STRONG HOMOMORPHISMS, CATEGORY THEORY, AND SEMANTIC PARADOX

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Abstract. In this essay we introduce a new tool for studying the patterns of sentential reference within the framework introduced in [2] and known as the language of paradox $\mathcal{L}_P$: strong $\mathcal{L}_P$-homomorphisms. In particular, we show that (i) strong $\mathcal{L}_P$-homomorphisms between $\mathcal{L}_P$ constructions preserve paradoxicality, (ii) many (but not all) earlier results regarding the paradoxicality of $\mathcal{L}_P$ constructions can be recast as special cases of our central result regarding strong $\mathcal{L}_P$-homomorphisms, and (iii) that we can use strong $\mathcal{L}_P$-homomorphisms to provide a simple demonstration of the paradoxical nature of a well-known paradox that has not received much attention in this context: the McGee paradox. In addition, along the way we will highlight how strong $\mathcal{L}_P$-homomorphisms highlight novel connections between the graph-theoretic analyses of paradoxes mobilized in the $\mathcal{L}_P$ framework and the methods and tools of category theory.

§1. Introduction. A body of recent work on the semantic paradoxes has focused on the patterns of sentential reference that do, or do not, generate paradoxes. One powerful tool for studying these inferential patterns is a graph-theoretic approach pioneered in [2] (hereafter referred to as the "Language of Paradox" or $\mathcal{L}_P$), which allows us to prove a number of interesting and important theorems regarding what sorts of patterns of reference give rise to paradoxes and other problematic semantic phenomena.[1] Three of the most powerful such results are:

1. A transitivity theorem: This result shows that any pattern of sentential reference whose dependency graph within $\mathcal{L}_P$ is transitive is paradoxical.
2. The unwinding theorem [2]: This result allows us to transform any finite, circular pattern of sentential reference into a corresponding infinite, non-looping pattern with the exact same status vis-a-vis paradoxicality, determinacy, or indeterminacy.
3. The Duality Theorem [2]: This theorem states that the result of replacing conjunctions within an $\mathcal{L}_P$ construction with disjunctions (obtaining a "dual" $\mathcal{L}_P^D$

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[1] Notable contributions to this approach include [1, 3, 8, 10].

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construction) has exactly the same status vis-a-vis paradoxicality, determinacy, or indeterminacy as the original (and vice versa).

While these results have provided deep insights into the structures that underlie semantic paradox and similar phenomena, they are limited. There remain a great many paradoxical constructions—including paradoxical infinitary constructions—that are neither transitive, nor unwindings of simpler finite structures within $\mathcal{L}_P$, nor duals of such finite structures or their unwindings, and hence resist explanation via these results. One such paradox is the $\mathcal{L}_P$ version of the McGee paradox (and, more convenient for our purposes, the dual of this paradox).

Thus, one task that will be undertaken within this paper is providing an analysis of (the dual of) the McGee paradox, thus filling this lacuna. In order to carry out this task, however, we will need to introduce a new and powerful tool within the $\mathcal{L}_P$ framework: strong $\mathcal{L}_P$-homomorphisms. The applications of strong $\mathcal{L}_P$-homomorphisms are not limited to the study of this single paradox, however, and thus we will take the time to explore various ways that strong $\mathcal{L}_P$-homomorphisms can be used to broaden our understanding of the patterns of sentential reference that generate paradoxes and other puzzles. In particular, we will see that we can give a very simple proof of a powerful theorem regarding $\mathcal{L}_P$-homomorphisms by re-formulating parts of the $\mathcal{L}_P$ framework within category theory, and this leads to the final task undertaken in the paper: to identify novel and potentially fruitful connections between existing methods for studying paradox such as $\mathcal{L}_P$ and the powerful methods and tools of contemporary category theory.

The essay will proceed as follows. In Section 2 we review the basics of the $\mathcal{L}_P$ approach; we introduce strong $\mathcal{L}_P$-homomorphisms; and we state, but not prove, the main result of this paper—the *strong $\mathcal{L}_P$-homomorphism theorem* (hereafter the S$\mathcal{L}_P$H theorem)—a result that entails that strong $\mathcal{L}_P$-homomorphisms preserve paradoxicality. Then, in Section 3, we recast the central notions involved in the $\mathcal{L}_P$ framework—including the notion of a strong $\mathcal{L}_P$-homomorphism—in category-theoretic terms, and provide an extremely simple category theoretic proof of our main theorem. In this section we will also show how a number of extant results are now merely applications of the S$\mathcal{L}_P$H theorem. In Section 4 we will show how to represent (the dual of) the McGee paradox within $\mathcal{L}_P$, and show that, although (the dual of) the McGee paradox is not transitive, it does exhibit another heretofore unrecognized condition that is sufficient for paradoxicality. In Section 5 we introduce a novel $\mathcal{L}_P$ construction—the *Infinite Flower*—that also exhibits this paradox-sufficient condition, but whose paradoxicality is much more obvious than that of (the dual of) the McGee construction, and we show how to apply the S$\mathcal{L}_P$H theorem to give an extremely simple alternative proof of the paradoxicality of (the dual of) the McGee paradox. The Appendix extends the examination of the connections between $\mathcal{L}_P$ and category theory via providing an alternative category-theoretic proof of the duality theorem.

§2. **The language of paradox $\mathcal{L}_P$.** Our first task is a brief review of a powerful framework for studying the patterns of sentential reference that generate semantic paradoxes: the language of paradox $\mathcal{L}_P$. $\mathcal{L}_P$ contains conjunction $\wedge$, a proper class of sentence names $S_\alpha, \alpha \in On$, and a falsity predicate F.[2] We allow arbitrarily long

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[2] On is the (proper) class of all ordinals.

infinitary conjunction—hence, given any non-empty class $A \subseteq \mathsf{On}$,

$$\wedge \{\mathsf{F}(\mathsf{S}_\alpha) : \alpha \in A\}$$

is a well-formed formula. This follows the treatment in [2, 3]. For present purposes, however, we will have no need for anything beyond countably infinite conjunctions. The well-formed formulas $\mathsf{WFF}_{\mathcal{L}_\mathsf{P}}$ are all and only the conjunctions of expressions of the form $\mathsf{F}(\mathsf{S}_\alpha)$, where a formula of the form $\mathsf{F}(\mathsf{S}_\alpha)$ alone is a treated as a one-conjunct conjunction and hence is in $\mathsf{WFF}_{\mathcal{L}_\mathsf{P}}$.

A quick note on notation: Officially, only the S_αs ($\alpha \in \mathsf{On}$) are sentence names in $\mathcal{L}_\mathsf{P}$. To improve clarity, however, we'll sometimes use other letters in the same sans serif font for sentences letters (e.g., L for the sentence name in the Liar paradox, Y_αs for sentence names in the Yablo paradox, and M_αs and $\mathsf{M}_\alpha^\mathsf{D}$s for sentence names in the McGee construction and its dual), and we will also use ordered n-tuples and other convenient constructions for indexing sentence letters. These more convenient notations should be understood as merely abbreviations for official $\mathcal{L}_\mathsf{P}$ constructions.

We obtain interesting (e.g., paradoxical) constructions within $\mathcal{L}_\mathsf{P}$ by applying a *denotation function* to $\mathcal{L}_\mathsf{P}$. Denotation functions assign well-formed formulas to sentence names:

DEFINITION 2.1. *A denotation function is any function*

$$\delta : \{\mathsf{S}_\alpha\}_{\alpha \in \mathsf{On}} \to \mathsf{WFF}_{\mathcal{L}_\mathsf{P}},$$

mapping each sentence name in $\mathcal{L}_\mathsf{P}$ to a well-formed formula of $\mathcal{L}_\mathsf{P}$.

Denotation functions allow us to reconstruct familiar semantic puzzles within $\mathcal{L}_\mathsf{P}$. For example, the Liar paradox is obtained via selecting any denotation function δ_L such that

$$\delta_\mathsf{L}(\mathsf{L}) = \mathsf{F}(\mathsf{L}).$$

Loosely put: Sentence name L names the sentence stating that (the sentence named by) L is false. We can express this more colloquially as

$$\mathsf{L} : \mathsf{F}(\mathsf{L}).$$

Similarly, we can reconstruct the Yablo paradox by selecting a denotation function δ_Y where, for all $n \in \mathbb{N}$,

$$\delta_\mathsf{Y}(\mathsf{Y}_n) = \wedge \{\mathsf{F}(\mathsf{Y}_m) : m \in \mathbb{N} \ \& \ m > n\},$$

and we can express this a bit more informally as

$$\mathsf{Y}_0 : \mathsf{F}(\mathsf{Y}_1) \wedge \mathsf{F}(\mathsf{Y}_2) \wedge \mathsf{F}(\mathsf{Y}_3) \wedge \cdots,$$
$$\mathsf{Y}_1 : \mathsf{F}(\mathsf{Y}_2) \wedge \mathsf{F}(\mathsf{Y}_3) \wedge \mathsf{F}(\mathsf{Y}_4) \wedge \cdots,$$
$$\mathsf{Y}_2 : \mathsf{F}(\mathsf{Y}_3) \wedge \mathsf{F}(\mathsf{Y}_4) \wedge \mathsf{F}(\mathsf{Y}_5) \wedge \cdots,$$
$$\text{etc.}$$

It will be useful to have some notation that tracks how a particular denotation function "behaves" with respect to a particular sentence or collection of sentences. Thus:
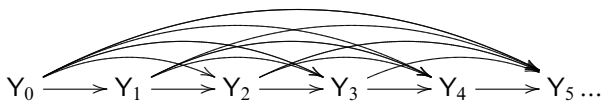
Fig. 1. An initial segment of Dep_{δ_Y}.

