell, X) \times \ell \rightarrow X$. It is known that the tangent space to $\text{Hom}(\ell, X)$ is naturally identified with the space of sections in the pull-back of the tangent bundle

$$T_{\text{Hom}(\ell, X)}|_f \cong H^0(\ell, f^*(T_X)).$$

In the most intuitive setup, this identification takes the following form.

FACT B.1. *Let Δ be the unit disc with coordinate t and let*

$$\begin{aligned} f : \Delta &\rightarrow \text{Hom}(\ell, X) \\ t &\mapsto f_t \end{aligned}$$

be a family of morphisms. If $\mu : \Delta \times \ell \rightarrow X$ is the induced universal morphism, then

$$Tf \left(\frac{\partial}{\partial t} \Big|_{t=0} \right) \in f^*(T_{\text{Hom}(\ell, X)}|_{f_0}) \cong T_{\text{Hom}(\ell, X)}|_{f_0}$$

is naturally identified with

$$T\mu \left(\frac{\partial}{\partial t} \right) \Big|_{\{0\} \times \ell} \in H^0(\{0\} \times \ell, \mu^*(T_X)|_{\{0\} \times \ell}) \cong H^0(\ell, f_0^*(T_X)).$$

The identification has become so standard that we often wrongly write ‘equal’ rather than ‘naturally isomorphic’.

B.2 The pull-back of line bundles

In this paper we need to consider morphisms of polarized varieties. More precisely, we fix a line bundle $L \in \text{Pic}(X)$ and wish to understand the tangent space to fibers of the natural morphism

$$\begin{aligned} P : \text{Hom}(\ell, X) &\rightarrow \text{Pic}(\ell) \\ g &\mapsto g^*(L). \end{aligned}$$

It seems folklore among a handful of experts that the tangent map

$$TP|_f : \underbrace{T_{\text{Hom}(\ell, X)}|_f}_{=H^0(\ell, f^*(T_X))} \rightarrow \underbrace{T_{\text{Pic}(\ell)}|_{f^*(L)}}_{=H^1(\ell, \mathcal{O}_\ell)}$$

can be expressed in terms of the first jet sequence of L in the following way. Dualize sequence (A.1) and twist by L to obtain

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \text{Jet}^1(L)^\vee \otimes L \longrightarrow T_X \longrightarrow 0. \tag{B.1}$$

The tangent map TP is then the first connecting morphism in the long exact sequence associated to the f -pull-back of (B.1),

$$\dots \longrightarrow H^0(\ell, f^*(\text{Jet}^1(L)^\vee \otimes L)) \longrightarrow H^0(\ell, f^*(T_X)) \xrightarrow{TP} H^1(\ell, \mathcal{O}_\ell) \longrightarrow \dots$$

For lack of a reference, we will prove the following weaker statement here which is sufficient for our purposes. More details will appear in a forthcoming survey.

THEOREM B.2. *Let Δ be the unit disc with coordinate t and*

$$\begin{aligned} f : \Delta &\rightarrow \text{Hom}(\ell, X) \\ t &\mapsto f_t \end{aligned}$$

be a family of morphisms. Assume that there exists a line bundle $H \in \text{Pic}(\ell)$ such that for all $t \in \Delta$ we have $f_t^*(L) \cong H$. Then the tangent vector

$$Tf \left(\left. \frac{\partial}{\partial t} \right|_{t=0} \right) \in T_{\text{Hom}(\ell, X)}|_{f_0} = H^0(\ell, f_0^*(T_X))$$

is contained in the image of the morphism

$$H^0(\ell, f_0^*(\text{Jet}^1(L)^\vee \otimes L)) \rightarrow H^0(\ell, f_0^*(T_X))$$

which comes from the dualized and twisted jet sequence (B.1).

The proof of Theorem B.2 may look rather involved at first glance, but with the results of Appendix A, its proof takes little more than a good choice of coordinates on the projectivized line bundles and an unwinding of the definitions.

Proof of Theorem B.2.