DEFINITION 2.2. *Given a denotation function δ, the dependency relation on δ is:*[3]

$$\mathsf{Dep}_\delta = \{\langle \mathsf{S}_\alpha, \mathsf{S}_\beta \rangle : \delta(\mathsf{S}_\alpha) = \wedge \{\mathsf{F}(\mathsf{S}_\gamma) : \gamma \in \Gamma\} \text{ and } \beta \in \Gamma\}.$$

The dependency relation Dep_δ tracks dependencies amongst the sentences in $\mathcal{L}_\mathsf{P}$ induced by the denotation function δ: if $\langle \mathsf{S}_\alpha, \mathsf{S}_\beta \rangle \in \mathsf{Dep}_\delta$, then the truth value of S_α depends (in part) on the truth value of S_β, since S_α refers to a conjunction containing S_β (according to δ). We will write $\mathsf{Dep}_\delta(\mathsf{S}_\alpha, \mathsf{S}_\beta)$ as shorthand for $\langle \mathsf{S}_\alpha, \mathsf{S}_\beta \rangle \in \mathsf{Dep}_\delta$. Figure 1 depicts (an initial segment of) the dependency relation for the $\mathcal{L}_\mathsf{P}$ version of the Yablo paradox (i.e., $\{\mathsf{Y}_n\}_{n\in\mathbb{N}}$ with δ_Y).[4]

It will be convenient to restrict our attention to collections of sentence names (with a denotation function) that are closed in the following sense:

DEFINITION 2.3. *Given a denotation function δ and a class of ordinals A, $\{\mathsf{S}_\alpha\}_{\alpha\in A}$ is closed under δ if and only if, for all $\alpha, \beta \in \mathsf{On}$, if $\alpha \in A$ and $\mathsf{Dep}_\delta(\mathsf{S}_\alpha, \mathsf{S}_\beta)$, then $\beta \in A$.*

Informally: A class of sentence names (with a denotation function) is closed if, whenever it contains a sentence name S_α, it contains all sentence names that S_α depends on. All examples discussed in the remainder of this essay are closed.

We evaluate the status of various (usually closed) collections of sentence names relative to a particular denotation function via considering assignments:

DEFINITION 2.4. *An assignment is any (total) function σ mapping the sentence names to truth values:*

$$\sigma : \{\mathsf{S}_\alpha\}_{\alpha\in\mathsf{On}} \to \{\top, \bot\}.$$

$\mathcal{L}_\mathsf{P}$-acceptability of an assignment is straightforward:

DEFINITION 2.5. *Given a denotation function δ, an assignment σ, and a class of ordinals A, σ is $\mathcal{L}_\mathsf{P}$-acceptable on $\{\mathsf{S}_\alpha\}_{\alpha\in A}$ with δ if and only if, for any $\alpha \in A$, $\sigma(\mathsf{S}_\alpha) = \top$ if and only if, for all β such that $\mathsf{Dep}_\delta(\mathsf{S}_\alpha, \mathsf{S}_\beta)$, $\sigma(\mathsf{S}_\beta) = \bot$.*

An assignment is $\mathcal{L}_\mathsf{P}$-acceptable when it assigns *truth* to a sentence name S_α if and only if it assigns *falsity* to every sentence name occurring in the conjunction denoted by S_α (and hence, intuitively, assigns *truth* to the conjunction itself).

We can now define what it means to say that an $\mathcal{L}_\mathsf{P}$ construction is paradoxical, determinate, or indeterminate:

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[3] Note that, since denotation functions are total, and we do not allow empty conjunctions, the dependency relation for a denotation function δ is always serial.

[4] Of course, since denotation functions are total, this only depicts (an initial segment of) the portion of the dependency relation in question that corresponds to the Yablo paradox.

DEFINITION 2.6.

- *A set of sentence names $\{S_\alpha\}_{\alpha \in A}$ with denotation function δ is paradoxical if there is no assignment σ that is $\mathcal{L}_P$-acceptable on $\{S_\alpha\}_{\alpha \in A}$ with δ.*
- *A set of sentence names $\{S_\alpha\}_{\alpha \in A}$ with denotation function δ is determinate if there is an assignment σ that is $\mathcal{L}_P$-acceptable on $\{S_\alpha\}_{\alpha \in A}$ with δ, and every σ that is $\mathcal{L}_P$-acceptable on $\{S_\alpha\}_{\alpha \in A}$ with δ agrees on the values assigned to each $S_\beta \in \{S_\alpha\}_{\alpha \in A}$.*
- *A set of sentence names $\{S_\alpha\}_{\alpha \in A}$ with denotation function δ is indeterminate if there exist assignments σ_1, σ_2 such that σ_1 and σ_2 are both $\mathcal{L}_P$-acceptable on $\{S_\alpha\}_{\alpha \in A}$ with δ and there is an $S_\beta \in \{S_\alpha\}_{\alpha \in A}$ such that $\sigma_1(S_\beta) \neq \sigma_2(S_\beta)$.*

It is now straightforward to prove that the $\mathcal{L}_P$ version of the Yablo paradox is paradoxical:

THEOREM 2.7 [2]. *$\{Y_n\}_{n \in \mathbb{N}}$ with δ_Y is paradoxical.*

Proof. Assume for *reductio* that there is an assignment function σ that is $\mathcal{L}_P$-acceptable on $\{Y_n\}_{n \in \mathbb{N}}$ with δ_Y.

Let $n \in \mathbb{N}$ be arbitrary. Assume for *reductio* that $\sigma(Y_n) = \top$. Then, for all Y_m such that $\mathsf{Dep}_{\delta_Y}(Y_n, Y_m)$, $\sigma(Y_m) = \bot$. $\mathsf{Dep}_{\delta_Y}(Y_n, Y_m)$ if and only if $n < m$. Let $j > n$. Then $\delta(Y_j) = \bot$. So, there is a Y_k such that $\mathsf{Dep}_{\delta_Y}(Y_j, Y_k)$ (i.e., $k > j$) and $\sigma(Y_k) = \top$. But then $\mathsf{Dep}_{\delta_Y}(Y_n, Y_k)$. So $\sigma(Y_k) = \bot$. Contradiction.

Since n was arbitrary, it follows that, for all $m \in \mathbb{N}$, $\sigma(Y_m) = \bot$. Let $j \in \mathbb{N}$ be arbitrary. Then $\sigma(Y_j) = \bot$. Then there is a Y_k such that $\mathsf{Dep}_{\delta_Y}(Y_j, Y_k)$ and $\sigma(Y_k) = \top$. Contradiction.   □

Actually, the paradoxicality of the Yablo paradox is merely a special case of the following phenomenon:[5]

THEOREM 2.8 [3]. *If Dep_δ is transitive on $\{S_\alpha\}_{\alpha \in A}$, then $\{S_\alpha\}_{\alpha \in A}$ (with δ) is paradoxical.*

Proof. Assume for *reductio* that there is an assignment function σ that is $\mathcal{L}_P$-acceptable on $\{S_\alpha\}_{\alpha \in A}$.

Let $j \in A$ be arbitrary. Assume for *reductio* that $\sigma(S_j) = \top$. Then, for all S_k such that $\mathsf{Dep}_\delta(S_j, S_k)$, $\sigma(S_k) = \bot$. Since Dep_δ is serial, there is a $k \in A$ such that $\mathsf{Dep}_\delta(S_j, S_k)$. So $\sigma(S_k) = \bot$. Again, by seriality, there is an $m \in A$ such that $\mathsf{Dep}_\delta(S_k, S_m)$ and $\sigma(S_m) = \top$. By transitivity, $\mathsf{Dep}_\delta(S_j, S_m)$. So $\sigma(S_m) = \bot$. Contradiction.

Since j was arbitrary, it follows that, for all $j \in A$, $\sigma(S_j) = \bot$. Let $k \in A$ be arbitrary. Then $\sigma(S_k) = \bot$. Then there is an $m \in A$ such that $\mathsf{Dep}_\delta(S_k, S_m)$ and $\sigma(S_m) = \top$. Contradiction.   □

Note that this result is unifying, in the sense that it identifies a property shared by many (but not all—see below!) paradoxes. For example, the $\mathcal{L}_P$ version of the Liar paradox is also transitive. Thus, $\mathcal{L}_P$ is a powerful tool for identifying general properties of paradoxes, allowing us to shift our attention away from isolated examples, and towards general types.

The connections between the Liar paradox and the Yablo paradox are actually much deeper than the mere fact that both are transitive, however, since the Yablo paradox is (isomorphic to) the "unwinding" of the Liar paradox. First, a definition:

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[5] A reminder: We are assuming that $\{S_\alpha\}_{\alpha \in A}$ with δ is closed.

DEFINITION 2.9. $\ll$ *is the lexicographic ordering on pairs of ordinals*:

$$\langle \alpha_1, \alpha_2 \rangle \ll \langle \beta_1, \beta_2 \rangle \text{ iff either: } \qquad \alpha_1 < \beta_1,$$
$$\text{or: } \qquad \alpha_1 = \beta_1 \text{ and } \alpha_2 < \beta_2.$$

Given an $\mathcal{L}_P$ construction, we can "unwind" it, obtaining an infinite analogue that contains no loops in its dependency relation:

DEFINITION 2.10. *The unwinding of* $\{S_\alpha\}_{\alpha \in A}$ *with* δ *is* $\{S_\alpha\}^U_{\alpha \in A}$ *with* δ^U *where*:

$$\{S_\alpha\}^U_{\alpha \in A} = \{S_{\langle n, \alpha \rangle} : n \in \mathbb{N} \text{ and } \alpha \in A\}$$

and

$$\delta^U(S_{\langle n, \alpha \rangle}) = \wedge\{S_{\langle m, \beta \rangle} : \langle n, \alpha \rangle \ll \langle m, \beta \rangle \text{ and } \mathsf{Dep}_\delta(S_\alpha, S_\beta)\}.$$

Given an assignment σ and a set of sentence names $\{S_\alpha\}_{\alpha \in A}$ with denotation function δ, the following construction provides us with a new assignment that treats each sentence name in $\{S_\alpha\}^U_{\alpha \in A}$ with δ^U analogously to the way σ treats its correlate in the original $\{S_\alpha\}_{\alpha \in A}$ with δ:

DEFINITION 2.11. *The* $\mathbb{N}$*-image of an assignment* σ *is* $\sigma^\mathbb{N}$ *where* (*for any* $\alpha \in \mathsf{On}, n \in \mathbb{N}$):

$$\sigma^\mathbb{N}(S_{\langle n, \alpha \rangle}) = \sigma(S_\alpha).$$

We now obtain the following striking result:[6]

THEOREM 2.12. *For any* $\{S_\alpha\}_{\alpha \in A}$ *with* δ *and assignment* σ, σ *is* $\mathcal{L}_P$*-acceptable on* $\{S_\alpha\}_{\alpha \in A}$ *with* δ *if and only if* $\sigma^\mathbb{N}$ *is* $\mathcal{L}_P$*-acceptable on* $\{S_\alpha\}^U_{\alpha \in A}$ *with* δ^U.

Proof. See [3, p. 143]. □

Further, it turns out that the assignments provided by Definition 2.11 are the only assignments (if any) that can be $\mathcal{L}_P$-acceptable on any unwinding:

DEFINITION 2.13. *An assignment* σ *is recurrent on* $\{S_\alpha\}^U_{\alpha \in A}$ *with* δ^U *if and only if, for any* $\alpha \in A$ *and* $n, m \in \mathbb{N}$:

$$\sigma(S_{\langle n, \alpha \rangle}) = \sigma(S_{\langle m, \alpha \rangle}).$$

THEOREM 2.14. *Any assignment* $\mathcal{L}_P$*-acceptable on* $\{S_\alpha\}^U_{\alpha \in A}$ *with* δ^U *is recurrent on* $\{S_\alpha\}^U_{\alpha \in A}$ *with* δ^U.

Proof. See [3, p. 142]. □

Thus, the following corollary is immediate:[7]

COROLLARY 2.15. $\{S_\alpha\}_{\alpha \in A}$ *with* δ *is paradoxical if and only if* $\{S_\alpha\}^U_{\alpha \in A}$ *with* δ^U *is paradoxical*.

This result allows us to further unify our explanation of at least some semantic paradoxes: Not only are the Liar and the Yablo paradox both instances of the transitivity-entails-paradoxicality phenomenon, but in addition, the latter is the

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[6] Cook [2] proves a version of this theorem, and the theorem regarding recurrent assignments discussed below, for unwindings of *finite* $\mathcal{L}_P$ constructions.

[7] And further, the number of acceptable assignments on $\{S_\alpha\}_{\alpha \in A}$ with δ is identical to the number of acceptable assignments on $\{S_\alpha\}^U_{\alpha \in A}$ with δ^U.

unwinding of the former, and hence will have the same status vis-a-vis paradoxicality, determinacy, or indeterminacy. As we shall see, there are other well-known semantic paradoxes that can be easily re-formulated within $\mathcal{L}_P$, but which are neither transitive nor the unwindings of any simpler $\mathcal{L}_P$ constructions.

Our next result concerns $\mathcal{L}_P^D$—the *dual* of the language of paradox $\mathcal{L}_P$. We obtain $\mathcal{L}_P^D$ by replacing the conjunctions in $\mathcal{L}_P$ with disjunctions. The definitions of denotation function, dependency relation, closure, and assignments remain the same, and the only modification required is adopting the following definition of $\mathcal{L}_P^D$-acceptability, which reverses the role of truth and falsity:

DEFINITION 2.16. *Given a denotation function δ, an assignment σ, and a class of ordinals A, σ is $\mathcal{L}_P^D$-acceptable on $\{S_\alpha\}_{\alpha \in A}$ with δ if and only if, for any $\alpha \in A$, $\sigma(S_\alpha) = \bot$ if and only if, for all β such that $\mathsf{Dep}_\delta(S_\alpha, S_\beta)$, $\sigma(S_\beta) = \top$.*

Thus, an assignment is $\mathcal{L}_P^D$-acceptable when it assigns *falsity* to a sentence name S_α if and only if it assigns *truth* to every sentence name occurring in the disjunction denoted by S_α (and hence, intuitively, assigns *falsity* to the disjunction itself).

The following definition and theorem provide the connection between constructions in $\mathcal{L}_P$ and constructions in $\mathcal{L}_P^D$.

DEFINITION 2.17. *Given an assignment σ, σ^D is the assignment*:

$$\sigma^D(S_\alpha) = \top, \qquad\qquad\qquad if\ \sigma(S_\alpha) = \bot,$$
$$= \bot, \qquad\qquad\qquad if\ \sigma(S_\alpha) = \top.$$

THEOREM 2.18. *Given a denotation function δ and assignment σ, σ is $\mathcal{L}_P$-acceptable on $\{S_\alpha\}_{\alpha \in A}$ with δ if and only if σ^D is $\mathcal{L}_P^D$-acceptable on $\{S_\alpha\}_{\alpha \in A}$ with δ.*

Simply put, replacing the conjunctions in an $\mathcal{L}_P$ construction with disjunctions (resulting in an $\mathcal{L}_P^D$ construction) preserves the status of the construction with respect to the paradoxicality/determinacy/indeterminacy trichotomy (and vice versa).

So far, we have only discussed results that are already well-known from the literature on $\mathcal{L}_P$. We now state a novel result, which we shall prove in the next section. First, some additional definitions:

DEFINITION 2.19. *A unary function*:

$$f : \{S_\alpha\}_{\alpha \in A} \to \{S_\beta\}_{\beta \in B}$$

is a strong $\mathcal{L}_P$-homomorphism from $\{S_\alpha\}_{\alpha \in A}$ with δ_1 to $\{S_\beta\}_{\beta \in B}$ with δ_2 if and only if:

- Homo: *For every $S_{\alpha_1}, S_{\alpha_2} \in \{S_\alpha\}_{\alpha \in A}$, if $\mathsf{Dep}_{\delta_1}(S_{\alpha_1}, S_{\alpha_2})$, then $\mathsf{Dep}_{\delta_2}(f(S_{\alpha_1}), f(S_{\alpha_2}))$.*
- Strong: *For every $S_{\alpha_1} \in \{S_\alpha\}_{\alpha \in A}, S_{\beta_1} \in \{S_\beta\}_{\beta \in B}$, if $\mathsf{Dep}_{\delta_2}(f(S_{\alpha_1}), S_{\beta_1})$, then there is an $S_{\alpha_2} \in \{S_\alpha\}_{\alpha \in A}$ such that $\mathsf{Dep}_{\delta_1}(S_{\alpha_1}, S_{\alpha_2})$ and $f(S_{\alpha_2}) = S_{\beta_1}$.*

DEFINITION 2.20. *A strong $\mathcal{L}_P$-homomorphism f from $\{S_\alpha\}_{\alpha \in A}$ with δ_1 to $\{S_\beta\}_{\beta \in B}$ with δ_2 is surjective if and only if*:

- Surj: *For every $S_{\beta_1} \in \{S_\beta\}_{\beta \in B}$, there is an $S_{\alpha_1} \in \{S_\alpha\}_{\alpha \in A}$ such that $f(S_{\alpha_1}) = S_{\beta_1}$.*

We will write:

$$\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle \xrightarrow{\ \mathsf{Str}\mathcal{L}_P\ } \!\!\!\!\twoheadrightarrow \langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$$

if there is a strong $\mathcal{L}_P$-homomorphism from $\{S_\alpha\}_{\alpha \in A}$ with δ_1 to $\{S_\beta\}_{\beta \in B}$ with δ_2, and

$$\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle \xrightarrow{\mathsf{StrSur}\mathcal{L}_P} \langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$$

if there is a strong surjective $\mathcal{L}_P$-homomorphism from $\{S_\alpha\}_{\alpha \in A}$ with δ_1 to $\{S_\beta\}_{\beta \in B}$ with δ_2.[8]

Given a strong $\mathcal{L}_P$-homomorphism f from $\{S_\alpha\}_{\alpha \in A}$ with δ_1 to $\{S_\beta\}_{\beta \in B}$ with δ_2, any assignment σ on $\{S_\beta\}_{\beta \in B}$ determines a corresponding assignment σ^f on $\{S_\alpha\}_{\alpha \in A}$:

DEFINITION 2.21. *Given a strong $\mathcal{L}_P$-homomorphism f from $\{S_\alpha\}_{\alpha \in A}$ with δ_1 to $\{S_\beta\}_{\beta \in A}$ with δ_2,*

$$\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle \xrightarrow{\mathsf{Str}\mathcal{L}_P} \langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle,$$

and any assignment σ, the assignment induced by f on $\{S_\alpha\}_{\alpha \in A}$ (with σ) is σ^f where:

$$\sigma^f(S_\alpha) = \sigma(f(S_\alpha)), \quad \text{if } \alpha \in A,$$
$$= \top, \qquad\qquad \text{otherwise.}$$

We now state (but do not yet prove) what we shall call the *strong $\mathcal{L}_P$-homomorphism theorem* (or $S\mathcal{L}_PH$ theorem):

THEOREM 2.22. *If*

$$\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle \xrightarrow{\mathsf{Str}\mathcal{L}_P} \langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle,$$

then (where f is any function witnessing the strong $\mathcal{L}_P$-homomorphism), for any assignment σ, if σ is $\mathcal{L}_P$-acceptable on $\{S_\beta\}_{\beta \in B}$ with δ_2 then σ, σ^f are $\mathcal{L}_P$-acceptable on $\{S_\alpha\}_{\alpha \in A}$ with δ_1.

We can strengthen the conditional to a biconditional by requiring that the strong $\mathcal{L}_P$-homomorphism be surjective:[9]

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[8] It is worth noting that $\xrightarrow{\mathsf{StrSur}\mathcal{L}_P}$ induces a partial order on equivalent classes of isomorphic $\mathcal{L}_P$ constructions. That is, if

$$[\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle]_\simeq = \{\langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle : \langle \{S_\beta\}_{\beta \in B}, \delta_1 \rangle \simeq \langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle\}$$

and

$$[\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle]_\simeq \xrightarrow{\mathsf{Str}\mathcal{L}_P[\simeq]} [\langle \{S_\beta\}_{\alpha \in A}, \delta_2 \rangle]_\simeq$$

if and only if there are

$$\langle \{S_\gamma\}_{\gamma \in G}, \delta_2 \rangle \in [\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle]_\simeq,$$
$$\langle \{S_\varepsilon\}_{\varepsilon \in E}, \delta_2 \rangle \in [\langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle]_\simeq,$$

such that

$$\langle \{S_\gamma\}_{\gamma \in G}, \delta_1 \rangle \xrightarrow{\mathsf{StrSur}\mathcal{L}_P} \langle \{S_\varepsilon\}_{\varepsilon \in E}, \delta_2 \rangle,$$

then $\xrightarrow{\mathsf{StrSur}\mathcal{L}_P[\simeq]}$ is a partial order on the space of $[\{S_\alpha\}_{\alpha \in A}, \delta]_\simeq$s. Additionally, this partial order contains least upper bounds. Proof of these facts is left to the reader.

[9] Note that Theorem 2.23 does not entail that, for any assignment σ_1 that is $\mathcal{L}_P$-acceptable on $\{S_\alpha\}_{\alpha \in A}$ with δ_1, there is corresponding assignment σ_2 that is $\mathcal{L}_P$-acceptable on $\{S_\beta\}_{\beta \in B}$ with δ_2. See the discussion of the Open Pair in Section 3 for a counterexample.