Step 1. Consider the following diagram.

$$\begin{array}{ccc} \Delta \times \ell & \xrightarrow[\text{universal morphism}]{\mu} & X \\ \text{projection } \pi_2 \downarrow & & \\ \ell & & \end{array}$$

As a first step, we will find convenient coordinates on the pull-back of the \mathbb{P}^1 -bundle over X ,

$$\mu^*\mathbb{P}(L \oplus \mathcal{O}_X) \cong \mathbb{P}(\mu^*(L) \oplus \mathcal{O}_{\Delta \times \ell}).$$

Since $\text{Pic}(\Delta) = \{e\}$, there exists an isomorphism $\mu^*(L) \cong \pi_2^*(H)$ which induces an isomorphism

$$\mu^*\mathbb{P}(L \oplus \mathcal{O}_X) \cong \Delta \times \mathbb{P}(H \oplus \mathcal{O}_\ell).$$

We use these coordinates to write the base change diagram as follows.

$$\begin{array}{ccc} \Delta \times \mathbb{P}(H \oplus \mathcal{O}_\ell) & \xrightarrow{\tilde{\mu}} & \mathbb{P}(L \oplus \mathcal{O}_X) \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ \Delta \times \ell & \xrightarrow{\mu} & X \end{array} \tag{B.2}$$

For convenience of notation, write $Y := \mathbb{P}(L \oplus \mathcal{O}_X)$ and $Y' := \Delta \times \mathbb{P}(H \oplus \mathcal{O}_\ell)$. Let $\Sigma_\ell \subset Y'$ and $\Sigma_X \subset Y$ be the disjoint union of the sections that come from the direct sum decompositions. It is clear from the construction that $\tilde{\mu}(\Sigma_\ell) \subset \Sigma_X$.

Step 2. Recall from [Har77, Remark III.9.3.1] that there exists a natural morphism of sheaves

$$\alpha : \mu^*\pi_*(T_Y(-\log \Sigma_X)) \rightarrow \tilde{\pi}_*\tilde{\mu}^*(T_Y(-\log \Sigma_X)).$$

Although μ is not flat, we claim the following.

CLAIM B.3. *The map α is an isomorphism.*

Proof. Because the claim is local on the base, we can assume without loss of generality that the locally trivial \mathbb{P}^1 -bundle π is actually trivial. For trivial \mathbb{P}^1 -bundles, however, [Del70, Proposition II.3.2(iii)] shows that the logarithmic tangent sheaf decomposes as

$$T_Y(-\log \Sigma_X) \cong \mathcal{O}_Y \oplus \pi^*(T_X).$$

For these two sheaves, Claim B.3 follows easily from the commutativity of diagram (B.2) and from the projection formula. □

Step 3 (Conclusion). To avoid confusion, we name the canonical liftings of vector fields on Δ

$$\begin{aligned}\tau_{\text{up}} &:= \frac{\partial}{\partial t} \in H^0(Y', T_{Y'}), \\ \tau_{\text{down}} &:= \frac{\partial}{\partial t} \in H^0(\Delta \times \ell, T_{\Delta \times \ell}).\end{aligned}$$

In view of Fact B.1, to prove Theorem B.2, it suffices to show the following stronger statement.

CLAIM B.4. *The vector field*

$$T\mu(\tau_{\text{down}}) \in H^0(\Delta \times \ell, \mu^*(T_X))$$

is in the image of

$$\beta : H^0(\Delta \times \ell, \mu^*(\text{Jet}^1(L)^\vee \otimes L)) \rightarrow H^0(\Delta \times \ell, \mu^*(T_X)).$$

Use Corollary A.2 and the results of Step 2 to identify

$$\begin{aligned}H^0(\Delta \times \ell, \mu^*(\text{Jet}^1(L)^\vee \otimes L)) &\cong H^0(Y', \tilde{\mu}^*(T_Y(-\log \Sigma_X))) \\ H^0(\Delta \times \ell, \mu^*(T_X)) &\cong H^0(Y', \tilde{\mu}^* \pi^*(T_X)).\end{aligned}$$

These identifications make it easier to write down β . Namely, by Corollary A.2, β becomes nothing but the pull-back of the tangent map of π , i.e. $\beta = \tilde{\mu}^*(T\pi)$. Claim B.4 is thus reformulated as follows.

CLAIM B.5. *The vector field*

$$T\tilde{\mu}(\tau_{\text{up}}) \in H^0(Y', \tilde{\mu}^*(T_X))$$

is in the image of

$$\tilde{\mu}^*(T\pi) : H^0(Y', \tilde{\mu}^*(T_Y(-\log \Sigma_X))) \rightarrow H^0(Y', \tilde{\mu}^* \pi^*(T_X)).$$

In this formulation, the proof of Claim B.4, and hence of Theorem B.2, becomes trivial. The only thing to note is that

$$T\tilde{\mu}(\tau_{\text{up}}) \in H^0(Y', \tilde{\mu}^*(T_Y(-\log \Sigma_X))) \subset H^0(Y', \tilde{\mu}^*(T_Y)).$$

That, however, follows from the facts that $\tau_{\text{up}} \in H^0(Y', T_{Y'}(-\log \Sigma_\ell))$ and that $\tilde{\mu}(\Sigma_\ell) \subset \Sigma_X$. Theorem B.2 is thus shown. \square

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