THEOREM 2.23. *If*

$$\langle\{S_\alpha\}_{\alpha\in A}, \delta_1\rangle \stackrel{\mathsf{StrSur}\mathcal{L}_\mathsf{P}}{\longrightarrow} \langle\{S_\beta\}_{\beta\in B}, \delta_2\rangle,$$

then (*where f is any function witnessing the strong $\mathcal{L}_\mathsf{P}$-homomorphism*), *for any assignment σ, σ^f is $\mathcal{L}_\mathsf{P}$-acceptable on $\{S_\alpha\}_{\alpha\in A}$ with δ_1 if and only if σ is $\mathcal{L}_\mathsf{P}$-acceptable on $\{S_\beta\}_{\beta\in B}$ with δ_2.*

We could prove the S$\mathcal{L}_\mathsf{P}$H theorem (and its strengthening Theorem 2.23) within the framework just described.[10] Here we shall take a somewhat different path. It turns out that these theorems can be easily reformulated and proven with very basic category-theoretic tools. Doing so is the task of the next section.

§3. $\mathcal{L}_\mathsf{P}$-homomorphisms and category theory. In this section we will reconstruct some of the machinery of the language of paradox $\mathcal{L}_\mathsf{P}$ within the category of sets Set, and use this reconstruction to prove the results given at the end of the previous section.

Before doing so, a historical note is in order. The results given in this section, and applied in the sections that follow, are not the first time that investigations of paradoxicality and related phenomenon have mobilized category-theoretic tools. Of particular note, in this regard, is [6], which proves a number of fixed-point theorems (including versions of Tarski's Theorem and Cantor's Theorem) within a category-theoretic setting. Thus, the idea that category theory has something to teach us about paradoxes is not new. The particular methods deployed below, however, are novel, and no doubt merely scratch the surface of what can be learned about semantic paradoxes via adopting a category-theoretic perspective.[11]

Back to the mathematics! Understood category-theoretically, a denotation function δ is just a $\mathcal{P}$-coalgebra from S to $\mathcal{P}(S)$ (where $\mathcal{P}$ is the (covariant) powerset functor $\mathcal{P}$ in Set) for some set S (understood as our set of sentence names).

Given this understanding of a denotation function, we can define acceptability as follows: Let Ω be the set of truth values (i.e., $\{\top, \bot\}$), and $\downarrow : \mathcal{P}(\Omega) \to \Omega$ be the following morphism from $\mathcal{P}(\Omega)$ to Ω:

$$\downarrow(\{\bot\}) = \downarrow(\varnothing) = \top,$$
$$\downarrow(\{\top\}) = \downarrow(\{\top, \bot\} = \Omega) = \bot,$$

$\downarrow$ is the category-theoretic analogue of the Pierce arrow expressing, in the present context (and put somewhat loosely), something like "all of the sentences are false."[12] Then, given a denotation function $\delta : S \to \mathcal{P}(S)$, an assignment is just a morphism

---

[10] In fact, in an early draft of this paper, we did just this. Thanks are owed to an anonymous referee for pointing out the connection to category theory, and urging us to explore the resulting connections between paradoxical patterns of reference and category-theoretic notions.

[11] In more detail: we do not claim that these results, viewed purely as theorems regarding the category Set, are either particularly novel or particularly deep. The novelty and importance of the material in the present section is tied to the *applications* of category theory to the study of paradoxes, not to the category-theoretic results taken in-and-of-themselves.

[12] Actually, our category-theoretic reconstruction is a bit more general than the original notion of a denotation function, since it allows one to map a sentence name to the empty conjunction (i.e., it might map a member of S to $\varnothing$). This is harmless, since the intended cases are covered.

$$
\begin{array}{ccc}
S_1 & \xrightarrow{\;\sigma\;} & \Omega \\
\delta \downarrow & & \uparrow \downarrow \\
\mathcal{P}(S_1) & \xrightarrow[\mathcal{P}\sigma]{} & \mathcal{P}(\Omega)
\end{array}
$$

Fig. 2. σ is $\mathcal{L}_{\mathsf{P}}$-acceptable on δ.

$$
\begin{array}{ccc}
S_1 & \xrightarrow{\;f\;} & S_2 \\
\delta_1 \downarrow & & \downarrow \delta_2 \\
\mathcal{P}(S_1) & \xrightarrow[\mathcal{P}f]{} & \mathcal{P}(S_2)
\end{array}
$$

Fig. 3. f is a strong homomorphism from δ_1 to δ_2.

$\sigma : S \to \Omega$, and an assignment σ is $\mathcal{L}_{\mathsf{P}}$-acceptable relative to a denotation function δ if and only if Figure 2 commutes.

Next, a strong homomorphism between denotation functions δ_1 and δ_2 is, in the category-theoretic setting, just a coalgebra morphism between the $\mathcal{P}$-coalgebras $\delta_1 : S_1 \to \mathcal{P}(S_1)$ and $\delta_2 : S_2 \to \mathcal{P}(S_2)$. $f : S_1 \to S_2$ is a coalgebra morphism from δ_1 to δ_2 if and only if Figure 3 commutes.

Finally, the additional requirement that a strong homomorphism be surjective, needed for the strengthened version of the $\mathsf{S}\mathcal{L}_{\mathsf{P}}\mathsf{H}$ theorem (i.e., Theorem 2.23), becomes, in the present context, the requirement that the coalgebra morphism f is an epimorphism.

We can now provide a very simple category-theoretic proof of the $\mathsf{S}\mathcal{L}_{\mathsf{P}}\mathsf{H}$ theorem:

THEOREM 3.1. *If $\delta_1 : S_1 \to \mathcal{P}(S_1)$ and $\delta_2 : S_2 \to \mathcal{P}(S_2)$ are $\mathcal{P}$-coalgebras in* Set, *and $f : S_1 \to S_2$ is a coalgebra morphism from δ_1 to δ_2, then, for any $\sigma : S_2 \to \Omega$, if σ is $\mathcal{L}_{\mathsf{P}}$-acceptable on δ_2 then $\sigma \circ f$ is $\mathcal{L}_{\mathsf{P}}$-acceptable on δ_1.*

Proof. Assume $f : S_1 \to S_2$ is a coalgebra morphism from $\delta : S_1 \to \mathcal{P}(S_1)$ to $\delta_2 : S_2 \to \mathcal{P}(S_2)$, and hence

$$
\begin{array}{ccc}
S_1 & \xrightarrow{\;f\;} & S_2 \\
\delta_1 \downarrow & & \downarrow \delta_2 \\
\mathcal{P}(S_1) & \xrightarrow[\mathcal{P}f]{} & \mathcal{P}(S_2)
\end{array}
$$

commutes. Now, assume σ is acceptable on $\langle S_2, \delta_2 \rangle$, and hence

$$
\begin{array}{ccc}
S_2 & \xrightarrow{\;\sigma\;} & \Omega \\
\delta_2 \downarrow & & \uparrow \downarrow \\
\mathcal{P}(S_2) & \xrightarrow[\mathcal{P}\sigma]{} & \mathcal{P}(\Omega)
\end{array}
$$

commutes. Then

$$S_1 \xrightarrow{\ f\ } S_2 \xrightarrow{\ \sigma\ } \Omega$$

(diagram: $S_1 \xrightarrow{f} S_2 \xrightarrow{\sigma} \Omega$, with vertical maps δ_1, δ_2, $\downarrow$ to $\mathcal{P}(S_1) \xrightarrow{\mathcal{P}f} \mathcal{P}(S_2) \xrightarrow{\mathcal{P}\sigma} \mathcal{P}(\Omega)$)

commutes, and hence

(diagram: $S_1 \xrightarrow{\sigma \circ f} \Omega$, with vertical maps δ_1, $\downarrow$ to $\mathcal{P}(S_1) \xrightarrow{\mathcal{P}\sigma \circ \mathcal{P}f = \mathcal{P}(\sigma \circ f)} \mathcal{P}(\Omega)$)

commutes. Hence $\sigma \circ f$ is $\mathcal{L}_\mathsf{P}$-acceptable on $\langle S_1, \delta_1 \rangle$. $\square$

Adding in surjectivity, we obtain the following category-theoretic version of Theorem 2.23:

THEOREM 3.2. *If* $\delta_1 : S_1 \to \mathcal{P}(S_1)$ *and* $\delta_2 : S_2 \to \mathcal{P}(S_2)$ *are* $\mathcal{P}$-*coalgebras in* Set, *and* $f : S_1 \to S_2$ *is a coalgebra epimorphism from* δ_1 *to* δ_2, *then, for any* $\sigma : S_2 \to \Omega$, σ_2, *if* $\sigma \circ f$ *is* $\mathcal{L}_\mathsf{P}$-*acceptable on* δ_1 *then* σ *is* $\mathcal{L}_\mathsf{P}$-*acceptable on* δ_2.

Proof. Assume $f : S_1 \to S_2$ is a coalgebra morphism from $\delta : S_1 \to \mathcal{P}(S_1)$ to $\delta_2 : S_2 \to \mathcal{P}(S_2)$, and hence

(diagram: $S_1 \xrightarrow{f} S_2$, with vertical maps δ_1, δ_2 to $\mathcal{P}(S_1) \xrightarrow{\mathcal{P}f} \mathcal{P}(S_2)$)

commutes, and assume further that f is an epimorphism, and hence, for any two $g, h : S_2 \to X$, if $gf = hf$, then $g = h$. Assume $\sigma \circ f$ is $\mathcal{L}_\mathsf{P}$-acceptable on $\langle S_1, \delta_1 \rangle$, and hence

(diagram: $S_1 \xrightarrow{\sigma \circ f} \Omega$, with vertical maps δ_1, $\downarrow$ to $\mathcal{P}(S_1) \xrightarrow{\mathcal{P}\sigma \circ \mathcal{P}f} \mathcal{P}(\Omega)$)

commutes. The second diagram gives us

$$\downarrow \circ\, \mathcal{P}\sigma \circ \mathcal{P}f \circ \delta_1 = \sigma \circ f,$$

while the first diagram provides

$$\delta_2 \circ f = \mathcal{P}f \circ \delta_1.$$

A simple substitution then gives us

$$\downarrow \circ\, \mathcal{P}\sigma \circ \delta_2 \circ f = \sigma \circ f,$$

which, since f is an epimorphism, provides

$$\downarrow \circ \mathcal{P}\sigma \circ \delta_2 = \sigma.$$

Hence,

$$
\begin{array}{ccc}
S_2 & \xrightarrow{\ \sigma\ } & \Omega \\
{\scriptstyle \delta_2}\downarrow & & \uparrow{\scriptstyle \downarrow} \\
\mathcal{P}(S_2) & \xrightarrow[\ \mathcal{P}\sigma\ ]{} & \mathcal{P}(\Omega)
\end{array}
$$

commutes, and σ is $\mathcal{L}_\mathsf{P}$-acceptable on $\langle S_2, \delta_2 \rangle$. $\qquad\square$

The following corollary is straightforward:

COROLLARY 3.3. *If* $\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle \xrightarrow{\ \mathsf{StrSur}\mathcal{L}_\mathsf{P}\ } \langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$ *then*

$$|\{\sigma : \sigma \text{ is } \mathcal{L}_\mathsf{P}\text{-acceptable on } \{S_\alpha\}_{\alpha \in A} \text{ with } \delta_1\}|$$
$$\geq |\{\sigma : \sigma \text{ is } \mathcal{L}_\mathsf{P}\text{-acceptable on } \{S_\beta\}_{\beta \in B} \text{ with } \delta_2\}|.$$

Proof. Assume that $\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle \xrightarrow{\ \mathsf{StrSur}\mathcal{L}_\mathsf{P}\ } \langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$, and let

$$f : \{S_\alpha\}_{\alpha \in A} \to \{S_\beta\}_{\beta \in B}$$

be any $\mathcal{L}_\mathsf{P}$-homomorphism witnessing this fact. Then, if σ_1 and σ_2 are distinct assignments that are $\mathcal{L}_\mathsf{P}$-acceptable on $\{S_\beta\}_{\beta \in B}$, then there is an $S_{\beta_1} \in \{S_\beta\}_{\beta \in B}$ such that

$$\sigma_1(S_{\beta_1}) \neq \sigma_2(S_{\beta_1}).$$

But then, where $S_{\alpha_1} \in \{S_\alpha\}_{\alpha \in A}$ is such that $f(S_{\alpha_1}) = S_{\beta_1}$, we have:[13]

$$\sigma_1^f(f(S_{\alpha_1})) = \sigma_1(S_{\beta_1}) \neq \sigma_2(S_{\beta_1}) = \sigma_2^f(f(S_{\alpha_1})).$$

Thus, the mapping $(\xi)^f$ (induced by the strong $\mathcal{L}_\mathsf{P}$ homomorphism f) from $\mathcal{L}_\mathsf{P}$-acceptable assignments on $\{S_\beta\}_{\beta \in B}$ with δ_2 to $\mathcal{L}_\mathsf{P}$-acceptable assignments on $\{S_\alpha\}_{\alpha \in A}$ with δ_1 is one-to-one (but not necessarily onto). $\qquad\square$

As hinted in the final line of this proof, the inequality between the cardinalities of the classes of $\mathcal{L}_\mathsf{P}$-acceptable assignments cannot be strengthened to an identity, as the following simple example shows. Select a denotation function δ that provides an instance of the Liar

$$\delta(\mathsf{L}) = \mathsf{F}(\mathsf{L})$$

and an instance of the Open Pair[14]

$$\delta(\mathsf{OP}_1) = \mathsf{F}(\mathsf{OP}_2),$$
$$\delta(\mathsf{OP}_2) = \mathsf{F}(\mathsf{OP}_1),$$

---

[13] Note that it is at this step that we require surjectivity.

[14] The Open Pair was first introduced (as the No-No Paradox) in [9]. Our choice of terminology reflects the fact that, as understood here, the Open Pair is not paradoxical (of course, this does not count against Sorensen's compelling argument that, despite its formal consistency, the Open Pair is pathological in a different sense).
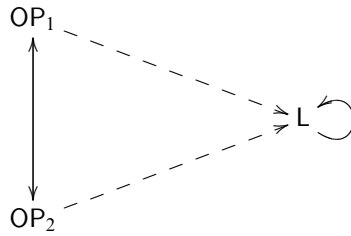
Fig. 4. A simple strong (surjective) $\mathcal{L}_P$-homomorphism.

then $f : \{OP_1, OP_2\} \to \{L\}$ where

$$f(OP_1) = f(OP_2) = L,$$

which is pictured in Figure 4, is a strong surjective $\mathcal{L}_P$-homomorphism from $\{OP_1, OP_2\}$ with δ to $\{L\}$ with δ. But the Liar is paradoxical, while the Open Pair is not (it is indeterminate, with two $\mathcal{L}_P$-acceptable truth value assignments, each of which assigns $\top$ to one sentence name and $\bot$ to the other).

This example of a strong surjective $\mathcal{L}_P$-homomorphism from a non-paradoxical $\mathcal{L}_P$ construction to a paradoxical one is an instance of the following fact:

THEOREM 3.4. *For any $\{S_\alpha\}_{\alpha \in A}$ with denotation function δ,*

$$\langle \{S_\alpha\}_{\alpha \in A}, \delta \rangle \overset{\mathsf{StrSur}\mathcal{L}_P}{\longrightarrow\!\!\!\rightarrow} \langle \{L\}, \delta_L \rangle.$$

Proof. Let

$$f(S_{\alpha_1}) = L$$

for any $S_{\alpha_1} \in \{S_\alpha\}_{\alpha \in A}$.

If $\mathrm{Dep}_\delta(S_{\alpha_1}, S_{\alpha_2})$, then, since $f(S_{\alpha_1}) = f(S_{\alpha_2}) = L$, and $\mathrm{Dep}_{\delta_L}(L, L)$, Homo is satisfied.

For Strong, the only case is $\mathrm{Dep}_{\delta_L}(f(S_{\alpha_1}), L)$ for some $S_{\alpha_1} \in \{S_\alpha\}_{\alpha \in A}$. Let $S_{\alpha_2} \in \{S_\alpha\}_{\alpha \in A}$ be such that $\mathrm{Dep}_\delta(S_{\alpha_1}, S_{\alpha_2})$ (the existence of such is guaranteed by the seriality of Dep_δ and the non-emptiness of $\{S_\alpha\}_{\alpha \in A}$). Then $f(S_{\alpha_2}) = L$.

f is clearly surjective. $\qquad \square$

Perhaps the simplest and most useful corollary of the $\mathsf{S}\mathcal{L}_P\mathsf{H}$ theorem is the following corollary:[15]

COROLLARY 3.5. *If $\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle \overset{\mathsf{Str}\mathcal{L}_P}{\longrightarrow\!\!\!\rightarrow} \langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$ and $\{S_\alpha\}_{\alpha \in A}$ with δ_1 is paradoxical, then $\{S_\beta\}_{\beta \in B}$ with δ_2 is paradoxical.*

Given this, we can use strong $\mathcal{L}_P$-homomorphisms to categorize $\mathcal{L}_P$ constructions in two (ultimately equivalent) ways:

---

[15] Note this corollary follows directly from the $\mathsf{S}\mathcal{L}_P\mathsf{H}$ theorem, and does not require surjectivity.

- We can show that $\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle$ is paradoxical by identifying an $\langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$ such that

$$\langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle \xrightarrow{\mathsf{Str}\mathcal{L}_\mathsf{P}} \langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle,$$

  where $\langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$ is known to be paradoxical.
- We can show that $\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle$ is non-paradoxical by identifying an $\langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$ such that

$$\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle \xrightarrow{\mathsf{Str}\mathcal{L}_\mathsf{P}} \langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle,$$

  where $\langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$ is known to be non-paradoxical.[16]

We can further demonstrate the usefulness of strong $\mathcal{L}_\mathsf{P}$-homomorphisms by noting that we can re-construe part of the content of the unwinding theorem as a mere special case of the $\mathsf{S}\mathcal{L}_\mathsf{P}\mathsf{H}$ theorem. We begin with the following observation:

THEOREM 3.6. *Given any* $\{S_\alpha\}_{\alpha \in A}$ *with* δ,

$$\langle \{S_\alpha\}_{\alpha \in A}^\mathsf{U}, \delta^\mathsf{U} \rangle \xrightarrow{\mathsf{Str}\mathcal{L}_\mathsf{P}} \langle \{S_\alpha\}_{\alpha \in A}, \delta \rangle.$$

Proof. The reader is invited to verify that

$$f : \{S_\alpha\}_{\alpha \in A}^\mathsf{U} \to \{S_\alpha\}_{\alpha \in A},$$

where

$$f(S_{\langle n, \alpha \rangle}) = S_\alpha$$

is a strong $\mathcal{L}_\mathsf{P}$-homomorphism from $\{S_\alpha\}_{\alpha \in A}^\mathsf{U}$ with δ^U to $\{S_\alpha\}_{\alpha \in A}$ with δ. □

Thus, given any $\mathcal{L}_\mathsf{P}$ construction $\{S_\alpha\}_{\alpha \in A}$ (with δ), there is a strong $\mathcal{L}_\mathsf{P}$-homomorphism from the unwinding of $\{S_\alpha\}_{\alpha \in A}$ (with δ) to $\{S_\alpha\}_{\alpha \in A}$ (with δ) itself. As a result, the right-to-left direction of Corollary 2.15 is also an immediate corollary of the $\mathsf{S}\mathcal{L}_\mathsf{P}\mathsf{H}$ theorem.[17]

Before moving on to our examination of the McGee paradox (and its dual), we further strengthen the connections between the $\mathcal{L}_\mathsf{P}$ approach and category theory by noting that the Duality Theorem can also be given a simple category-theoretic proof. The proof is given in the Appendix.

§4. The McGee paradox in $\mathcal{L}_\mathsf{P}$. The McGee paradox was originally formulated within the language of arithmetic with a truth predicate [7], and in that context the main point of interest in the construction is that it was the central component in a

---

[16] Note that the existence of a strong $\mathcal{L}_\mathsf{P}$-homomorphism from $\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle$ to $\langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$, where $\langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$ is indeterminate, entails that $\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle$ is indeterminate, but the existence of a strong $\mathcal{L}_\mathsf{P}$-homomorphism from $\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle$ to $\langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$, where $\langle \{S_\beta\}_{\beta \in B}, \delta_2 \rangle$ is determinate, only guarantees that $\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle$ is not paradoxical, but on its own is insufficient to determine whether $\langle \{S_\alpha\}_{\alpha \in A}, \delta_1 \rangle$ is determinate or indeterminate.

[17] Recall that, as we noted in Section 2, the left-to-right direction of Corollary 2.15 depends on a fact particular to unwindings: any assignment $\mathcal{L}_\mathsf{P}$-acceptable on the unwinding of an $\mathcal{L}_\mathsf{P}$ construction must be recurrent. Thus, we should not expect the other half of Corollary 2.15 to follow merely from the existence of the strong $\mathcal{L}_\mathsf{P}$-homomorphism.

proof that certain axiomatic theories of truth (in particular, FS; see [4]) are consistent but ω-inconsistent. Here our interests will be quite different, since we will be looking at the McGee Paradox in the context of $\mathcal{L}_P$, and we will, as a result, be implicitly assuming that we are working in the standard model of arithmetic (and hence that the McGee construction is genuinely paradoxical). Nevertheless, a quick reminder of McGee's original construction is in order.

Let $\mathcal{L}^T$ be any language sufficient for Peano Arithmetic (PA) supplemented with a unary truth predicate $T(\xi)$. $\ulcorner\Phi\urcorner$ is the name (or *Gödel code*) of Φ. We abbreviate iterated applications of the truth predicate as follows:

$$T^1(\xi) =_{df} T(\xi),$$
$$T^{n+1}(\xi) =_{df} T(\ulcorner T^n(\xi)\urcorner).$$

We define falsity in terms of truth and negation:

$$F(\xi) =_{df} \neg T(\xi).$$

And we abbreviate the iteration of falsity predicates along the same lines:

$$F^1(\xi) =_{df} F(\xi),$$
$$F^{n+1}(\xi) =_{df} F(\ulcorner F^n(\xi)\urcorner).$$

We now obtain the McGee Paradox via an application of Gödel's celebrated *diagonalization theorem* [5] to the predicate

$$\neg(\forall n \geq 1)T^n(\xi)$$

obtaining the *McGee sentence* M, where

$$\vdash_{PA} M \leftrightarrow \neg(\forall n \geq 1)T^n(\ulcorner M\urcorner).$$

Loosely put, the McGee Sentence M is true if and only if it is not the case that all sentences of the form:

> It is true that ...it is true that M

are true. Equivalently, M is true if and only if some sentence of that form is false—that is,

$$\vdash_{PA} M \leftrightarrow (\exists n \geq 1)\neg T^n(\ulcorner M\urcorner).$$

McGee's construction has a dual version that will be more convenient for our purposes, since it can be reconstructed relatively straightforwardly within $\mathcal{L}_P$ (McGee's original construction is more naturally reconstructed in the $\mathcal{L}_P^D$). The *Dual McGee Sentence* is M^D where (again via Gödelian diagonalization)

$$\vdash_{PA} M^D \leftrightarrow (\forall n \geq 1)\neg T^n(\ulcorner M^D\urcorner).$$

Loosely put, the Dual McGee Sentence M^D is true if and only if all sentences of the form:

> It is true that ...it is true that M^D

are false.

Calling this the "dual" of the original McGee construction is justified via noting that the above is equivalent to

$$\vdash_{\mathsf{PA}} \mathsf{M}^\mathsf{D} \leftrightarrow \neg(\exists n \geq 1)\mathsf{T}^n(\ulcorner\mathsf{M}^\mathsf{D}\urcorner).$$

In short, just as the move from $\mathcal{L}_\mathsf{P}$ to $\mathcal{L}_\mathsf{P}^\mathsf{D}$ involves switching conjunctions and disjunctions, the move from the original McGee construction to its dual involves replacing a universal quantifier (akin to an infinitary conjunction) with an existential quantifier (akin to an infinitary disjunction).

Given that we want to represent the McGee Paradox within $\mathcal{L}_\mathsf{P}$, it will be somewhat more convenient to work with the Dual of the McGee Paradox formulated in terms of our (defined) falsity predicate, rather than the (primitive) truth predicate.[18] The following is equivalent to the formulation of the McGee Sentence given above:

$$\vdash_{\mathsf{PA}} \mathsf{M}^\mathsf{D} \leftrightarrow \neg(\exists n)\mathsf{T}(\ulcorner\mathsf{T}^n(\ulcorner\mathsf{M}^\mathsf{D}\urcorner)\urcorner).$$

Replacing each occurrence of $\mathsf{T}(\xi)$ with $\mathsf{F}(\ulcorner\mathsf{F}(\xi)\urcorner)$, we obtain

$$\vdash_{\mathsf{PA}} \mathsf{M}^\mathsf{D} \leftrightarrow \neg(\exists n)\mathsf{F}(\ulcorner\mathsf{F}(\ulcorner\mathsf{F}^{2n}(\ulcorner\mathsf{M}^\mathsf{D}\urcorner)\urcorner)\urcorner).$$

Commuting the initial negation and the quantifier, we obtain

$$\vdash_{\mathsf{PA}} \mathsf{M}^\mathsf{D} \leftrightarrow (\forall n)\neg\mathsf{F}(\ulcorner\mathsf{F}(\ulcorner\mathsf{F}^{2n}(\ulcorner\mathsf{M}^\mathsf{D}\urcorner)\urcorner)\urcorner).$$

Finally, we can simplify this a bit, via "cancelling out" the initial negation-falsity prefix, obtaining the following *falsity variant* of the Dual McGee Sentence:[19]

$$\mathsf{M}^\mathsf{D} \equiv (\forall n)\mathsf{F}(\ulcorner\mathsf{F}^{2n}(\ulcorner\mathsf{M}^\mathsf{D}\urcorner)\urcorner).$$

Reconstructing the Dual of the McGee Paradox in $\mathcal{L}_\mathsf{P}$ is now straightforward: Our sentence names are $\{\mathsf{M}_n^\mathsf{D}\}_{n\in\mathbb{N}}$, and our denotation function $\delta_{\mathsf{M}^\mathsf{D}}$ is such that, for all $n \in \mathbb{N}$,

$$\delta_{\mathsf{M}^\mathsf{D}}(\mathsf{M}_0^\mathsf{D}) = \wedge\{\mathsf{F}(\mathsf{M}_{2m}^\mathsf{D}) : m \in \mathbb{N}\},$$
$$\delta_{\mathsf{M}^\mathsf{D}}(\mathsf{M}_{n+1}^\mathsf{D}) = \mathsf{F}(\mathsf{M}_n^\mathsf{D}).$$

We can write this a bit more colloquially as

$$\mathsf{M}_0^\mathsf{D} = \mathsf{F}(\mathsf{M}_0^\mathsf{D}) \wedge \mathsf{F}(\mathsf{M}_2^\mathsf{D}) \wedge \mathsf{F}(\mathsf{M}_4^\mathsf{D}) \wedge \mathsf{F}(\mathsf{M}_6^\mathsf{D}) \wedge \cdots,$$
$$\mathsf{M}_1^\mathsf{D} = \mathsf{F}(\mathsf{M}_0^\mathsf{D}),$$
$$\mathsf{M}_2^\mathsf{D} = \mathsf{F}(\mathsf{M}_1^\mathsf{D}),$$
$$\mathsf{M}_3^\mathsf{D} = \mathsf{F}(\mathsf{M}_2^\mathsf{D}),$$
$$\text{etc.}$$

---

[18] Results analogous to all those demonstrated with respect to the Dual of the McGee follow for McGee's original construction via the Duality Theorem.

[19] We can perform a similar transformation to obtain a falsity variant of the original McGee sentence

$$\mathsf{M} \equiv (\exists n)\mathsf{F}(\ulcorner\mathsf{F}^{2n}(\ulcorner\mathsf{M}\urcorner)\urcorner).$$

Details are left to the reader.
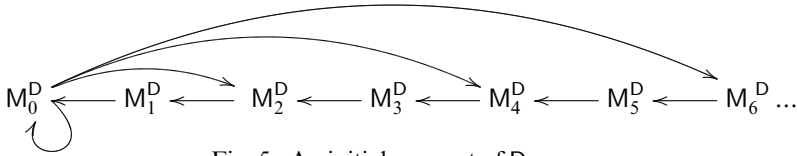
Fig. 5. An initial segment of $\mathsf{Dep}_{\delta_{\mathsf{M}^{\mathsf{D}}}}$.

Loosely speaking, $\mathsf{M}_0^{\mathsf{D}}$ "is" M^{D} in the informal construction from Section 1, and (again, loosely speaking) each $\mathsf{M}_{n+1}^{\mathsf{D}}$ "is" $\mathsf{F}(\mathsf{M}_n^{\mathsf{D}})$. Hence, for $n \geq 1$,

$$\mathsf{M}_n^{\mathsf{D}} \text{ "is" } \mathsf{F}^n(\mathsf{M}_0^{\mathsf{D}}),$$

and we can thus "rewrite" $\mathsf{M}_0^{\mathsf{D}}$ as

$$\mathsf{M}_0^{\mathsf{D}} \equiv \mathsf{F}(\mathsf{F}^0(\mathsf{M}_0^{\mathsf{D}})) \wedge \mathsf{F}(\mathsf{F}^2(\mathsf{M}_0^{\mathsf{D}})) \wedge \mathsf{F}(\mathsf{F}^4(\mathsf{M}_0^{\mathsf{D}})) \wedge \mathsf{F}(\mathsf{F}^6(\mathsf{M}_0^{\mathsf{D}})) \wedge \cdots$$

and hence as

$$\mathsf{M}_0^{\mathsf{D}} \equiv \mathsf{F}(\mathsf{M}_0^{\mathsf{D}}) \wedge \mathsf{F}^3(\mathsf{M}_0^{\mathsf{D}}) \wedge \mathsf{F}^5(\mathsf{M}_0^{\mathsf{D}}) \wedge \mathsf{F}^7(\mathsf{M}_0^{\mathsf{D}}) \wedge \cdots .$$

This is just (modulo some unimportant notational changes, and replacing an infinite conjunction with the corresponding universal quantification) the Dual of the McGee Sentence given above:

$$\mathsf{M} \equiv (\forall n)\mathsf{F}(\ulcorner \mathsf{F}^{2n}(\ulcorner \mathsf{M} \urcorner) \urcorner)$$

Figure 5 depicts (an initial segment of) the dependency relation for the $\mathcal{L}_{\mathsf{P}}$ version of the Dual of the McGee paradox (i.e., $\{\mathsf{M}_n^{\mathsf{D}}\}_{n \in \mathbb{N}}$ with $\delta_{\mathsf{M}^{\mathsf{D}}}$). Note that the dependency relation in the $\mathcal{L}_{\mathsf{P}}$ version of the Dual of the McGee paradox is not transitive. $\{\mathsf{M}_n^{\mathsf{D}}\}_{n \in \mathbb{N}}$ with $\delta_{\mathsf{M}^{\mathsf{D}}}$ is, however, paradoxical. In this section we give a direct proof of this fact—one that does not depend on the mobilization of strong $\mathcal{L}_{\mathsf{P}}$-homomorphisms:

LEMMA 4.1. *For any assignment function σ that is $\mathcal{L}_{\mathsf{P}}$-acceptable on $\{\mathsf{M}_n^{\mathsf{D}}\}_{n \in \mathbb{N}}$ with $\delta_{\mathsf{M}^{\mathsf{D}}}$, and any $n, m \in \mathbb{N}$, $\sigma(\mathsf{M}_n^{\mathsf{D}}) = \sigma(\mathsf{M}_{n+2m}^{\mathsf{D}})$.*

Proof. Straightforward induction (or via direct inspection of Figure 5), left to the reader. □

THEOREM 4.2. $\{\mathsf{M}_n^{\mathsf{D}}\}_{n \in \mathbb{N}}$ *with $\delta_{\mathsf{M}^{\mathsf{D}}}$ is paradoxical.*

Proof. Assume for *reductio* that there is an assignment σ that is $\mathcal{L}_{\mathsf{P}}$-acceptable on $\{\mathsf{M}_n^{\mathsf{D}}\}_{n \in \mathbb{N}}$ with $\delta_{\mathsf{M}^{\mathsf{D}}}$.
Assume that $\sigma(\mathsf{M}_0^{\mathsf{D}}) = \top$. Then, for all $\mathsf{M}_m^{\mathsf{D}}$ such that $\mathsf{Dep}_{\delta_{\mathsf{M}^{\mathsf{D}}}}(\mathsf{M}_0^{\mathsf{D}}, \mathsf{M}_m^{\mathsf{D}})$, $\sigma(\mathsf{M}_m^{\mathsf{D}}) = \bot$. In particular, $\sigma(\mathsf{M}_0^{\mathsf{D}}) = \bot$. Contradiction.
Thus, $\sigma(\mathsf{M}_0^{\mathsf{D}}) = \bot$. So there is an $m \in \mathbb{N}$ such that $\sigma(\mathsf{M}_{2m}^{\mathsf{D}}) = \top$. This contradicts our lemma. Contradiction.
Hence there is no such assignment. □

The paradoxicality of the Dual of the McGee paradox is, like the Yablo Paradox, a special case of a more general phenomenon. First, some additional definitions:[20]

DEFINITION 4.3. *Given a denotation function δ, an infinite sequence of ordinals*

$$\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \dots \alpha_n, \alpha_{n+1} \dots \rangle$$

is a δ-path if and only if, for all $m \in \mathbb{N}$, $\mathsf{Dep}_\delta(\mathsf{S}_{\alpha_m}, \mathsf{S}_{\alpha_{m+1}})$.

DEFINITION 4.4. *A δ-path*

$$\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \dots \alpha_n, \alpha_{n+1} \dots \rangle$$

is an n-repeater if and only if

$$\alpha_0 = \alpha_n$$

and there is no $m \in \mathbb{N}$ such that $0 < m < n$ and

$$\alpha_0 = \alpha_m.$$

THEOREM 4.5. *If $\{\mathsf{S}_\alpha\}_{\alpha \in A}$ (with δ) contains an S_β such that every δ-path beginning with β is an n-repeater for odd n, then $\{\mathsf{S}_\alpha\}_{\alpha \in A}$ (with δ) is paradoxical.*

Proof. Assume that $\{\mathsf{S}_\alpha\}_{\alpha \in A}$ (with δ) contains an S_β such that every δ-path beginning with β is an *n*-repeater for odd *n*. Assume for *reductio* $\{\mathsf{S}_\alpha\}_{\alpha \in A}$ (with δ) is not paradoxical. Then there is an assignment σ that is $\mathcal{L}_\mathsf{P}$-acceptable on $\{\mathsf{S}_\alpha\}_{\alpha \in A}$ (with δ). If $\sigma(\mathsf{S}_\beta) = \top$, then there is a δ-path

$$\beta, \alpha_1, \alpha_2, \alpha_3, \dots$$

such that

$$\sigma(\mathsf{S}_{\alpha_n}) = \top, \quad \text{if } n \text{ is even},$$
$$= \bot, \quad \text{if } n \text{ is odd}.$$

Since every δ-path beginning with β is an *n*-repeater for odd *n*, there is an *m* such that *m* is odd and $\mathsf{S}_\beta = \mathsf{S}_{\alpha_m}$. But then

$$\top = \sigma(\mathsf{S}_\beta) = \sigma(\mathsf{S}_{\alpha_m}) = \bot.$$

Contradiction. Hence $\sigma(\mathsf{S}_\beta) \neq \top$. A similar argument shows that $\sigma(\mathsf{S}_\beta) \neq \bot$. Hence, there is no such $\mathcal{L}_\mathsf{P}$-acceptable assignment. $\square$

This result, like the theorem regarding transitivity from the previous section, is also unifying, in the sense that it also identifies a property shared by many (but again, not all!) paradoxes. Note that the $\mathcal{L}_\mathsf{P}$ version of the Liar paradox is also a construction involving a sentence where every δ-path beginning with that sentence is an *n*-repeater for odd *n*. But, just as the denotation relation on the Dual of the McGee paradox is not transitive, the dependency relation on the $\mathcal{L}_\mathsf{P}$ version of the Yablo paradox does not contain any such *n*-repeaters (since it contains no loops at all!).

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[20] Note that we only need to consider infinite paths, since, given seriality (i.e., the fact that any denotation function δ is a total function on sentence names), any path can be extended to an infinite one.
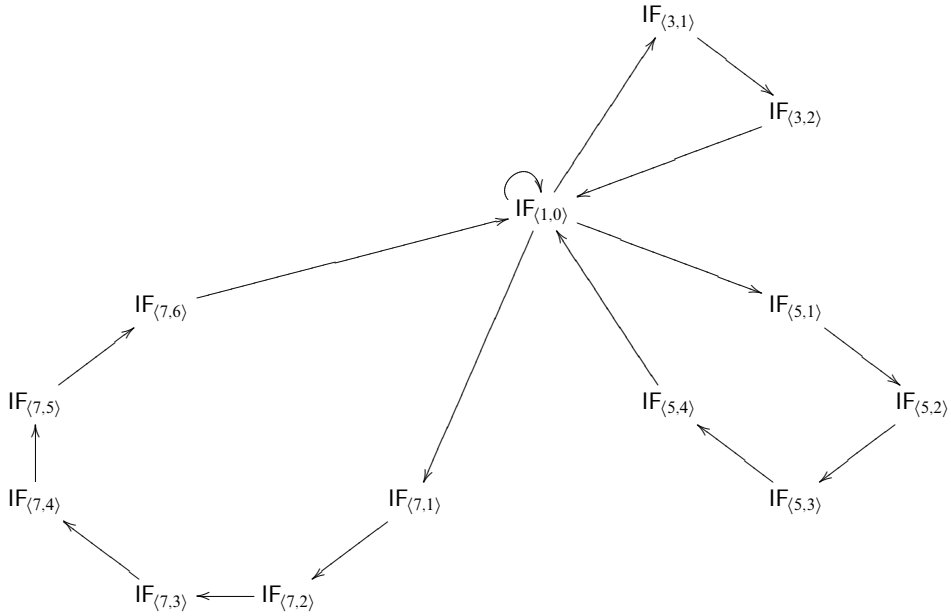
Fig. 6. An initial segment of $\mathrm{Dep}_{\delta_{\mathsf{IF}}}$.

Now that we have given a proof of the paradoxicality of (the Dual of) the McGee paradox using methods mobilized in extant work within the $\mathcal{L}_{\mathsf{P}}$ framework, it is time to see what can be done using our new tool—strong $\mathcal{L}_{\mathsf{P}}$-homomorphisms.

§5. The infinite flower and $\mathcal{L}_{\mathsf{P}}$-homomorphisms. We will begin this section by constructing a novel paradox: The Infinite Flower. First, we use the following pairs as indexes for the sentence names:

$$A = \{\langle n, m\rangle : 0 < m < n \text{ and } n \text{ odd, or } n = 1 \text{ and } m = 0\}.$$

We now define the denotation function as follows (using IF rather than S merely for aesthetics—it's a flower, after all):

$$\delta_{\mathsf{IF}}(\mathsf{IF}_{\langle n,m\rangle}) = \begin{cases} \{\mathsf{IF}_{\langle j,1\rangle} : j \text{ odd}\} \cup \{\mathsf{IF}_{\langle 1,0\rangle}\}, & \text{if } n = 1 \text{ and } m = 0, \\ \{\mathsf{IF}_{\langle n,m+1\rangle} : n \in \mathbb{N}\}, & \text{if } n \text{ odd and } 0 < m < n - 1, \\ \{\mathsf{IF}_{\langle 1,0\rangle}\}, & \text{if } n \text{ odd and } m = n - 1. \end{cases}$$

The structure of the Infinite Flower can be seen a bit more clearly in Figure 6, which represents its first four "petals."

Clearly, any path in the Infinite Flower beginning with $\mathsf{IF}_{\langle 0,1\rangle}$ is an n repeater for some odd n (the n is given by the first value of the pairs labeling the sentence names in the relevant "petal"). Hence the Infinite Flower is paradoxical, and is another instance of the sort of paradoxicality delineated by Theorem 4.5.

But we don't need all of the fancy machinery involved in Theorem 4.5 in order to see the paradoxicality of the Infinite Flower. There is a much more straightforward,

rather obvious route to this knowledge. First, we merely need to note the familiar fact that any "cyclic" $\mathcal{L}_P$ construction of odd length is paradoxical:

DEFINITION 5.1. *Given a denotation function δ, an finite sequence of n-many ordinals*

$$\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \ldots \alpha_{n-1} \rangle$$

is an n-cycle (with δ) if and only if:

$$\mathsf{Dep}_\delta = \{\langle \mathsf{S}_{\alpha_m}, \mathsf{S}_{\alpha_{m+1}} \rangle : 0 \le m < n-1\} \cup \{\langle \mathsf{S}_{\alpha_{n+1}}, \mathsf{S}_{\alpha_0} \rangle\}.$$

THEOREM 5.2. [3] *Given $\{\mathsf{S}_\alpha\}_{\alpha \in A}$ (with δ), if A (with δ) is an n-cycle for odd n, then $\{\mathsf{S}_\alpha\}_{\alpha \in A}$ (with δ) is paradoxical.*

The Liar sentence L (with δ_L) is a 1-repeater (hence paradoxical), while the Open Pair $\mathsf{OP}_1, \mathsf{OP}_2$ (with δ_{OP}) is a 2-cycle, hence not covered by this result.[21] Examples of *n*-cycles for lengths 3, 5, and 7 are given by the respective "petals" in Figure 6.

Now, once we have noted this very basic fact about odd cycles, the reader might fairly ask what it has to do with the topic at hand: the McGee paradox and strong $\mathcal{L}_P$-homomorphisms. The answer is simple: We can give an alternative argument to show that (the dual of) the McGee paradox is indeed paradoxical by noting two facts:

1. The Infinite Flower is paradoxical.
2. There is a strong $\mathcal{L}_P$-homomorphism from the Infinite Flower to dual of the McGee paradox.

The first fact is easily verified once one notes that the Infinite Flower consists merely of a bunch of *n*-cycles pasted together (one for each odd *n*)—filling in the rather trivial details is left to the reader. For the second fact:[22]

THEOREM 5.3.

$$\langle \{\mathsf{IF}_{\langle m,n \rangle} : m, n \in \mathbb{N}, m \text{ odd}, 0 < n < m\} \cap \{\mathsf{IF}_{\langle 1,0 \rangle}\}, \delta_{\mathsf{IF}} \rangle \xrightarrow{\mathsf{Str}\mathcal{L}_P} \langle \{\mathsf{M}_n^{\mathsf{D}} : n \in \mathbb{N}\}, \delta_{\mathsf{M}^{\mathsf{D}}} \rangle.$$

Proof. The reader is invited to verify that

$$f\left(\mathsf{IF}_{\langle m,n \rangle}\right) = \mathsf{M}_n$$

is a strong $\mathcal{L}_P$-homomorphism from the Infinite Flower to the Dual of the McGee paradox. $\square$

This result, plus the observation just made regarding the paradoxicality of the Infinite Flower, provide a simple proof of the paradoxicality of (the dual of) the McGee paradox via a quick application of the $\mathsf{S}\mathcal{L}_P\mathsf{H}$ theorem (and, in particular, of Corollary 3.5). Further, this proof is arguably a good bit simpler than the more direct proofs given in the previous section (via either Theorem 4.2 or an application of Theorem 4.5).

Thus, strong $\mathcal{L}_P$-homomorphisms are a powerful tool for showing that novel $\mathcal{L}_P$ constructions are paradoxical, as advertised. In order to show that an $\mathcal{L}_P$ construction $\{\mathsf{S}_\alpha\}_{\alpha \in A}$ (with δ) is paradoxical, we need only show some familiar or simpler

---

[21] Any *n*-cycle where *n* is even and non-zero is indeterminate, with exactly two $\mathcal{L}_P$-acceptable assignments. Again, see [3].

[22] The mapping in Theorem 5.3 is surjective, but we do not need that fact for the result.

construction whose paradoxicality has already been (or can easily be) established can be mapped to it by a strong $\mathcal{L}_P$-homomorphism.

§6. Conclusion. In this essay, we have demonstrated that strong $\mathcal{L}_P$-homomorphisms are a powerful tool within the $\mathcal{L}_P$ framework. They have allowed us to identify straightforward characterizations of, and connections between, constructions that have not been dealt with previously within the $\mathcal{L}_P$ approach, such as (the dual of) the McGee paradox and the Infinite Flower. In addition, strong $\mathcal{L}_P$-homomorphisms have illuminated new connections between various more familiar $\mathcal{L}_P$ constructions, and have allowed us to generalize and unify extant results such as the unwinding theorem (and the duality theorem, handled in the Appendix). Finally, they have provided us with the tools to begin to draw important connections between the patterns of sentential reference that generate paradox and the methods and tools of category theory. No doubt, however, much more work remains to be done.

Appendix: Duality and category theory. Here, we give a novel category-theoretic proof of the Duality Theorem (Theorem 2.18). Recall that the only distinction between $\mathcal{L}_P$ and $\mathcal{L}_P^D$ is that the former uses conjunctions whereas the latter uses disjunctions, and hence $\mathcal{L}_L^D$ mobilizes $\mathcal{L}_P^D$-acceptability instead of $\mathcal{L}_P$-acceptability. Thus, our first task is to formulate the notion of $\mathcal{L}_P^D$ in the category-theoretic setting. Let $| : \mathcal{P}(\Omega) \to \Omega$ be the morphism

$$|(\{\bot\}) = |(\{\top, \bot\} = \Omega) = \top,$$
$$|(\{\top\}) = |(\varnothing) = \bot.$$

$|$ is the category-theoretic analogue of the Sheffer stroke (expressing something like "at least one of the sentences is false"). Thus, given a denotation function δ (i.e., a coalgebra from S to $\mathcal{P}(S)$), an assignment σ (i.e., a morphism from S to Ω) is $\mathcal{L}_P^D$ acceptable on S relative to δ if and only if Figure 7 commutes.

Now, let $\neg : \Omega \to \Omega$ be the mapping:

$$\neg(\top) = \bot,$$
$$\neg(\bot) = \top.$$

Note that $\neg \circ \neg = id_\Omega$, and that $\neg$ is a monomorphism.[23] Given an assignment σ, the dual assignment σ^D given in Definition 2.17 is just $\neg \circ \sigma$. Recall that we defined the functor $\downarrow : \mathcal{P}(\Omega) \to \Omega$ as

$$\downarrow(\{\bot\}) = \downarrow(\varnothing) = \top,$$
$$\downarrow(\{\top\}) = \downarrow(\{\top, \bot\} = \Omega) = \bot.$$

The reader is invited to verify that

$$\neg \circ \downarrow \circ \mathcal{P}\neg = |,$$
$$\neg \circ | \circ \mathcal{P}\neg = \downarrow,$$

---

[23] $\neg$ is also an epimorphism, but we will not need this fact.

$$S_1 \xrightarrow{\sigma} \Omega$$
$$\delta \downarrow \qquad \uparrow |$$
$$\mathcal{P}(S_1) \xrightarrow[\mathcal{P}\sigma]{} \mathcal{P}(\Omega)$$

Fig. 7. σ is $\mathcal{L}_\mathsf{P}^\mathsf{D}$-acceptable on δ.

that is, the following diagrams commute:

$$\mathcal{P}(\Omega) \xrightarrow{|} \Omega \qquad\qquad \mathcal{P}(\Omega) \xrightarrow{\downarrow} \Omega$$
$$\mathcal{P}_\neg \downarrow \quad \uparrow \neg \qquad\qquad \mathcal{P}_\neg \downarrow \quad \uparrow \neg$$
$$\mathcal{P}(\Omega) \xrightarrow{\downarrow} \Omega \qquad\qquad \mathcal{P}(\Omega) \xrightarrow{|} \Omega$$

Now, in order to prove the duality theorem, we need to merely prove

> Given a denotation function δ and assignment σ, σ is $\mathcal{L}_\mathsf{P}$-acceptable on $\{S_\alpha\}_{\alpha\in A}$ with δ if and only if σ^D is $\mathcal{L}_\mathsf{P}^\mathsf{D}$-acceptable on $\{S_\alpha\}_{\alpha\in A}$ with δ.

Which, in the present context, amounts to proving that the diagram on the left commutes if and only if the diagram on the right commutes:

$$\mathcal{P}(\Omega) \xrightarrow{\sigma} \Omega \qquad\qquad \mathcal{P}(\Omega) \xrightarrow{\neg\circ\sigma} \Omega$$
$$\delta \downarrow \quad \uparrow \downarrow \qquad\qquad \delta \downarrow \quad \uparrow |$$
$$\mathcal{P}(\Omega) \xrightarrow[\mathcal{P}\sigma]{} \Omega \qquad\qquad \mathcal{P}(\Omega) \xrightarrow[\mathcal{P}(\neg\circ\sigma)]{} \Omega$$

But the diagram on the left commutes if and only if

$$S_2 \xrightarrow{\sigma} \Omega \xrightarrow{\neg} \Omega$$
$$\delta \downarrow \qquad\qquad \uparrow \neg$$
$$\mathcal{P}(S_2) \xrightarrow[\mathcal{P}\sigma]{} \mathcal{P}(\Omega) \xrightarrow[\downarrow]{} \Omega$$

commutes (since $\neg$ is a monomorphism), which commutes if and only if

$$S_2 \xrightarrow{\sigma} \Omega \xrightarrow{\neg} \Omega$$
$$\delta \downarrow \qquad\qquad\qquad\qquad\qquad \uparrow \neg$$
$$\mathcal{P}(S_2) \xrightarrow[\mathcal{P}\sigma]{} \mathcal{P}(\Omega) \xrightarrow[\mathcal{P}\neg]{} \mathcal{P}(\Omega) \xrightarrow[|]{} \Omega \xrightarrow{\neg} \Omega$$

commutes (since $\neg \circ | \circ \mathcal{P}\neg = \downarrow$), which commutes if and only if

$$
\begin{array}{ccc}
S_2 & \xrightarrow{\ \sigma\ } \Omega \xrightarrow{\ \neg\ } \Omega \\
\downarrow{\scriptstyle\delta_2} & & \uparrow{\scriptstyle |} \\
\mathcal{P}(S_2) & \xrightarrow[\mathcal{P}\sigma]{} \mathcal{P}(\Omega) \xrightarrow[\mathcal{P}\neg]{} \mathcal{P}(\Omega)
\end{array}
$$

commutes (since $\neg \circ \neg = id_\Omega$), which is just

$$
\begin{array}{ccc}
S_2 & \xrightarrow{\ \neg\circ\sigma\ } \Omega \\
\downarrow{\scriptstyle\delta_2} & & \uparrow{\scriptstyle |} \\
\mathcal{P}(S_2) & \xrightarrow[\mathcal{P}(\neg\circ\sigma)]{} \mathcal{P}(\Omega)
\end{array}
$$

Thus, the Duality Theorem is proven.

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